MTG 4302-5316

Introduction to Topology 1

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Administrative

See the course syllabus and webpage.

Overview of course topics.

What is topology?
What is point-set topology?
What is algebraic topology?

Who are the students in our class?
What are your reasons for enrolling?
Which graduate students may take the first year exam?

Book: "Topology" by James Munkres, 2nd Edition.
I will be reading it; please do the same.

Thanks to Peter Bubenik, whose notes were extremely helpful when preparing my notes!

Chapter 1: Set Theory and Logic

Section 1: Sets

We'll do a "naive" but accurate treatment of sets.

Set	A
Element	$a \in A$
Subset	$B \subset A$ (equivalently $B \subseteq A$)
Proper subset	$B \subsetneq A$

A set can be an element of another set

$A \in \mathcal{A} \leftarrow$ collection

Power set $\mathcal{P}(A)$ is the set of all subsets of A.

(Can't consider the set of all sets, instead the class of all sets.)

Cartesian product

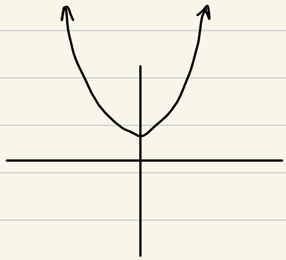
$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

Formally, $(a,b) = \{ \{a\}, \{a,b\} \}$

Section 2: Functions

A <u>function</u> $f: X \to Y$ is a subset of $X \times Y$ with each $x \in X$ appearing exactly once as the first coordinate of an ordered pair in this subset.

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2+1$



$\{(x, x^2+1) \in \mathbb{R} \times \mathbb{R}\}$

Formally, $g: \mathbb{R} \to \mathbb{R}_+$ is a different function, where
$x \longmapsto x^2+1$
$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

Section 3: Relations

Def A *relation* on a set A is a subset $C \subset A \times A$.

If $(x,y) \in C$, we write xCy and say
"x is related to y" or
"x is in the relation C to y".

An *equivalence relation* $\sim \subset A \times A$ is a relation satisfying

(1) $x \sim x$ (reflexive)

(2) $x \sim y \Rightarrow y \sim x$ (symmetric)

(3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ (transitive)

Equivalence classes form a *partition* of A

$\uparrow$

disjoint nonempty sets whose union is A

An *order relation* $\leq \subset A \times A$ is a relation satisfying

(1) Either $x \leq y$ or $y \leq x$

(2) $x \leq y$ and $y \leq x \Rightarrow x = y$

(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$

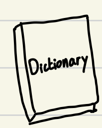
(called a *total order*)

Munkres uses $<$ as the primary symbol,
where $x < y$ when $x \leq y$ and $x \neq y$.

Ex $(\mathbb{R}, \leq)$

Ex $(\mathbb{R}^2, \leq)$ with the lexicographic order

$(a,b) \leq (c,d) \iff a < c$ or $a = c$ and $b \leq d$.



Section 4: Integers and real numbers

Def A <u>binary operation</u> on a set A is a function $f: A \times A \to A$.

There is a set of real numbers $\mathbb{R}$ with binary operations $+$, $\cdot$, and a linear order $\leq$ satisfying a list of axioms.

(Munkres assumes $\mathbb{R}$ exists.)

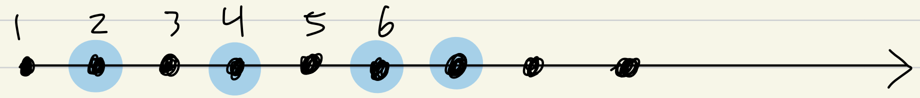
Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Positive integers $\mathbb{Z}_+$
Nonnegative integers $\mathbb{Z}_{\geq 0}$

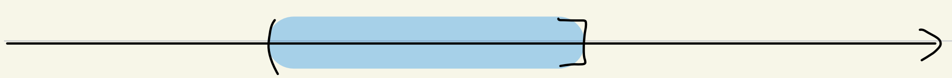
Def A set $A \subset \mathbb{R}$ is <u>inductive</u> if $1 \in A$, and if $x \in A \Rightarrow x+1 \in A$.

<u>Induction principle</u> If $A \subset \mathbb{Z}_+$ is inductive, then $A = \mathbb{Z}_+$.

Munkres uses the Inductive Principle to prove:

<u>Well-ordering property</u> Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.





