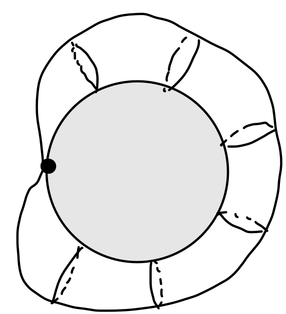
MTG 6346, Topology 1

Website: people.clas.ufl.edu/henry-adams/mtg6346-f2024

Syllabus (and office hours)

Chapter 0: Geometric notions

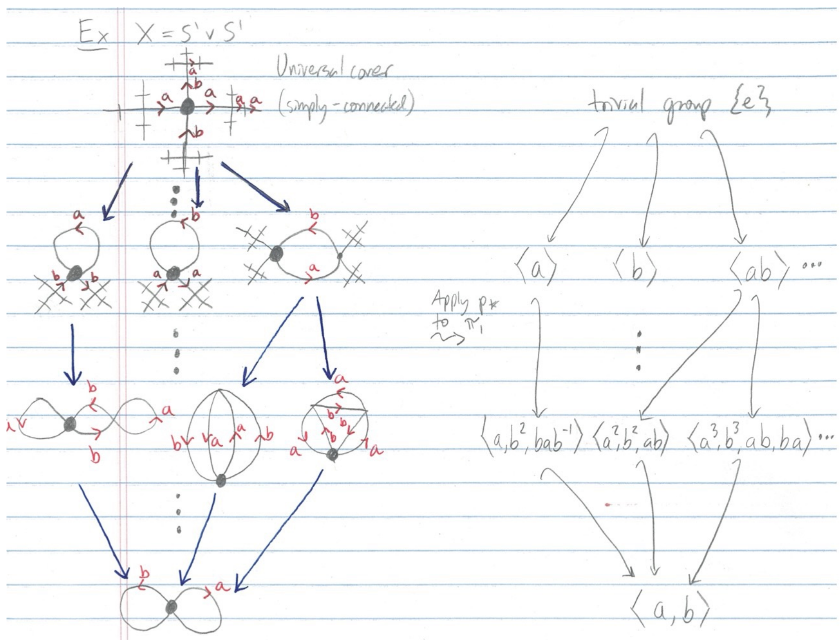
Cell complexes



Chapter 1: Fundamental group

Associates to each space X a group $\pi_1(X)$ measuring the 1-dimensional holes.

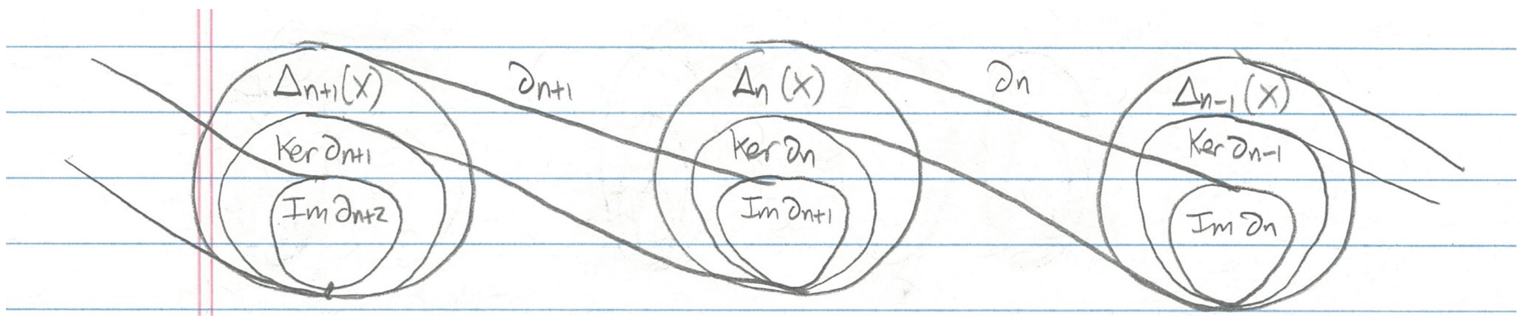
Covering spaces



Chapter 2: Homology

Associates to each space X the abelian groups (or vector spaces) $H_k(X)$ measuring the k-dimensional holes.

