Chapter 1: Fundamental group

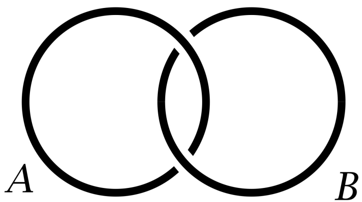
Associates to each space X a group $\pi_1(X)$
measuring the 1-dimensional holes.

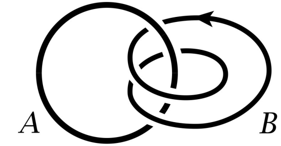
Section 1.1 Basic constructions

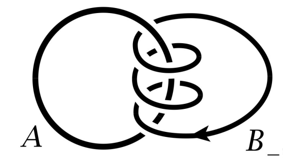
Section 1.2 Van Kampen's theorem

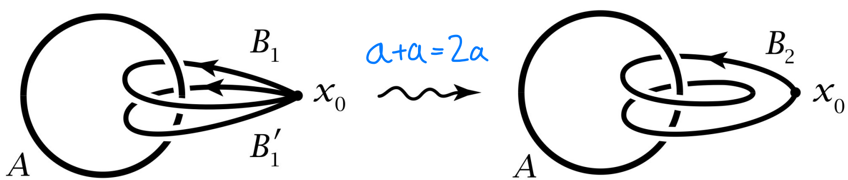
Section 1.3 Covering spaces

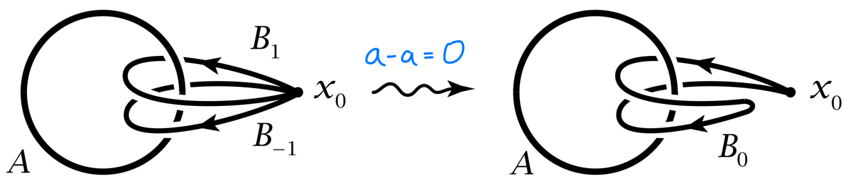
__The idea__ What is the algebraic structure of loops B in the complement of a single loop A?





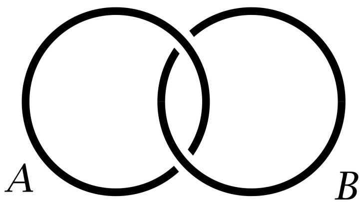


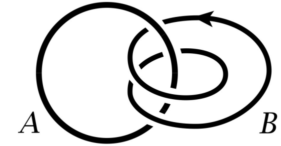


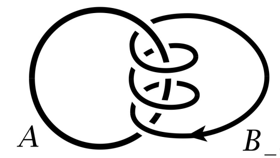


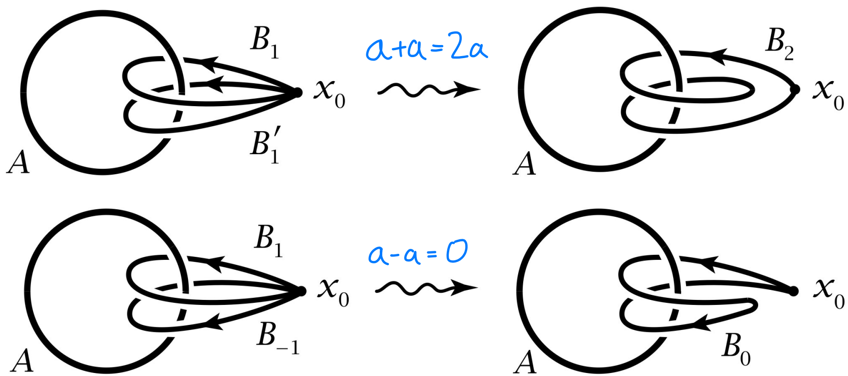
$\mathbb{R}^3 \setminus A \simeq$?

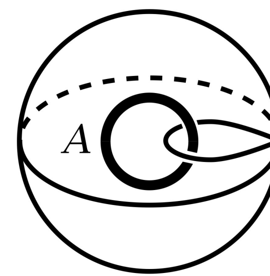
The idea What is the algebraic structure of loops B in the complement of a single loop A ?







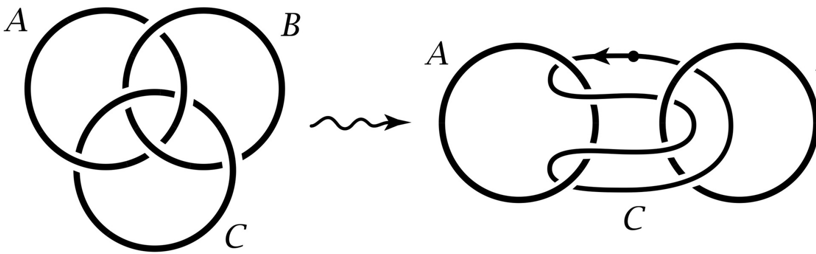




$\mathbb{R}^3 \setminus A \simeq S^1 \vee S^2$

$\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$

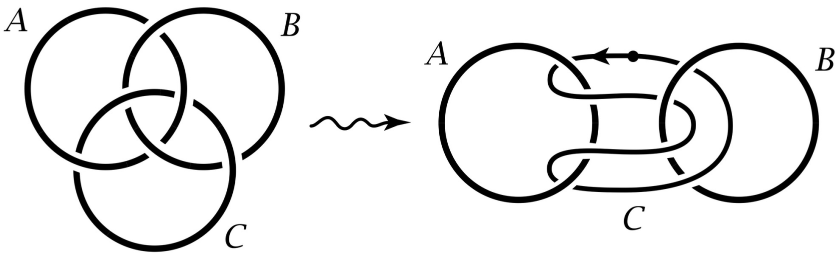
The idea What is the algebraic structure of loops C in the complement of two unlinked loops A and B?



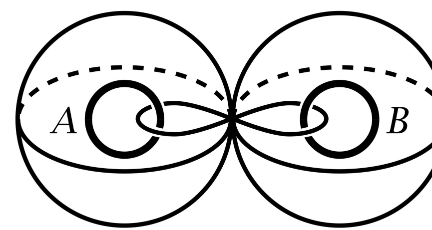
$aba^{-1}b^{-1} \neq 1$

$\mathbb{R}^3 \setminus (A \cup B) \simeq ?$

<u>The idea</u> What is the algebraic structure of loops C in the complement of two unlinked loops A and B ?



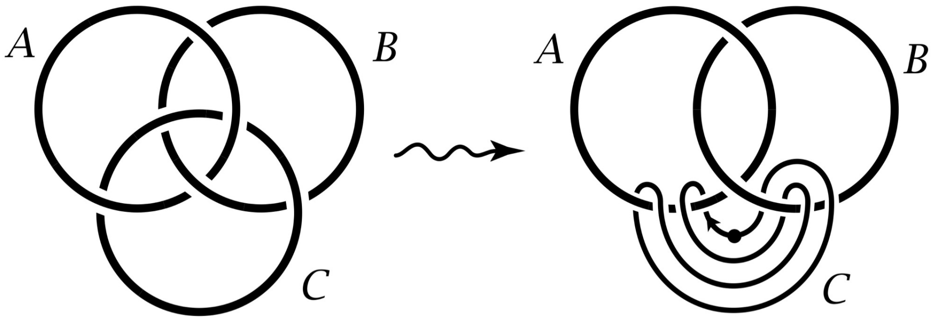
$aba^{-1}b^{-1} \neq 1$



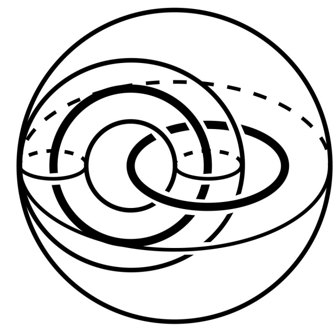
$$\mathbb{R}^3 \setminus (A \cup B) \simeq S^1 \vee S^1 \vee S^2 \vee S^2$$

$$\pi_1(S^1 \vee S^1 \vee S^2 \vee S^2) \cong \langle a, b \rangle \cong \mathbb{Z} * \mathbb{Z}$$

The idea What is the algebraic structure of loops C in the complement of two linked loops A and B?