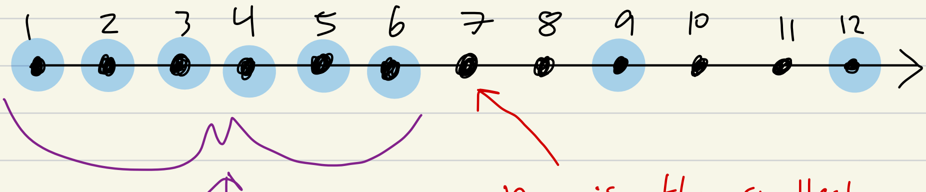
For $n \in \mathbb{Z}_+$, let S_n be the section
$S_n = \{1, 2, \ldots, n-1\}$.

So $S_{n+1} = \{1, 2, \ldots, n\}$, and $S_1 = \emptyset = \{\}$.

<u>Strong induction principle</u> Let $A \subset \mathbb{Z}_+$, and suppose $S_n \subset A$ implies $n \in A$ for all $n \in \mathbb{Z}_+$. Then $A = \mathbb{Z}_+$.

<u>Pf (using well-ordering)</u> If $A \subsetneq \mathbb{Z}_+$, then let n be the smallest integer in $\mathbb{Z}_+ - A \neq \emptyset$ (by well-ordering). So $S_n \subset A$, which implies $n \in A$, a contradiction. $\square$

$A \subsetneq \mathbb{Z}_+$



n is the smallest element not in A.

So $S_n \subset A$.
By assumption on A, this implies $n \in A$, a contradiction.

Section 5: Cartesian products

Main point: Define the notation $\{A_\alpha\}_{\alpha \in J}$, which is not a collection of sets since we allow repeats.

<u>Def</u> An <u>indexed family of sets</u> $\{A_\alpha\}_{\alpha \in J}$ consists of
- a nonempty collection of sets $\mathcal{A}$
- an <u>indexing set</u> J
- a surjective function $f: J \to \mathcal{A}$.

(for $\alpha \in J$, let $f(\alpha) = A_\alpha$)

We may have $A_\alpha = A_\beta$ for $\alpha \neq \beta$.

<u>Def</u> $\displaystyle\bigcup_{\alpha \in J} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in J\}$

Note this equals $\displaystyle\bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for some } A \in \mathcal{A}\}$
since f is surjective.

<u>Def</u> $\displaystyle\bigcap_{\alpha \in J} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in J\}$,

which equals $\displaystyle\bigcap_{A \in \mathcal{A}} A$ since f is surjective.

The two most important examples are when the index set is $J = \{1, 2, \dots, m\}$ or $J = \mathbb{Z}_+$.

<u>$J = \{1, 2, \dots, m\}$</u>

An <u>m-tuple</u> on a set X is a function $\{1, \dots, m\} \to X$, denoted $(x_1, \dots, x_m)$ with $x_i \in X$.

Let $\{A_1, \dots, A_m\}$ be a family of sets indexed by $\{1, \dots, m\}$.
Let $X = \bigcup_{i=1}^m A_i = A_1 \cup \dots \cup A_m$.
The <u>cartesian product</u> $\prod_{i=1}^m A_i = A_1 \times \dots \times A_m$ is

$$\{ \text{m-tuples } (x_1, \dots, x_m) \text{ of } X \mid x_i \in A_i \ \forall i \}.$$

<u>Ex</u> $\mathbb{R}^m$

<u>Ex</u> $S^1 \times [0, 1]$ is a cylinder

<u>$J = \mathbb{Z}_+$</u>

An <u>infinite sequence</u> or <u>ω-tuple</u> is a function $\mathbb{Z}_+ \to X$, denoted $(x_1, x_2, \dots)$ with $x_i \in X$.

Let $\{A_1, A_2, \dots\}$ be a family of sets indexed by $\mathbb{Z}_+$.
Let $X = \bigcup_{i \in \mathbb{Z}^+} A_i$.
The <u>cartesian product</u> $\prod_{i \in \mathbb{Z}^+} A_i$ is

$$\{ \omega\text{-tuples } (x_1, x_2, \dots) \text{ of } X \mid x_i \in A_i \ \forall i \}.$$

<u>Ex</u> $\mathbb{R}^\omega$

§5 Ex 3 Let $A = A_1 \times A_2 \times \dots$ and $B = B_1 \times B_2 \times \dots$

(a) Show if $B_i \subset A_i$ $\forall i$, then $B \subset A$.