Hard to define, easy to compute.



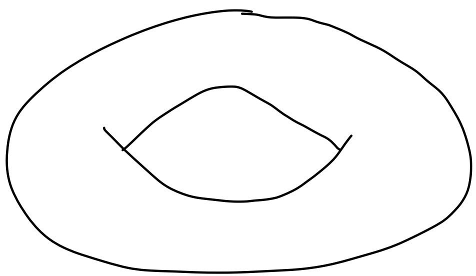
Singular vs simplicial homology.

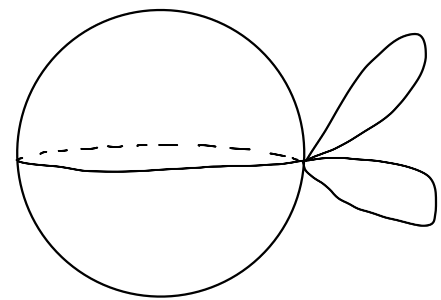
Homology with different coefficients.

Chapter 3: Cohomology

Associates to each space X a graded ring $H^*(X)$ measuring holes of all dimensions.

Hard to define, easy to compute.





Chapter 4: Homotopy groups

Associates to each space X the groups $\pi_k(X)$ measuring the k-dimensional holes.

Easy to define, hard to compute.

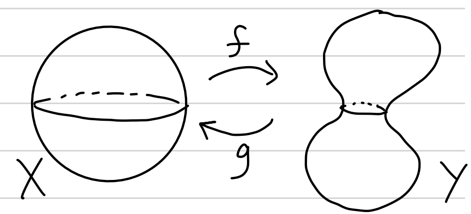
A positive result is Whitehead's Theorem:
If a (continuous) map $f: X \to Y$ between CW complexes induces isomorphisms $\pi_k(X) \longrightarrow \pi_k(Y)$ $\forall k$, then $X \simeq Y$.

Chapter 0: Geometric notions

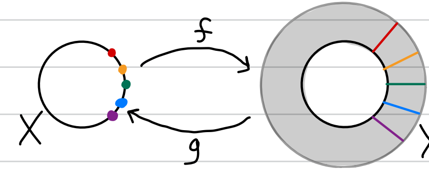
$\mathbb{1}: X \to X$ is the identity function on a set.
A map $f: X \to Y$ between topological spaces is a continuous function.

Homotopy and homotopy type

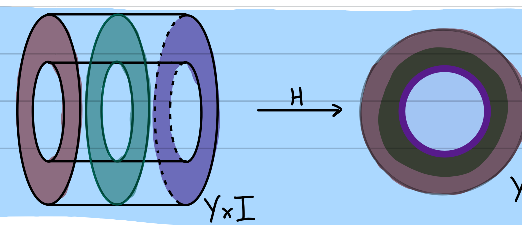
Maps $f: X \to Y$ and $g: Y \to X$ are homeomorphisms if $gf = \mathbb{1}_X$, $fg = \mathbb{1}_Y$.
Denoted $X \cong Y$, $X \approx Y$, or $X = Y$.



Def The maps f, g are homotopy equivalences if $gf \simeq \mathbb{1}_X$ and $fg \simeq \mathbb{1}_Y$.
We say X and Y are homotopy equivalent, denoted $X \simeq Y$.

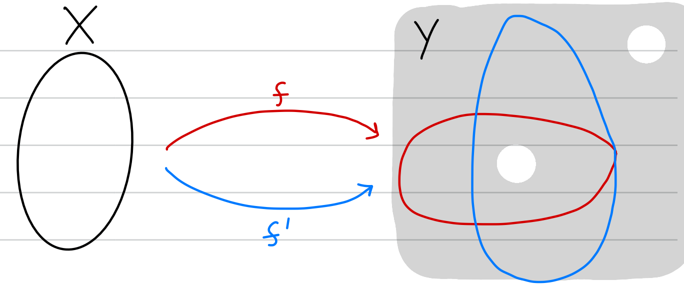


We still need to define homotopies between maps...