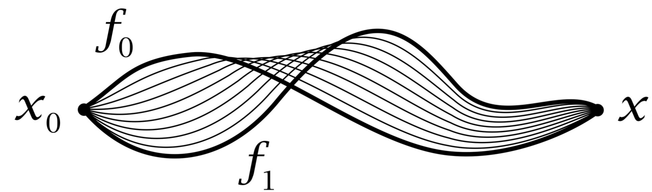


$aba^{-1}b^{-1} = 1$



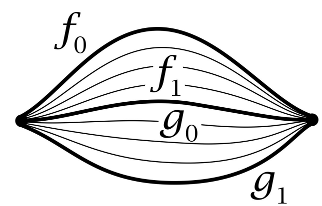
$$\mathbb{R}^3 \setminus (A \cup B) \simeq \text{torus} \vee S^2$$
$$\pi_1(\text{torus} \vee S^2) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

Section 1.1 Basic constructions



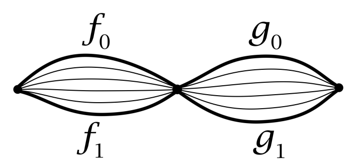
A *path* is a map $f: I \to X$

A *homotopy of paths* is a homotopy $f_t: I \to X$ rel $\{0,1\}$.
We say f_0 and f_1 are homotopic, denoted $f_0 \simeq f_1$,
or $[f_0] = [f_1]$ since this is an equivalence relation.

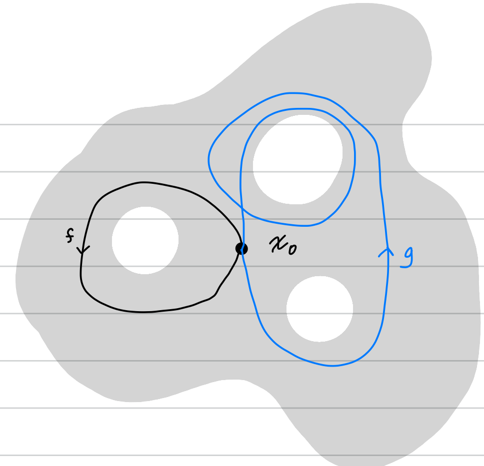


The *product* of paths $f, g: I \to X$
with $f(1) = g(0)$ is defined by $f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1. \end{cases}$

The product respects homotopy classes.

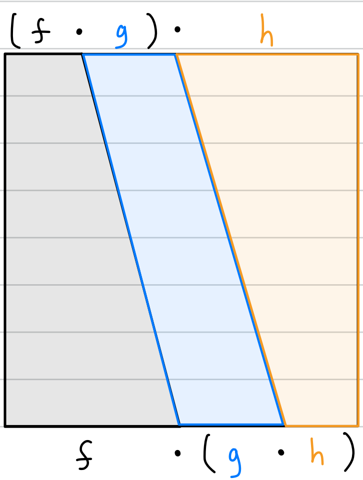


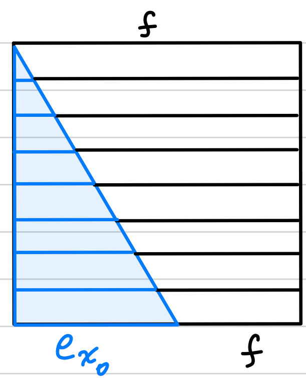
Path $f: I \to X$ is a <u>loop</u> if $f(0) = x_0 = f(1)$.
The set of all homotopy classes of loops based at x_0 is denoted $\pi_1(X, x_0)$.



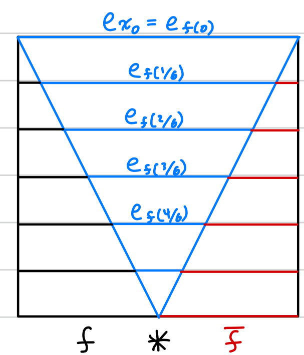
<u>Proposition 1.3</u> $\pi_1(X, x_0)$ is a group with respect to the product $[f] \cdot [g] = [f \cdot g]$.

<u>Pf</u> See how Hatcher uses reparametrizations to prove associativity, identity.





Inverses: $\bar{f}: I \to X$ by $\bar{f}(s) = f(1-s)$.



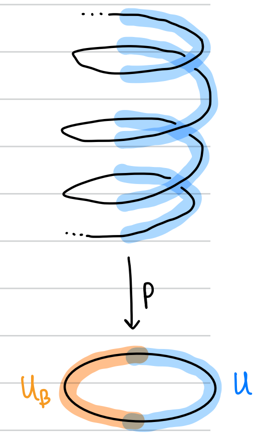
The fundamental group of the circle

We prove $\pi_1(S^1) \cong \mathbb{Z}$ using covering spaces.

Def (page 56) A covering space of X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$ such that

- there is an open cover $\{U_\alpha\}$ of X s.t. $\forall \alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ each mapped homeomorphically onto U_α by p.

Ex $p: \mathbb{R} \to S^1$ by $p(t) = (\cos 2\pi t, \sin 2\pi t)$



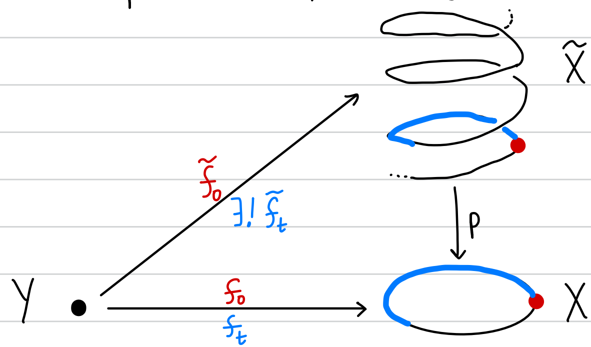
Def (page 60) A lift of a map $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ with $p\tilde{f} = f$.

$$
\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & X
\end{tikzcd}
$$

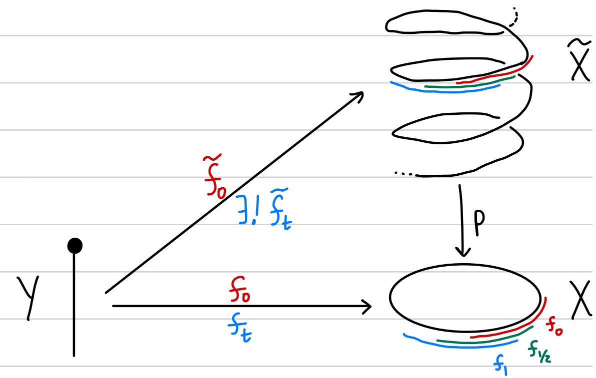
Prop 1.30 Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a lift $\tilde{f}_0: Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ that lifts f_t.

$$\left(\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "\{f_t\}"'] \arrow[ur, dashed, "\exists!\ \{\tilde{f}_t\}"] & X \end{tikzcd}\right)$$

Ex $Y = pt = \{0\}$ (path lifting)

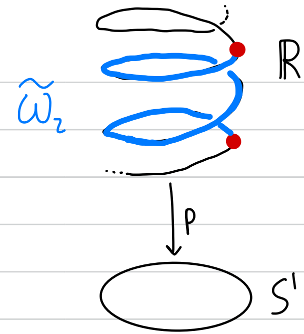


Ex $Y = I$ (homotopy lifting)



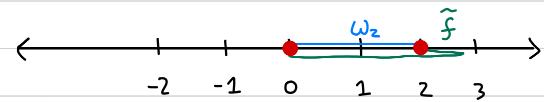
Pf idea Unique lifts over an open set exist by the homeomorphism property $(\bullet)$.
Piecing these together takes a page in Hatcher.