(b) Show the converse holds if B is nonempty.

(Let $(b_1, b_2, \dots) \in B$. Fix $i \in \mathbb{Z}_+$ and suppose $x \in B_i$.
Note $(b_1, b_2, \dots, b_{i-1}, x, b_{i+1}, \dots) \in B \subset A$.
Hence $x \in A_i$, and so $B_i \subset A_i$)

(c) Show that if A is nonempty, then each A_i is nonempty.
Does the converse hold ? (axiom of countable choice)

(d) How does $A \cup B$ relate to $\prod_i (A_i \cup B_i)$?
How does $A \cap B$ relate to $\prod_i (A_i \cap B_i)$?

Section 6: Finite sets

Def A set A is *finite* if there is a bijection $f: A \xrightarrow{\cong} \{1, \ldots, n\}$ for $n \in \mathbb{Z}_+$, "A has cardinality n"
or if A is empty "A has cardinality 0"

Goal: Show the cardinality of a finite set is unique.

Lemma Let A be finite and $a_0 \in A$.
Then $\exists f: A \xrightarrow{\cong} \{1, \ldots, n+1\} \iff \exists g: A - \{a_0\} \xrightarrow{\cong} \{1, \ldots, n\}$.

Theorem Suppose $f: A \xrightarrow{\cong} \{1, \ldots, n\}$ and $B \subsetneq A$.
Then $\nexists\ g: B \xrightarrow{\cong} \{1, \ldots, n\}$.

(Book also proves:
And if $B \neq \emptyset$, $\exists\ h: B \xrightarrow{\cong} \{1, \ldots, m\}$ for some $m < n$.)

Pf Let $C \subset \mathbb{Z}_+$ be the set of all n for which the theorem is true. We will show C is inductive.
If $n = 1$, then $B = \emptyset$, and $\nexists\ g: \emptyset \xrightarrow{\cong} \{1\}$.
If theorem is true for n, we'll show true for $n+1$.
Let $f: A \xrightarrow{\cong} \{1, \ldots, n+1\}$, let $B \subsetneq A$.
If $B = \emptyset$, same as before.
If $B \neq \emptyset$, choose $a_0 \in B$ and $a_1 \in A - B$.
Apply lemma to get $A - \{a_0\} \xrightarrow{\cong} \{1, \ldots, n\}$.
Note $B - \{a_0\} \subsetneq A - \{a_0\}$ (consider a_1).
Since the theorem is true for n, $\nexists\, g: B - \{a_0\} \xrightarrow{\cong} \{1, \ldots, n\}$.
By lemma, $\nexists$ bijection $B \xrightarrow{\cong} \{1, \ldots, n+1\}$. $\square$

Corollary 1 If A is finite, there is no bijection of A with a proper subset of itself.

Pf
$$\begin{tikzcd} A \arrow[r, "\cong", "f"'] \arrow[d, "\cong"', "g"] & B \arrow[dl, "\cong", "g \circ f^{-1}"] \\ \{1, \dots, n\} & \end{tikzcd}$$
$g \circ f^{-1}$ would contradict Theorem.

Corollary 2 The cardinality of a finite set A is unique.

Pf For $m < n$, suppose we had bijections
$$\begin{tikzcd} A \arrow[r, "\cong", "f"'] \arrow[d, "\cong"', "g"] & \{1, \dots, n\} \\ \{1, \dots, m\} \arrow[ur, "\cong", "g \circ f^{-1}"'] & \end{tikzcd}$$
$g \circ f^{-1}$ would contradict Corollary 1.

Corollary 3 $\mathbb{Z}_+$ is not finite.

$$\begin{aligned} f : \mathbb{Z}_+ &\longrightarrow \mathbb{Z}_+ - \{1\} \\ n &\longmapsto n+1 \end{aligned}$$
is a bijection of $\mathbb{Z}_+$ with a proper subset.

From now on, we'll freely use basic facts about finite sets, such as:

Corollary 4 Set $A \neq \emptyset$ is finite
$\Leftrightarrow$ $\exists$ surjection $\{1, \dots, n\} \twoheadrightarrow A$ for some $n \in \mathbb{Z}_+$.
$\Leftrightarrow$ $\exists$ injection $A \hookrightarrow \{1, \dots, k\}$ for some $k \in \mathbb{Z}_+$.