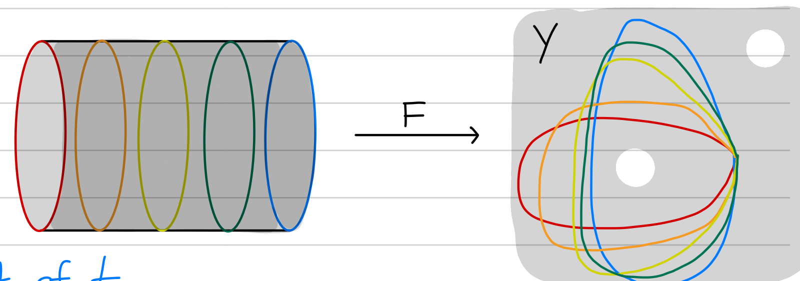
Let $I=[0,1]$ be the unit interval.

Def Maps $f_0, f_1 : X \to Y$ are *homotopic* $(f_0 \simeq f_1)$ if there is a map $F : X \times I \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ $\forall x \in X$.

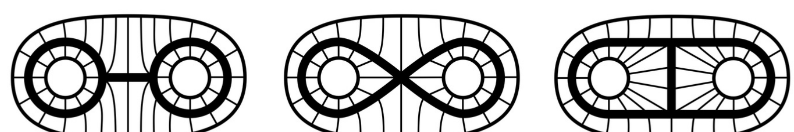


- Let $f_t : X \to Y$ via $f_t(x) = F(x,t)$

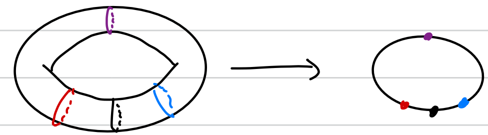
- f is *nullhomotopic* if it is homotopic to a constant map.



Let $A \subset X$. If $f_t|_A$ is independent of t, meaning $f_t(a) = f_{t'}(a)$ $\forall a \in A$ and $t, t' \in I$, then F is a *homotopy rel A*.



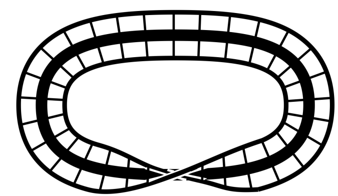
Let $A \subset X$. A retraction is a map $r: X \to X$ with $r(X) = A$ and $r|_A = \mathbb{1}_A$.



So $r^2 = r$. (Analogous to projections)

In algebraic topology, surjections b/w spaces need not induce surjections b/w groups, but retractions do.

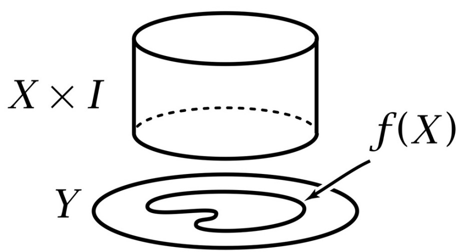
A deformation retraction of X onto A is a homotopy rel A from $\mathbb{1}_X$ to a retraction $r: X \to A$.

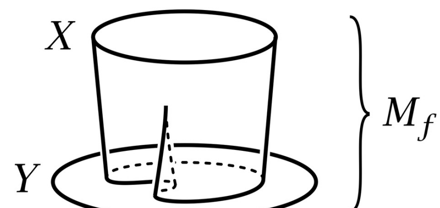


I.E., a deformation retraction is $F: X \times I \to X$ with $f_0 = \mathbb{1}_X$, $f_1(X) = A$, and $f_t|_A = \mathbb{1}_A \ \forall t$.