Thm 1.7 $\Phi : \mathbb{Z} \to \pi_1(S^1)$ via $\Phi(n) = [\omega_n]$, where $\omega_n(s) = (\cos 2\pi n s, \ \sin 2\pi n s)$, is an isomorphism.



Pf Note $\widetilde{\omega}_n : I \to \mathbb{R}$ via $\widetilde{\omega}_n(s) = ns$ lifts $\omega_n : I \to S^1$ (since $p\widetilde{\omega}_n = \omega_n$).

Note $\Phi(n) = [\omega_n] = [p\widetilde{\omega}_n] = [p\tilde{f}]$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n (since $\tilde{f} \simeq \widetilde{\omega}_n$ by a linear homotopy).



Φ is a homomorphism

Let $\tau_m : \mathbb{R} \to \mathbb{R}$ translate $\tau_m(x) = m + x$.

Note $\widetilde{\omega}_m \cdot (\tau_m \widetilde{\omega}_n)$ is a path in $\mathbb{R}$ from 0 to $m+n$

So $\Phi(m+n) = [p\,(\widetilde{\omega}_m \cdot (\tau_m \widetilde{\omega}_n))] = [\omega_m \cdot \omega_n] = \Phi(m) \cdot \Phi(n)$.

Φ is surjective

Let $f : I \to S^1$ be a loop based at $(1,0)$.

By Prop 1.30 (path lifting) $\exists!$ lift $\tilde{f} : I \to \mathbb{R}$ with $\tilde{f}(0) = 0$.

Necessarily $\tilde{f}_1(1) = n$ for some $n \in \mathbb{Z}$, giving $\Phi(n) = [p\tilde{f}] = [f]$.

Φ is injective

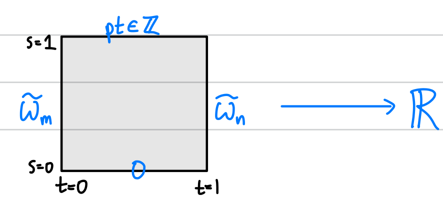
Suppose $\Phi(m) = \Phi(n)$, so $\underset{f_0}{\omega_m} \underset{f_t}{\simeq} \underset{f_1}{\omega_n}$.

By Prop 1.30 (homotopy lifting) $\exists$ lift $\tilde{f}_t : I \to \mathbb{R}$.

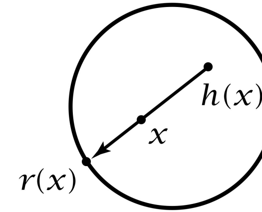
Necessarily $\tilde{f}_t(0) = 0 \ \forall t$ and $\tilde{f}_t(1) = pt \in \mathbb{Z} \ \forall t$.

By uniqueness of path lifting, $\tilde{f}_0 = \widetilde{\omega}_m$ and $\tilde{f}_1 = \widetilde{\omega}_n$.

Hence $m = \widetilde{\omega}_m(1) = \widetilde{\omega}_n(1) = n$.



Applications of $\pi_1(S^1) \cong \mathbb{Z}$



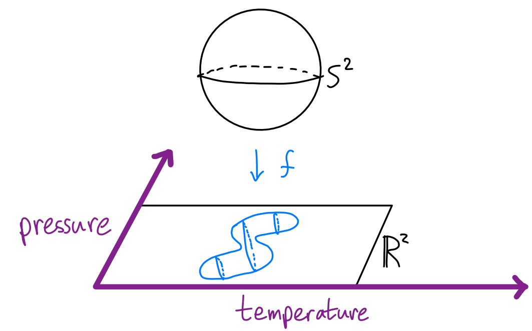
<u>Thm 1.8</u> Every nonconstant polynomial has a root in $\mathbb{C}$.

<u>Thm 1.9</u> $n=2$ case of Brouwer fixed point theorem (Cor 2.15):

"Every map $h: D^n \to D^n$ has a fixed point: $h(x)=x$."

<u>Thm 1.10</u> $n=2$ case of Borsuk-Ulam theorem (Cor 2B.7):

"Every map $f: S^n \to \mathbb{R}^n$ identifies some antipodal pair: $f(x)=f(-x)$."



<u>Cor 1.11</u> $n=2$ case of:

"If S^n is the union of $n+1$ closed sets, then some set contains an antipodal pair $\{x,-x\}$."

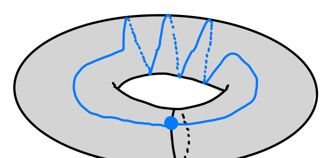
<u>Cor 1.16</u> $n=2$ case of Invariance of Dimension (Thm 2.26):

"$\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$."

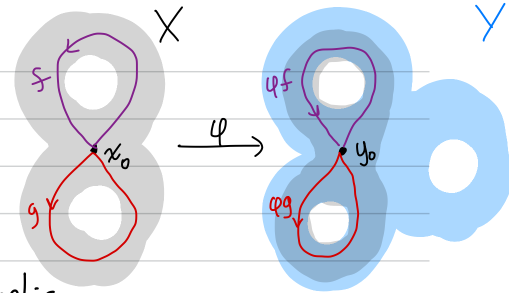
<u>Prop 1.12</u> $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

<u>Ex 1.13</u> $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

$\pi_1((S^1)^n) \cong \mathbb{Z}^n$



Induced homomorphisms



Def A map $\varphi : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $\varphi_*([f]) = [\varphi f]$.

$\varphi_*([f]\cdot[g]) = [\varphi(f \cdot g)] = [\varphi f \cdot \varphi g] = \varphi_*([f]) \cdot \varphi_*([g])$

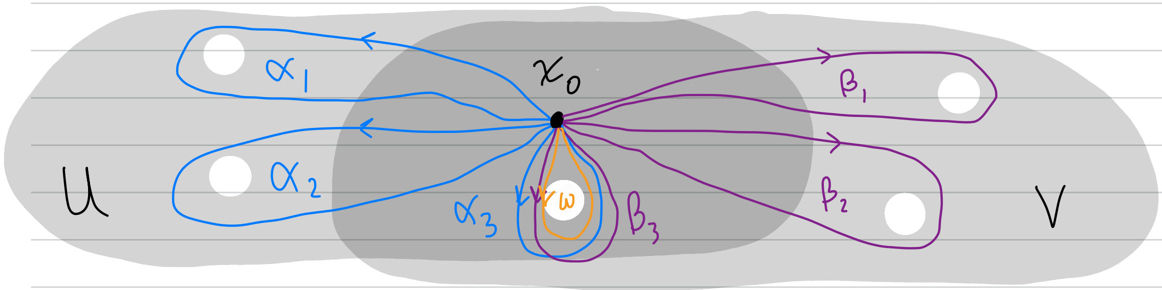
well-defined since $f_0 \simeq f_1$ via $f_t \Rightarrow \varphi f_0 \simeq \varphi f_1$ via φf_t