Section 7: Countable and uncountable sets

Def A set A is
- infinite if it is not finite
- countably infinite if $\exists$ bijection $A \xrightarrow{\cong} \mathbb{Z}_+$
- countable if it is finite or countably infinite
- uncountable if it is not countable.

Ex $\mathbb{Z}$ is countably infinite.

$$\begin{array}{cccccccccccc}
\mathbb{Z}_+ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 2i & 2i+1 \\
\cong \downarrow f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
\mathbb{Z} & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \cdots & i & -i
\end{array}$$

Lemma If $C \subset \mathbb{Z}_+$ is infinite, then C is countably infinite.

Pf Define $f : \mathbb{Z}_+ \xrightarrow{\cong} C$,

Let $f(1)$ = smallest element of C.

If $f(1), \ldots, f(n-1)$ have been defined, then let $f(n)$ = smallest element of $C - \{f(1), \ldots, f(n-1)\}$.

This is called a recursive definition. Must do things "in order": certainly can't define $f(n)$ = smallest element of $C - \{f(1), \ldots, f(n)\}$.

f injective: For $m < n$, note $C - \{f(1), \ldots, f(n-1)\}$ contains $f(n)$ but not $f(m)$. So $f(m) \neq f(n)$.

<u>f surjective</u>: Let $c \in C$.

Note $f(\mathbb{Z}_+) \not\subseteq \{1, \dots, c\}$ since f injective.

Hence $\exists\, n \in \mathbb{Z}_+$ with $f(n) \geq c$

Let $m \in \mathbb{Z}_+$ be the <u>smallest</u> integer with $f(m) \geq c$.

So $\forall i < m$, $f(i) < c \Rightarrow c \notin \{f(1), \dots, f(m-1)\}$

$\Rightarrow f(m) \leq c$ by def^{n} of f.

Hence $f(m) = c$, as desired. $\square$

<u>Thm</u> For $B \neq \emptyset$, the following are equivalent:

(1) B is countable

(2) $\exists$ surjection $f: \mathbb{Z}_+ \twoheadrightarrow B$

(3) $\exists$ injection $g: B \hookrightarrow \mathbb{Z}_+$.

<u>Pf</u> $(1) \Rightarrow (2)$ B countably infinite $\exists\, f: \mathbb{Z}_+ \xrightarrow{\cong} B$ ✓

B finite ✓

$$\begin{tikzcd}
\{1, \dots, n\} \arrow[r, "\cong"] & B \\
\mathbb{Z}_+ \arrow[u, twoheadrightarrow] \arrow[ur]
\end{tikzcd}$$

$(2) \Rightarrow (3)$ Given $f: \mathbb{Z}_+ \twoheadrightarrow B$, define $g: B \longrightarrow \mathbb{Z}_+$ by

$g(b) =$ smallest element of $f^{-1}(b)$. (nonempty since f surjective)

Note g is injective since $b \neq b' \Rightarrow f^{-1}(b) \cap f^{-1}(b') = \emptyset \Rightarrow g(b) \neq g(b')$.

$(3) \Rightarrow (1)$

$$\begin{tikzcd}
g: B \arrow[rr, hookrightarrow] \arrow[dr, "\cong"'] & & \mathbb{Z}_+ \\
& \operatorname{image}(B) \arrow[ur, hookrightarrow] &
\end{tikzcd}$$

$\operatorname{image}(B) := \{n \in \mathbb{Z}_+ : n = g(b) \text{ for some } b \in B\}$

image(B) finite ✓

image(B) infinite — apply last lemma ✓

Corollary $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.

Pf Define $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$

$$(n, m) \longmapsto 2^n \cdot 3^m$$

Note f is injective by the uniqueness of prime factorizations.

Thm A finite product of countable sets is countable

Pf Proceed by induction.

Thm $\{0,1\}^\omega$ is uncountable (So the countable product of countable sets need not be countable.)

Pf Recall an element of $\{0,1\}^\omega$ is an infinite tuple $(x_1, x_2, x_3, \ldots)$ with $x_i \in \{0,1\}$.

We show any $g: \mathbb{Z}_+ \to \{0,1\}^\omega$ is not surjective.

$$\begin{aligned}
g(1) &= (x_{11}, & x_{12}, & \quad x_{13}, & x_{14}, & \ldots) \\
g(2) &= (x_{21}, & x_{22}, & \quad x_{23}, & x_{24}, & \ldots) \\
g(3) &= (x_{31}, & x_{32}, & \quad x_{33}, & x_{34}, & \ldots) \\
g(4) &= (x_{41}, & x_{42}, & \quad x_{43}, & x_{44}, & \ldots)
\end{aligned}$$

Define $y = (y_1, y_2, y_3, \ldots) \in \{0,1\}^\omega$ by $y_i = \begin{cases} 0 & \text{if } x_{ii} = 1 \\ 1 & \text{if } x_{ii} = 0. \end{cases}$

Note y is not in the image of g.