<u>Def</u> For $f: X \to Y$, the <u>mapping cylinder</u> is the quotient space

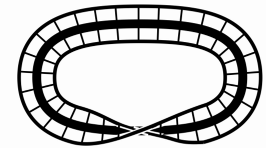
$$M_f = (X \times I) \amalg Y \Big/ (x,1) \sim f(x) \ \forall x \in X$$





- Clearly M_f deformation retracts onto Y.
- Corollary 0.21 shows if f is a homotopy equivalence, then M_f deformation retracts onto $X \times \{0\} = X$.
- Hence if $X \simeq Y$, then $\exists$ a third space Z that deformation retracts onto X and Y (choose $Z = M_f$).

Question Can the Möbius band be written as a mapping cylinder?

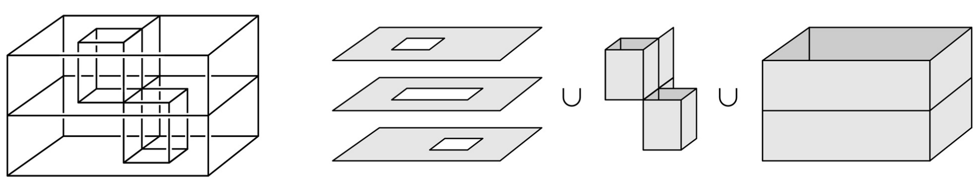


Answer: Yes, with f a 2-to-1 map from the circle to itself.

Question Let $A \subset X$.
If there is a retraction from X onto A, then is there a deformation retraction from X onto A?

Answer: Not in general. A deformation retraction is a homotopy equivalence but a retraction need not be.

Question Does Bing's house with two rooms deformation retract onto a point?



Answer: Yes, but it is not easy.

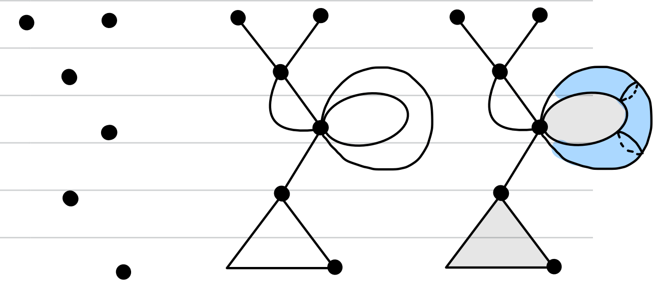
Cell complexes

Closed n-disk $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Boundary $(n-1)$-sphere $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$.

(Open) n-cell $e^n = D^n \setminus \partial D^n$.

(Note $e^0 = D^0 = pt$ since $\partial D^0 = \emptyset$.)



X^0 $\quad$ X^1 $\quad$ $X^2 = X$

Def A CW complex X is built by

(1) Starting with a discrete set X^0.

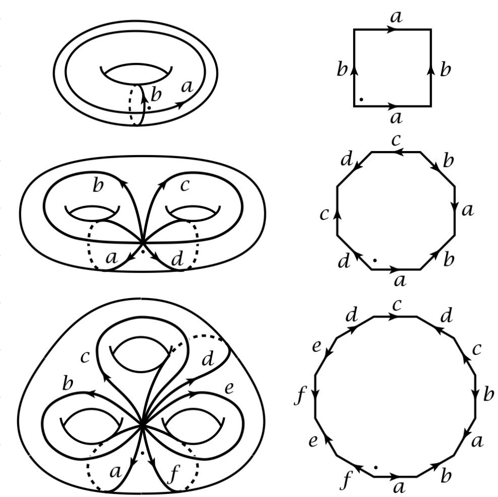
(2) Inductively forming the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via $\varphi_\alpha : S^{n-1} \to X^{n-1}$.

As a set, $\quad X^n = X^{n-1} \amalg_\alpha e^n_\alpha$.

As a space, $\quad X^n = \left(X^{n-1} \amalg_\alpha D^n_\alpha\right) / x \sim \varphi_\alpha(x) \ \forall x \in \partial D^n_\alpha$.