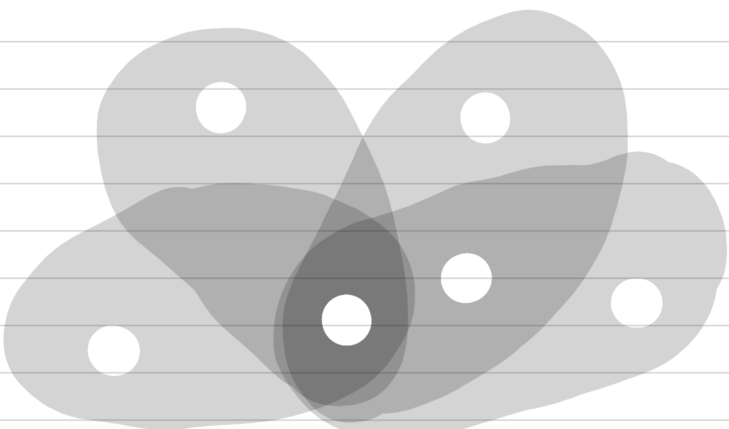
Functor π_1 is a functor (see §2.3) since

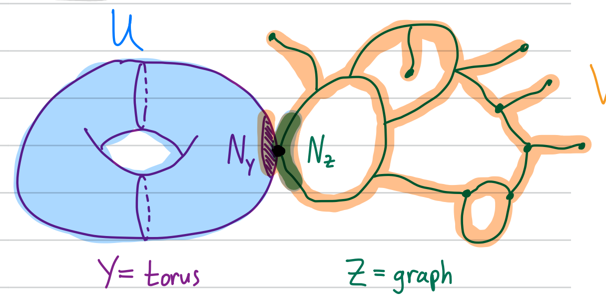
- $(\varphi \psi)_* = \varphi_* \psi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$.
- $\mathbb{1}_* = \mathbb{1}$, i.e., $\mathbb{1} : X \to X$ induces $\mathbb{1} : \pi_1(X, x_0) \to \pi_1(X, x_0)$.

Prop 1.18 If $\varphi : X \to Y$ is a homotopy equivalence, then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.

Section 1.2 Van Kampen's theorem (arbitrary unions)







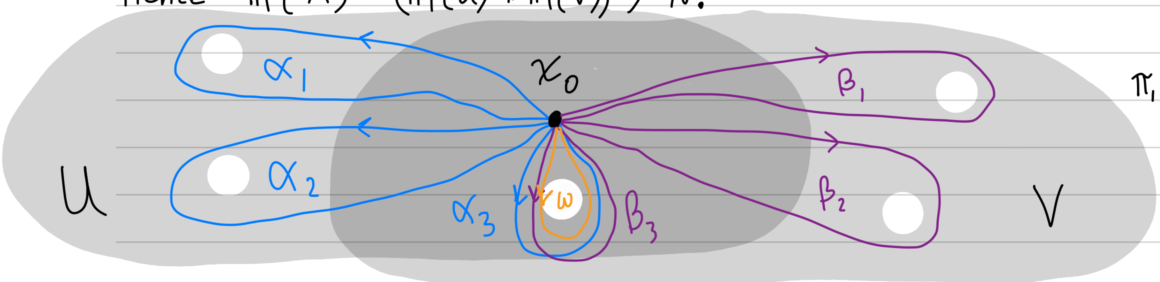
<u>RECALL</u> The Seifert-van Kampen theorem (Two set version, from Munkres!)

<u>Thm (Seifert-van Kampen)</u> Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_u(w)^{-1} i_v(w)$ for $w \in \pi_1(U \cap V, x_0)$.

Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.

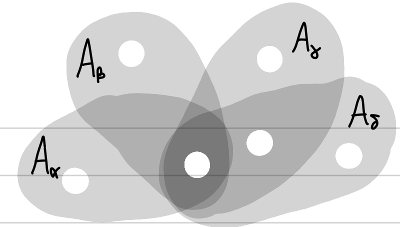


$$\begin{tikzcd} & \pi_1(U) \ni \alpha_3 \arrow[dr, "j_u"] & \\ \pi_1(U \cap V) \ni w \arrow[ur, "i_u"] \arrow[dr, "i_v"'] & & \pi_1(X) \\ & \pi_1(V) \ni \beta_3 \arrow[ur, "j_v"'] & \end{tikzcd}$$

$\pi_1(U) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ $\quad \pi_1(V) = \langle \beta_1, \beta_2, \beta_3 \rangle$ $\quad \pi_1(U \cap V) = \langle w \rangle$

$$\begin{aligned} \pi_1(X) &\cong \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid i_u(w)^{-1} i_v(w) \rangle \\ &= \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid \alpha_3^{-1} \beta_3 \rangle \\ &\cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2, w \rangle \end{aligned}$$

Note $i_u(w) = \alpha_3$ and $i_v(w) = \beta_3$



Section 1.2 Van Kampen's theorem (arbitrary unions)

Thm 1.20 Let $X = \bigcup_\alpha A_\alpha$ s.t. each A_α is open, path-connected, and contains x_0.

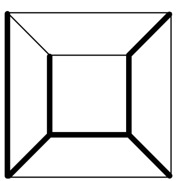
- If each $A_\alpha \cap A_\beta$ is path-connected, then $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ (defined using the j_α) is surjective.

  If furthermore each $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then $\ker(\Phi)$ is the normal subgroup generated by all elements $i_{\alpha\beta}(w)\, i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$.

(For $g_\alpha, \tilde{g}_\alpha \in \pi_1(A_\alpha)$, $g_\beta \in \pi_1(A_\beta)$, $g_\gamma \in \pi_1(A_\gamma)$, $\Phi(g_\alpha g_\beta^3 \tilde{g}_\alpha g_\gamma^{-2}) = j_\alpha(g_\alpha)\, j_\beta(g_\beta)^3\, j_\alpha(\tilde{g}_\alpha)\, j_\gamma(g_\gamma)^{-2}$.)

$$\begin{tikzcd} & \pi_1(A_\alpha) \arrow[dr, "j_\alpha"] & \\ \pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X) \\ & \pi_1(A_\beta) \arrow[ur, "j_\beta"'] & \end{tikzcd}$$

Ex



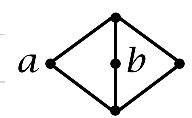
Write X as union of 5 open sets each containing the bold tree and one extra edge.

Double, triple intersections path-connected.

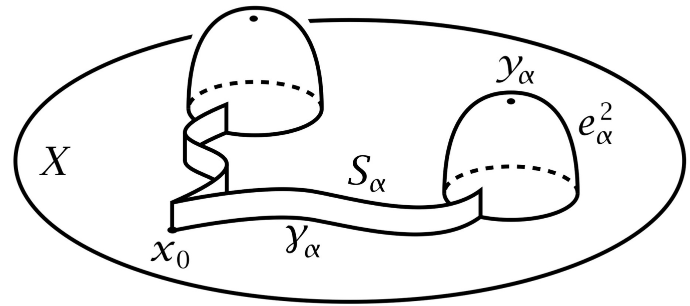
Double intersections contractible $\Rightarrow \ker(\Phi)$ trivial.

So Φ gives an isomorphism $\pi_1(X) \cong *_{i=1}^{5} \pi_1(A_i) = *_{i=1}^{5} \mathbb{Z}$.

Non-Ex To see the triple intersection assumption is necessary, consider $A_\alpha = X \setminus \{a\}$, $A_\beta = X \setminus \{b\}$, $A_\gamma = X \setminus \{c\}$.



(To get the right answer $\mathbb{Z} * \mathbb{Z}$, use only A_α and A_β.)



Applications to cell complexes

Y obtained from path-connected X by attaching 2-cells e^2_α via $\varphi_\alpha : S^1 \to X$.
Fix $x_0 \in X$. Choose paths γ_α to image(φ_α).
Let $N \subseteq \pi_1(X, x_0)$ be normal subgroup generated by all $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$.