Fact $\mathbb{R}$ is uncountable. (Munkres: decimal expanison proof unsatisfying. Later proof using order properties.)

For A a set, recall the <u>power set</u> $\mathcal{P}(A)$ is the set of all subsets of A.

<u>Thm</u> $\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

This follows from the following stronger theorem:

<u>Thm</u> For A a set,
$\nexists$ a surjection $g: A \twoheadrightarrow \mathcal{P}(A)$
and $\nexists$ an injection $f: \mathcal{P}(A) \hookrightarrow A$

<u>Pf</u> Let $g: A \to \mathcal{P}(A)$.
Let $B = \{a \in A \mid a \in A - g(a)\}$.
If we had $B = g(a_0)$ for some $a_0 \in A$, we'd have
$$a_0 \in B \iff a_0 \in A - g(a_0) \iff a_0 \in A - B.$$
This is a contradiction.
Hence g is not surjective.

(If $\exists f: B \hookrightarrow A$, then define $g: A \twoheadrightarrow B$ by letting
$$g(a) = f^{-1}(a) \text{ for } a \in \text{im}(f),$$
and defining g arbitrarily for $a \notin \text{im}(f)$.)

Ex The set $\mathbb{Q}_+$ of positive rationals is countable.

$$\begin{tikzcd} & (n, m) \arrow[r, mapsto] & m/n \\ & \mathbb{Z}_+ \times \mathbb{Z}_+ \arrow[r, two heads] & \mathbb{Q}_+ \\ \mathbb{Z}_+ \arrow[ur, two heads] \arrow[urr, bend right, two heads] & & \end{tikzcd}$$

Thm A countable union of countable sets is countable.

Pf Let $\{A_n\}_{n \in J}$ be an indexed family of countable sets with J countable.

Get surjections $\mathbb{Z}_+ \xrightarrow{f_n} A_n \quad \forall n$

$\mathbb{Z}_+ \xrightarrow{g} J$

Define surjection $\mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \bigcup_{n \in J} A_n$

$(k, m) \longmapsto f_{g(k)}(m)$

Def Sets A, B have the <u>same cardinality</u> if $\exists$ bijection $f : A \xrightarrow{\cong} B$.

(Section 7, Ex 6 + wiki)

Cantor - Schröder - Bernstein Thm

If $\exists$ injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, then A and B have the same cardinality.

Pf Assume WLOG A and B are disjoint.

For $a \in A$ consider

$$\cdots \longmapsto b_{-3} \overset{g}{\longmapsto} a_{-2} \overset{f}{\longmapsto} b_{-1} \overset{g}{\longmapsto} \underset{a_0}{\overset{a}{\shortparallel}} \overset{f}{\longmapsto} b_1 \overset{g}{\longmapsto} a_2 \overset{f}{\longmapsto} b_3 \overset{g}{\longmapsto} a_4 \overset{f}{\longmapsto} \cdots$$

Similarly for $b \in B$.

Three possibilities: The sequence

(1) Stops at some $b_{-k} \in B$ (<u>B-stopper</u>)

(2) Stops at some $a_{-k} \in A$ (<u>A-stopper</u>)

(3) Is bi-infinite or cyclic.

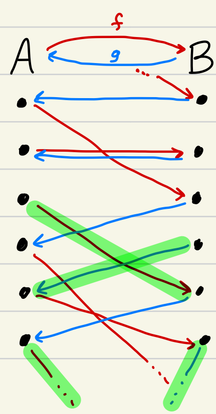
Since f, g injective, these sequences partition $A \amalg B$.

Define $h : A \xrightarrow{\cong} B$ via

$h(a) = f(a)$ if a is in A-stopper seq.

$h(a) = g^{-1}(a)$ if a is in B-stopper seq.

(either works) if a is in bi-infinite/cyclic seq.