(3) Let $X = \bigcup_n X^n$. Give X the Weak topology:

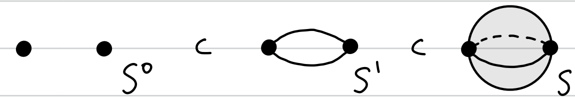
$A \subset X$ is open (resp. closed) in $X \Leftrightarrow A \cap X^n$ is open (resp. closed) in X^n $\forall n$.



- A consequence is Closure-finiteness: the closure of each cell intersects only finitely many cells.

Ex 0.3 The sphere S^n $(n \geq 1)$ has a CW structure with a 0-cell e^0 and an n-cell e^n attached via $S^{n-1} \to e^0$.

An alternate CW structure is two 0-cells, two 1-cells, two 2-cells, ..., two n-cells.



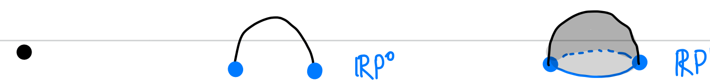
This allows us to define $S^\infty = \cup_n S^n$, which is contractible.
(Rmk: HW1. Could use Whitehead's theorem, but don't.)

Has a $\mathbb{Z}/2$ group action, whose orbit (quotient) space is:

Ex 0.4 Real projective n-space

$\mathbb{RP}^n = \{\text{all lines through origin in } \mathbb{R}^{n+1}\}$
$= (\mathbb{R}^{n+1} \setminus \{\vec{0}\}) \;/\; v \sim \lambda v \text{ for } 0 \neq \lambda \in \mathbb{R}$
$= S^n / (v \sim -v)$

$\mathbb{RP}^0 = e^0$ $\quad$ $\mathbb{RP}^1 = e^0 \cup e^1$ $\quad$ $\mathbb{RP}^2 = e^0 \cup e^1 \cup e^2$



It follows by induction that $\mathbb{RP}^n$ has a CW structure $e^0 \cup \ldots \cup e^n$ with one i-cell $\forall i \leq n$.

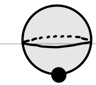
Ex 0.5 $\mathbb{RP}^\infty = \cup_n \mathbb{RP}^n$

Ex 0.6 Complex projective n-space

$\mathbb{C}P^n = \{\text{all complex lines through origin in } \mathbb{C}^{n+1}\}$
$= (\mathbb{C}^{n+1} \setminus \{\vec{0}\}) / v \sim \lambda v \text{ for } 0 \neq \lambda \in \mathbb{C}$
$= S^{2n+1} / v \sim \lambda v \text{ for } |\lambda| = 1.$

$\mathbb{C}P^0 = S^1/S^1$
$= pt$

$\mathbb{C}P^1 = S^3/$ points in certain circles identified
$= S^2$



Recall the Hopf fibration!

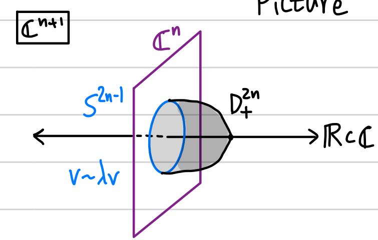
$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

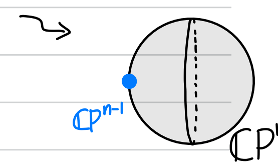
See Examples 4.44 and 4.45.

Note

- Each vector in S^{2n+1} is equivalent under $v \sim \lambda v$ with $|\lambda|=1$ to a vector $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative, and $|w| \leq 1$.
  (Indeed, if $v = (r_1 e^{i\theta_1}, \dots, r_{n+1} e^{i\theta_{n+1}}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, choose $\lambda = e^{-i\theta_{n+1}}$ to get $v \sim \lambda v = (r_1 e^{i(\theta_1 - \theta_n)}, \dots, r_{n+1}) \in \mathbb{C}^n \times \mathbb{R}$.)
- These vectors form a disk D^{2n}_+ bounded by the sphere $S^{2n-1} = \{(w,0) \mid |w| = 1\}$.
- When the last coordinate is zero, we have the remaining identifications $v \sim \lambda v$ for $v \in S^{2n-1}$ (i.e. $\mathbb{C}P^{n-1}$).