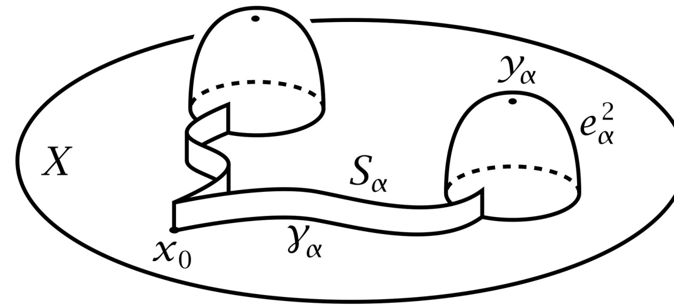
Prop 1.26

(a) $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ with kernel N, so $\pi_1(Y) \cong \pi_1(X)/N$.

(b) If instead Y were obtained by attaching n-cells for some $n > 2$, then $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.

(c) For X a path-connected CW complex, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \cong \pi_1(X)$.

Rmk Choice of path γ_α doesn't matter, since a different path η_α gives a conjugate element
$\eta_\alpha \varphi_\alpha \overline{\eta_\alpha} = (\eta_\alpha \overline{\gamma_\alpha}) \gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha} (\gamma_\alpha \overline{\eta_\alpha})$.



Pf (a)

Let $Z = Y \cup \{\text{rectangular strips}\} \simeq Y$.
Choose $y_\alpha \in e^2_\alpha$.
Note $A = Z - \bigcup_\alpha \{y_\alpha\} \simeq X$ and $B = Z - X \simeq *$
are open path-connected sets with union Z.
Note $A \cap B \simeq \bigvee_\alpha S^1$ with $\pi_1(A \cap B)$ generated
(loosely speaking) by $[\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha]$.

Van Kampen's says $\pi_1(Y) \cong \pi_1(Z)$ is isomorphic to the quotient of
$\pi_1(A) \cong \pi_1(X)$ by the normal subgroup generated by the image of $\pi_1(A \cap B) \longrightarrow \pi_1(A)$,
which corresponds to N.

(b) The only difference with the above proof is $A \cap B \simeq \bigvee_\alpha S^{n-1}$, with $n > 2$.
So $\pi_1(A \cap B)$ is trivial and van Kampen's gives $\pi_1(Y) \cong \pi_1(Z) \underset{\text{van Kampen}}{\cong} \pi_1(A) \cong \pi_1(X)$.

(c) If X is finite-dimensional ($X = X^n$ for some n), then (c) follows from (b) and induction.
(Add on 3-cells, then 4-cells, etc.)

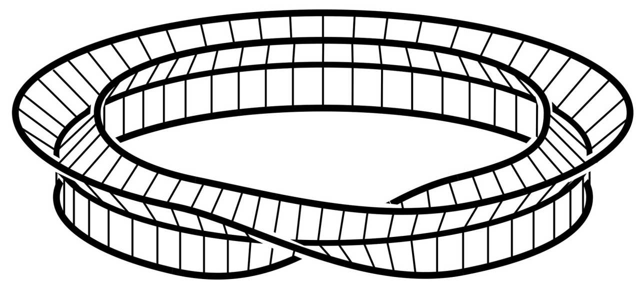
Otherwise, let $f: I \to X$ be a loop based at $x_0 \in X^2$.
$\mathrm{Im}(f)$ is compact and hence lives in a finite subcomplex of X by Proposition A.1, and hence in X^n for some n.
Since $\pi_1(X^2) \to \pi_1(X^n)$ is surjective by (b), f is homotopic to a loop in X^2.
So $\pi_1(X^2) \to \pi_1(X)$ is surjective.

To see it is also injective, suppose f is a loop in X^2 which is nullhomotopic in X via a nullhomotopy $F: I \times I \to X$.
$\mathrm{Im}(F)$ is compact, hence lies in X^n for some $n \geq 2$.
Since $\pi_1(X^2) \to \pi_1(X^n)$ is injective by (b), it follows that f is nullhomotopic in X^2.

Corollary For every group G there is a 2-dimensional CW complex X_G with $\pi_1(X_G) \cong G$

Pf Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$, which exists since every group is a quotient of a free group.

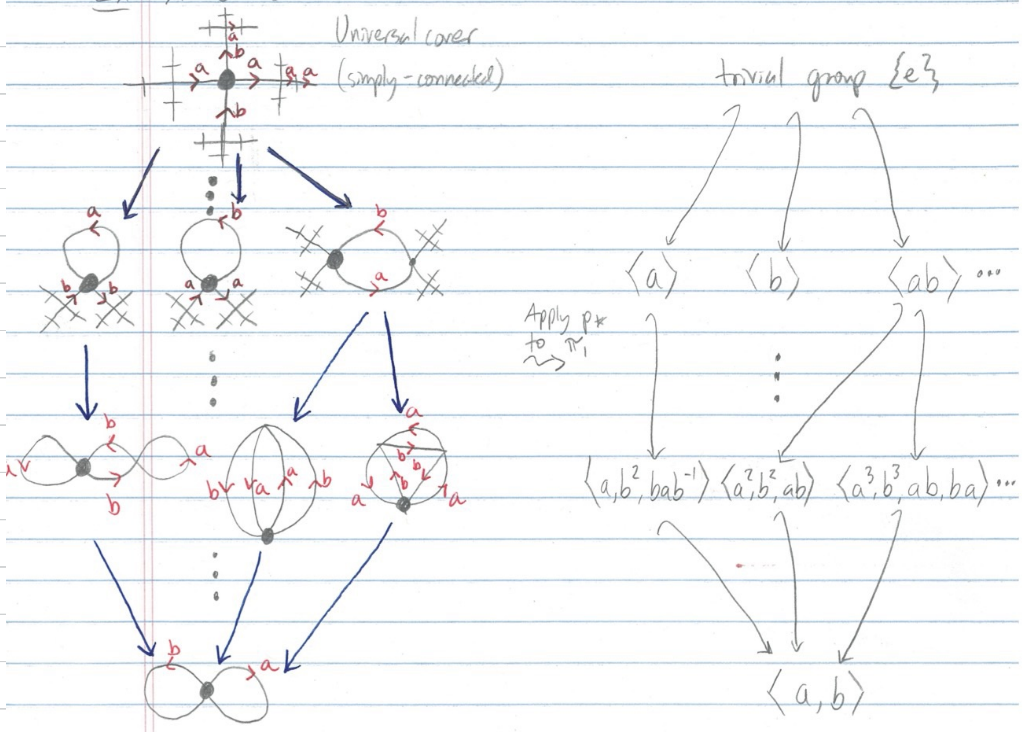
Construct X_G from $\vee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β via loops specified by the words r_β.



Ex $G = \mathbb{Z}/n\mathbb{Z}$

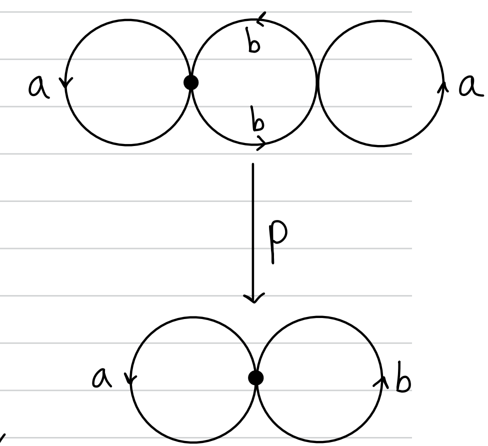
Section 1.3 Covering spaces

Ex $X = S^1 \vee S^1$



Prop 1.30 Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a lift $\tilde{f}_0: Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ that lifts f_t.