Sections 9-11

Perhaps the most commonly used axiomatic system for mathematics is Zermelo-Fränkel set theory

- ZFC with Axiom of Choice
- ZF without Axiom of Choice

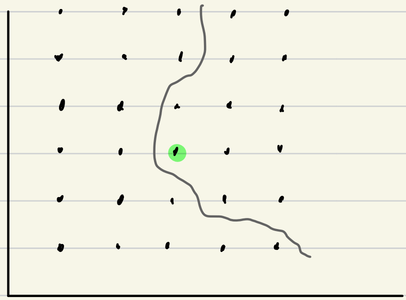
Thm In ZF, the following are equivalent:

- Axiom of Choice (§9)
- Well-ordering theorem (§10)
- Maximum principle (§11)
- Zorn's lemma (§11)

Section 10: Well-ordered sets

Def A well-order on a set A is an order relation (total order) s.t. every nonempty subset of A has a smallest element.

Ex $(\mathbb{Z}_+, \leq)$



Ex $(\mathbb{Z}_+ \times \mathbb{Z}_+, \leq \text{ lexicographic})$

Non-Ex $(\mathbb{Z}, \leq)$

Non-Ex $(\mathbb{R}_{\geq 0}, \leq)$ think $(0,1)$

Non-Ex $(\mathbb{Z}_+)^\omega := \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \times \ldots$, lexicographic order.
Indeed, consider the set of all sequences with a single entry 2 and all other entries 1: $(1,1,1,1,2,1,1,1,1,\ldots)$

Well-ordering theorem Every set has a well-ordering.

Proved by Zermelo in 1904.
Startled mathematical community.
Nobody has constructed specific well-ordering on $(\mathbb{Z}_+)^\omega$.
Proof uses Axiom of Choice.

Section 9: Axiom of Choice

Axiom of Choice Given a collection $\mathcal{A}$ of disjoint nonempty sets, $\exists$ a set C consisting of exactly one element from each set in $\mathcal{A}$.

(I.e., $C \subset \bigcup_{A \in \mathcal{A}} A$, and $|C \cap A| = 1$ for each $A \in \mathcal{A}$.)

What if the sets are not disjoint?

Def A choice function on a collection $\mathcal{B}$ of nonempty sets is a function $f : \mathcal{B} \longrightarrow \bigcup_{B \in \mathcal{B}} B$

such that $f(B) \in B$, for all $B \in \mathcal{B}$.

Consequence of Axiom of Choice For any collection of nonempty sets, there exists a choice function.

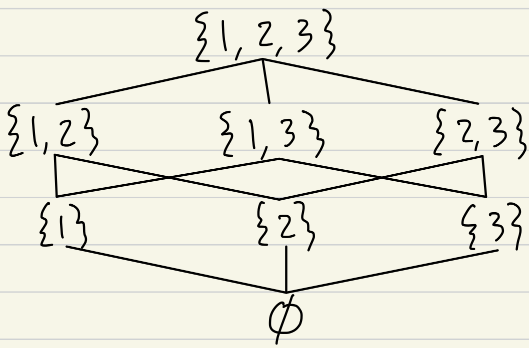
Section 11: Maximum principle and Zorn's lemma

Def A partial order $\leq$ on a set S (poset) satisfies

- $a \leq a$
- $a \leq b,\ b \leq a \implies a = b$
- $a \leq b,\ b \leq c \implies a \leq c$

Some pairs of elements may not be comparable ($a \nleq b$ and $b \nleq a$ is okay).

Ex Subsets of $\{1,2,3\}$ under inclusion.



Def A chain is a totally ordered subset of a poset.

Ex $\emptyset \subset \{2\} \subset \{1,2,3\}$ is a chain. It is contained in a maximal chain $\emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\}$, for example.

Maximum principle In a poset, every chain is contained in a maximal chain.

Zorn's lemma Let A be a poset. If every chain in A has an upper bound in A, then A has a maximal element.

u s.t. $c \leq u$ $\forall c$ in chain

m s.t. $m \leq a \Rightarrow m = a$ $\forall a \in A$

Maximum principle implies Zorn's lemma

Let A be a poset. By the Maximum principle, let $B \subset A$ be a maximal chain. By the hypothesis to Zorn's lemma, let $u \in A$ be an upper bound for B. To see u is maximal in A, note that if $u < v$, then the chain $B \cup \{v\}$ would contradict the maximality of B.

Thm In ZF, the following are equivalent:

- Axiom of Choice (§9)
- Well-ordering theorem (§10)
- Maximum principle (§11)
- Zorn's lemma (§11)