Picture $n=1$

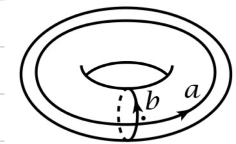




So $\mathbb{C}P^n = D^{2n}_+ / v \sim \lambda v$ for $v \in \partial D^{2n}_+ = S^{2n-1}$ and $|\lambda| = 1$
is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n} $(\varphi: S^{2n-1} \to \mathbb{C}P^{n-1})$.
By induction we have a CW structure $e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$ for $\mathbb{C}P^n$.

Operations on spaces

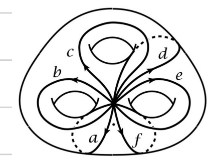
Let X, Y be CW complexes and $A \subset X$ be a subcomplex.

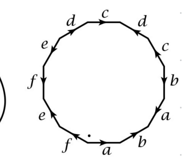


Product $X \times Y$ has CW structure with a $(n+m)$-cell $e^n \times e^m$ for each n-cell of X and m-cell of Y.

Quotient X/A has CW structure with one cell for each cell of $X \setminus A$, plus a 0-cell (for A).

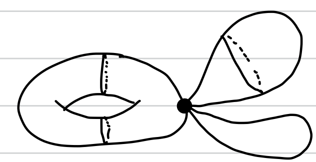
For example, the quotient of a surface by its 1-skeleton is S^2.





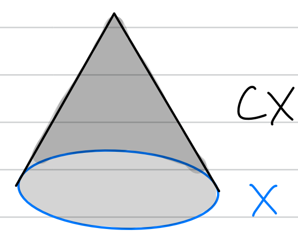
Wedge sum $X \vee Y$

For example, $X^n / X^{n-1} = \bigvee_\alpha S^n_\alpha$, with one n-sphere for each n-cell of X.



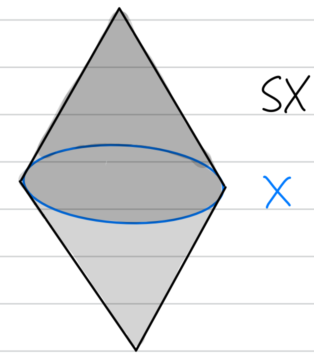
Cone $CX = (X \times I)/(X \times \{1\})$

CX is contractible.



<u>Suspension</u> $SX = C_+X \cup C_-X = (X \times [-1,1]) / {\scriptstyle (x,1)\sim(x',1) \text{ and } (x,-1)\sim(x',-1)\ \forall x,x'\in X}$

For example, $S(S^n) = S^{n+1}$.



Homology $H_{i+1}(SX) = H_i(X)$

Homotopy groups $\pi_{i+1}(SX) \neq \pi_i(X)$

Map $X \xrightarrow{f} Y$ gives map $SX \xrightarrow{Sf} SY$.

The homotopy classes $S^i \xrightarrow{f} S^n$ that remain after arbitrarily many suspensions $S^{i+k} \xrightarrow{S^k f} S^{n+k}$ are the "stable" homotopy groups of spheres.

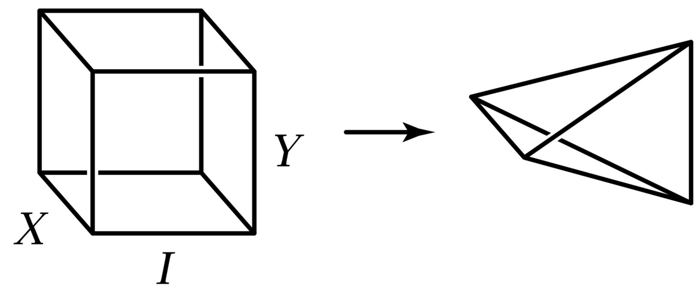
<u>Join</u> $X * Y = (X \times Y \times I) / {\scriptstyle (x,y,0)\sim(x,y',0),\ (x,y,1)\sim(x',y,1)}\ \forall x,x'\in X\ \forall y,y'\in Y$