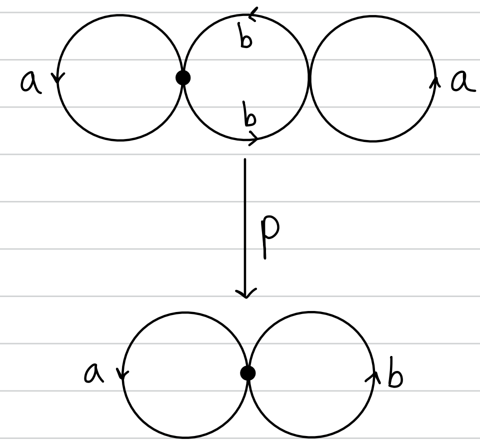
Prop 1.31 Let $p: \tilde{X} \to X$ be a covering space. Then $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. Also, $\mathrm{Image}(p_*)$ is all homotopy classes of loops in X that lift to loops (not paths) in $\tilde{X}$.



Pf If $[f] \in \ker(p_*)$, then pf is nullhomotipic in X. By Prop 1.30 we can lift to see f is nullhomotopic in $\tilde{X}$.

Clearly loops lifting to loops represent elements in $\mathrm{Image}(p_*)$. Conversely, $[g] \in \mathrm{Image}(p_*)$ implies $g \simeq g'$ with g' lifting to a loop, which by Prop 1.30 means g lifts to a loop.

Prop 1.32 Let $p: \tilde{X} \to X$ be a covering space with X and $\tilde{X}$ path-connected. The number of sheets $|p^{-1}(x_0)|$ is equal to the index $[\pi_1(X) : H]$, where $H = p_* \pi_1(\tilde{X})$.



Pf Define $\Phi : \{\text{cosets of } H\} \longrightarrow p^{-1}(x_0)$ by

$$H[g] \longmapsto \tilde{g}(1)$$

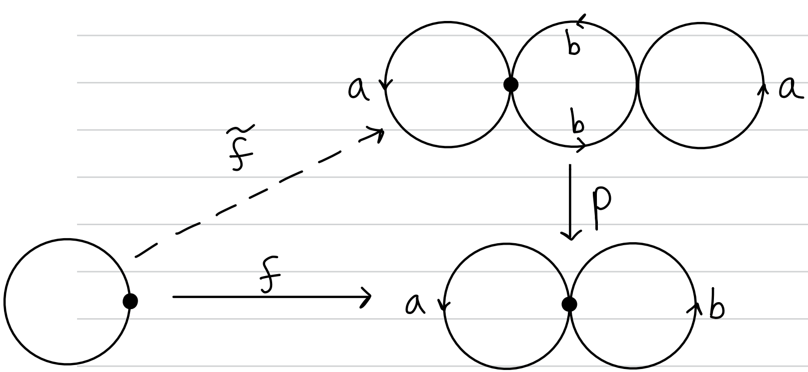
where $\tilde{g}$ is a lift of g starting at $\tilde{x}_0$.
Φ is well-defined since elements of H lift to loops.
Φ is surjective since $\tilde{X}$ is path-connected.
Φ is injective since $\Phi(H[g_1]) = \Phi(H[g_2])$ implies $g_1 \bar{g}_2$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ and $H[g_1] = H[g_2]$.

We care about lifts of general maps, not just of homotopies.

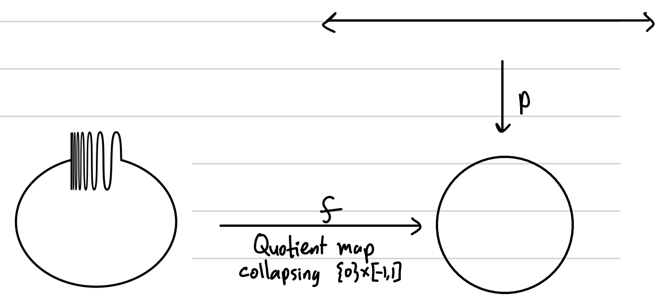
<u>Prop 1.33</u> (Lifting criterion) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space. Let $f:(Y,y_0)\to(X,x_0)$ be a map with Y connected and locally path-connected. Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

<u>Rmk</u> ($\Longrightarrow$) is obvious since $f_* = p_*\tilde{f}_*$.



<u>Ex 1.3.7: Necessity of Y locally path-connected</u>



$\pi_1(Y,y_0)$ is trivial group, but no lift exists.

We care about lifts of general maps, not just of homotopies.

<u>Prop 1.33</u> (Lifting criterion) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space.
Let $f:(Y,y_0)\to(X,x_0)$ be a map with Y connected and locally path-connected.
Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

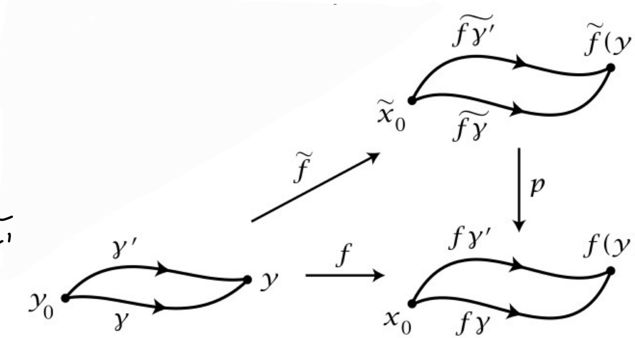
So our definition of $\tilde{f}$ will be well-defined.

So our definition of $\tilde{f}$ will be continuous.

<u>Pf</u> $(\Rightarrow)$ is obvious since $f_*=p_*\tilde{f}_*$.

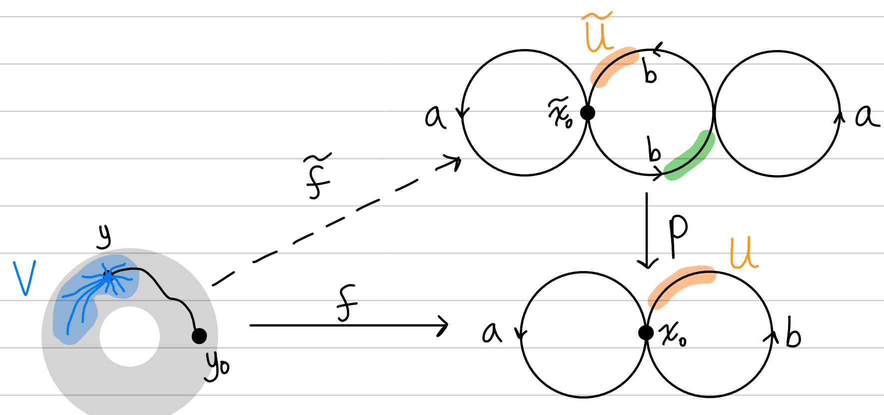
$(\Leftarrow)$ Y is path-connected since it is connected and locally path-connected.
For $y\in Y$, let γ be a path in Y from y_0 to y.
Path $f\gamma$ in X based at x_0 lifts uniquely to path $\widetilde{f\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$.
Define $\tilde{f}:Y\to\tilde{X}$ by $\tilde{f}(y)=\widetilde{f\gamma}(1)$.

<u>$\tilde{f}$ well-defined</u>: Given two such paths γ,γ', note
$[f\gamma'\cdot\overline{f\gamma}]\in f_*(\pi_1(Y,x_0))\subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$.
By Prop. 1.31, $f\gamma'\cdot\overline{f\gamma}$ lifts to a loop in $\tilde{X}$.
By uniqueness of path lifting, the first half of this loop is $\widetilde{f\gamma'}$
and the second half is $\widetilde{f\gamma}$ traversed backwards,
so $\widetilde{f\gamma'}(1)=\widetilde{f\gamma}(1)$.