<u>Ex</u> $X * pt = CX$

<u>Ex</u> $X * S^0 = SX$

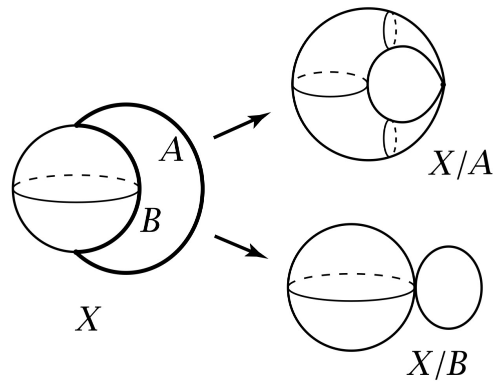
<u>Ex</u> $S^1 * S^1 = S^3$

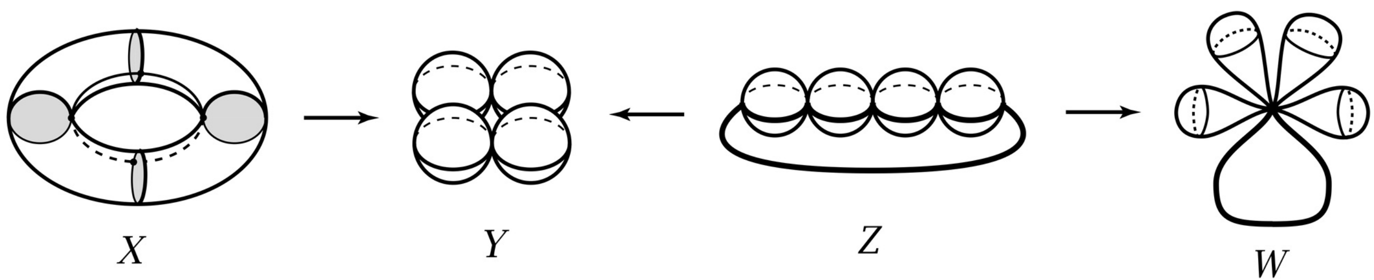
<u>Ex</u> $S^n * S^m = S^{n+m+1}$



Two criteria for homotopy equivalence

Collapsing subspaces: If (X,A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X/A$ is a homotopy equivalence.





Attaching spaces

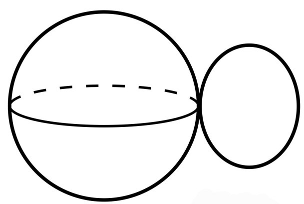
Let X, Y be spaces and $A \subset X$.

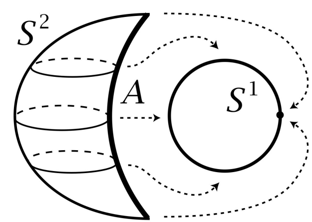
Let $f: A \to Y$ be a map.

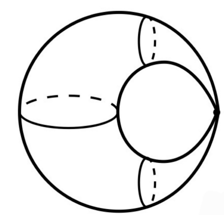
The *attaching space* (or *adjunction space*) is the quotient

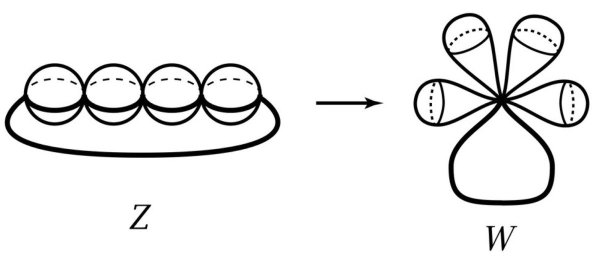
$$Y \cup_f X = (X \amalg Y) \,/\, a \sim f(a) \quad \forall a \in A.$$

Fact If (X, A) is a CW pair and $f, g: A \to Y$ are homotopic, then $Y \cup_f X \simeq Y \cup_g X$.









We are skipping the section on the homotopy extension property, though this important property is how many facts in Chapter 0 are proven.