$\tilde{f}$ continuous: Let $y \in Y$. Since $p : \tilde{X} \to X$ is a covering space,
let $f(y) \in U \underset{\text{open}}{\subseteq} X$ with $\tilde{f}(y) \in \tilde{U} \subset \tilde{X}$ and $p|_{\tilde{U}} : \tilde{U} \to U$ a homeomorphism.
Since $f^{-1}(U)$ is open in Y, choose a path-connected open set $y \in V \subset f^{-1}(U)$.
We will show $\tilde{f}|_V = (p|_U)^{-1} \circ f|_V$, hence $\tilde{f}$ is continuous at y.

Indeed, fix a path γ in Y from y_0 to y.
For each $y' \in V$, fix a path η in V from y to y'.
Each path $f\gamma \cdot f\eta$ in X has a lift $\widetilde{f\gamma} \cdot \widetilde{f\eta}$ with $\widetilde{f\eta} = (p|_U)^{-1} f\eta$ mapping to $\tilde{U}$.
Thus $\tilde{f}(V) \subset \tilde{U}$ and $\tilde{f}|_V = (p|_U)^{-1} \circ f|_V$.



We also have a unique lifting property.

<u>Prop 1.34</u> Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at a point and Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}_1, \tilde{f}_2"] \arrow[r, "f"'] & X \end{tikzcd}$$

<u>Pf</u> The main idea is to show $\{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is open and closed.

Classification of covering spaces

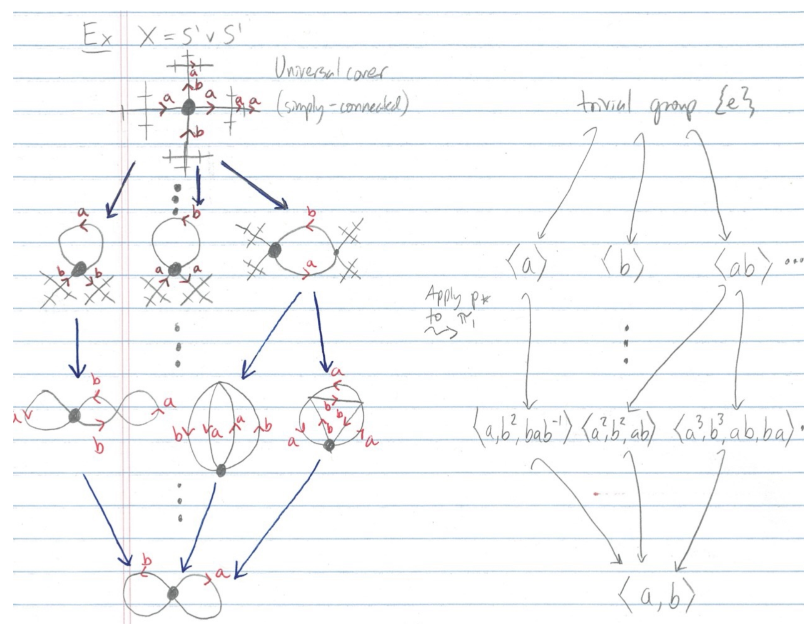
Thm 1.38 X path-connected, locally path-connected, semilocally simply-connected. Then

$$\left\{\begin{array}{l}\text{basepoint-preserving iso classes of path-connected}\\ \text{covering spaces } p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\} \xrightarrow{B} \left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(X,x_0)\end{array}\right\}$$

$$[p:(\tilde{X},\tilde{x}_0)\to(X,x_0)] \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

is a bijection.

Rmk If you ignore basepoints, then you map to conjugacy classes of subgroups.



Def Covering spaces $p_1:\tilde{X}_1\to X$ and $p_2:\tilde{X}_2\to X$ are <u>isomorphic</u> if there is a homeomorphism $f:\tilde{X}_1\to\tilde{X}_2$ with $p_1=p_2 f$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Prop 1.37 (B is well-defined and injective)

Let X be path-connected and locally path-connected.
Two connected covering spaces are basepoint-preserving isomorphic
iff $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2,\tilde{x}_2))$.

$$\begin{tikzcd}
(\tilde{X}_1,\tilde{x}_1) \arrow[dr, "p_1"'] & & (\tilde{X}_2,\tilde{x}_2) \arrow[dl, "p_2"] \\
& (X,x_0) &
\end{tikzcd}$$

Pf ($\Rightarrow$) $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$ induce (or imply) $\subseteq$ and $\supseteq$.

($\Leftarrow$) By the lifting criterion (Prop 1.33)
$\subseteq$ gives a lift $\tilde{p}_1 : \tilde{X}_1 \to \tilde{X}_2$ (so $p_2\tilde{p}_1 = p_1$), and
$\supseteq$ gives a lift $\tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_1$ (so $p_1\tilde{p}_2 = p_2$).

$$\begin{tikzcd}
& \tilde{X}_2 \arrow[d, "p_2"] \\
\tilde{X}_1 \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$

Since these lifts compose to fix basepoints,
unique lifting (Prop 1.34) gives $\tilde{p}_2\tilde{p}_1 = \mathbb{1}_{\tilde{X}_1}$ and $\tilde{p}_1\tilde{p}_2 = \mathbb{1}_{\tilde{X}_2}$.

$$\begin{tikzcd}
& \tilde{X}_1 \arrow[d, "p_1"] \\
\tilde{X}_1 \arrow[ur, dotted, "\mathbb{1}_{\tilde{X}_1} = \tilde{p}_2\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$

Note $p_1(\tilde{p}_2\tilde{p}_1) = (p_1\tilde{p}_2)\tilde{p}_1 = p_2\tilde{p}_1 = p_1$.

$$\begin{tikzcd}
(\tilde{X}_1,\tilde{x}_1) \arrow[rr, bend left, "\tilde{p}_1"] \arrow[dr, "p_1"'] & & (\tilde{X}_2,\tilde{x}_2) \arrow[ll, bend left, "\tilde{p}_2"] \arrow[dl, "p_2"] \\
& (X,x_0) &
\end{tikzcd}$$

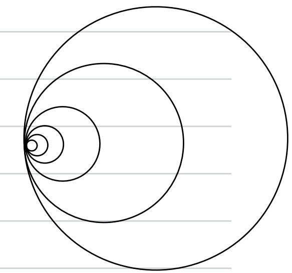
Classification of covering spaces

Thm 1.38 X path-connected, locally path-connected, semilocally simply-connected. Then

$$\left\{\begin{array}{c}\text{basepoint-preserving iso classes of path-connected}\\ \text{covering spaces } p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\} \xrightarrow{B} \left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(X,x_0)\end{array}\right\}$$

$$[p:(\tilde{X},\tilde{x}_0)\to(X,x_0)] \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

is a bijection.

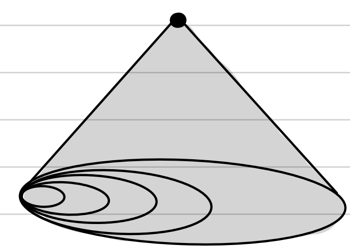


Def X is semilocally simply-connected (slsc) if $\forall x\in X$, $\exists$ open set $x\in V$ with $\pi_1(V)\to\pi_1(X)$ trivial.

Ex The Hawaiian earrings are not slsc.

To see this condition is necessary, consider the universal cover, and V small enough to be evenly-covered.

Ex The cone over the Hawaiian earrings is slsc but not lsc.



Recall X is locally simply-connected (lsc) if it has a basis with simply-connected sets.

Note lsc $\Rightarrow$ slsc.

Prop 1.36 (B is surjective)
X path-connected, locally path-connected, semilocally simply-connected.
Then $\forall$ subgroups $H \subset \pi_1(X, x_0)$, $\exists$ covering space $p: (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

Pf (1) Define the universal cover $p: \tilde{X} \to X$ with $\pi_1(\tilde{X})$ trivial.
(2) Define $\tilde{X}_H$ as a quotient of $\tilde{X}$.

(1) $\tilde{X} := \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$.
$p: \tilde{X} \to X$ via $p([\gamma]) = \gamma(1)$.
The slsc hypothesis is used to define the topology on $\tilde{X}$ via a basis.
Can check this is a covering space.

To see that $\tilde{X}$ is path-connected, form a path $I \to \tilde{X}$ with
$0 \mapsto [x_0]$ and $1 \mapsto [\gamma]$ via $t \mapsto [\gamma_t]$, where $\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t \le s \le 1. \end{cases}$

To see that $\tilde{X}$ is simply-connected, recall p_* injective. Let $[\gamma] \in \text{Image}(p_*)$.
$\text{Image}(p_*)$ is represented by loops lifting to loops.
Note $t \mapsto [\gamma_t]$ lifts γ, and for this to be a loop means $[x_0] = [\gamma_1] = [\gamma]$.

Before we define a topology on $\widetilde{X}$ we make a few preliminary observations. Let $\mathcal{U}$ be the collection of path-connected open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Note that if the map $\pi_1(U) \to \pi_1(X)$ is trivial for one choice of basepoint in U, it is trivial for all choices of basepoint since U is path-connected. A path-connected open subset $V \subset U \in \mathcal{U}$ is also in $\mathcal{U}$ since the composition $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ will also be trivial. It follows that $\mathcal{U}$ is a basis for the topology on X if X is locally path-connected and semilocally simply-connected.

Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{\, [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \,\}$$

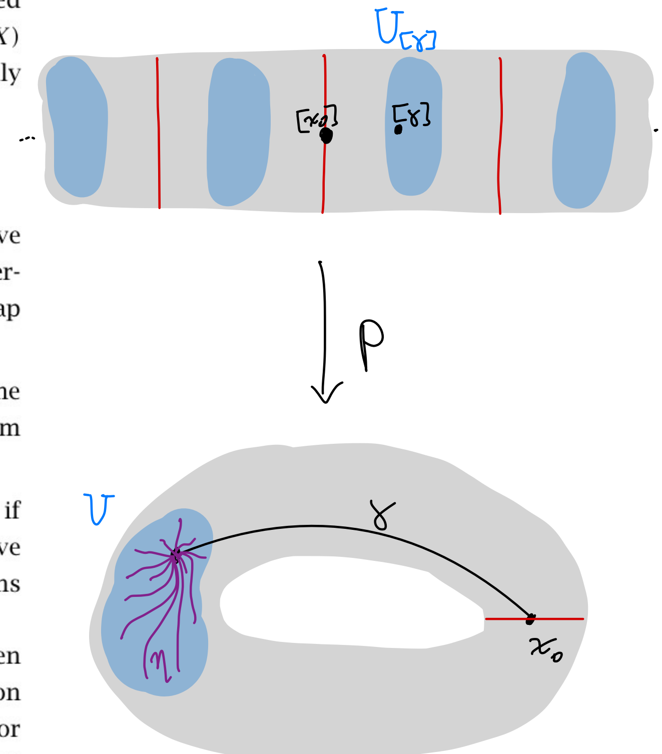
As the notation indicates, $U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$. Observe that $p : U_{[\gamma]} \to U$ is surjective since U is path-connected and injective since different choices of η joining $\gamma(1)$ to a fixed $x \in U$ are all homotopic in X, the map $\pi_1(U) \to \pi_1(X)$ being trivial. Another property is

(∗) $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. For if $\gamma' = \gamma \cdot \eta$ then elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ and hence lie in $U_{[\gamma]}$, while elements of $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \overline{\eta} \cdot \mu] = [\gamma' \cdot \overline{\eta} \cdot \mu]$ and hence lie in $U_{[\gamma']}$.

This can be used to show that the sets $U_{[\gamma]}$ form a basis for a topology on $\widetilde{X}$. For if we are given two such sets $U_{[\gamma]}, V_{[\gamma']}$ and an element $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$ by (∗). So if $W \in \mathcal{U}$ is contained in $U \cap V$ and contains $\gamma''(1)$ then $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$.

The bijection $p : U_{[\gamma]} \to U$ is a homeomorphism since it gives a bijection between the subsets $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U. Namely, in one direction we have $p(V_{[\gamma']}) = V$ and in the other direction we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V, since $V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$ and $V_{[\gamma']}$ maps onto V by the bijection p.

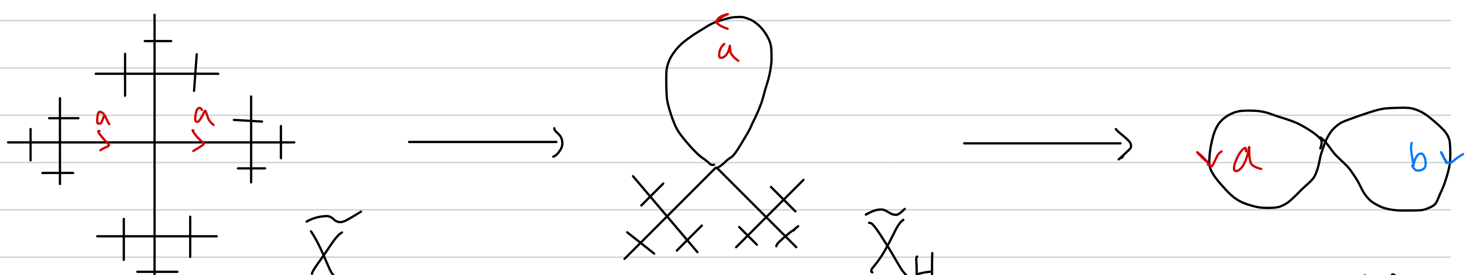
The preceding paragraph implies that $p : \widetilde{X} \to X$ is continuous. We can also deduce that this is a covering space since for fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$ partition $p^{-1}(U)$ because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$ by (∗).



(2) For $[\gamma],[\gamma'] \in \tilde{X}$, define $[\gamma]\sim[\gamma']$ if $\gamma(1)=\gamma'(1)$ and $[\gamma\cdot\bar{\gamma}'] \in H$.
This is an equivalence relation since H is a subgroup
- reflexive : identity
- symmetric : inverses
- transitive : H closed under multiplication

Define $\tilde{X}_H$ to be the quotient space $\tilde{X}_H = \tilde{X}/\sim$.
Can check the map $\tilde{X}_H \to X$ induced from $[\gamma]\to\gamma(1)$ gives a covering space.

We claim $\pi_1(\tilde{X}_H, \tilde{x}_0) \to \pi_1(X, x_0)$ has image H
(where $\tilde{x}_0$ is the equivalence class of $[x_0]$).
Indeed, a loop γ in X lifts to a loop in $X_H \Leftrightarrow [\gamma]\sim[x_0] \Leftrightarrow [\gamma]\in H$.



$\tilde{X}$ $\tilde{X}_H$ X

$H = \langle a \rangle$ $\pi_1(X) = \langle a,b \rangle$

Deck transformations and group actions

Let $p: \tilde{X} \to X$ be a covering space. The group of deck transformations is

$$G(\tilde{X}) = \left\{ \begin{array}{l} \text{covering space} \\ \text{isomorphisms} \end{array} \quad \begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \right\},$$

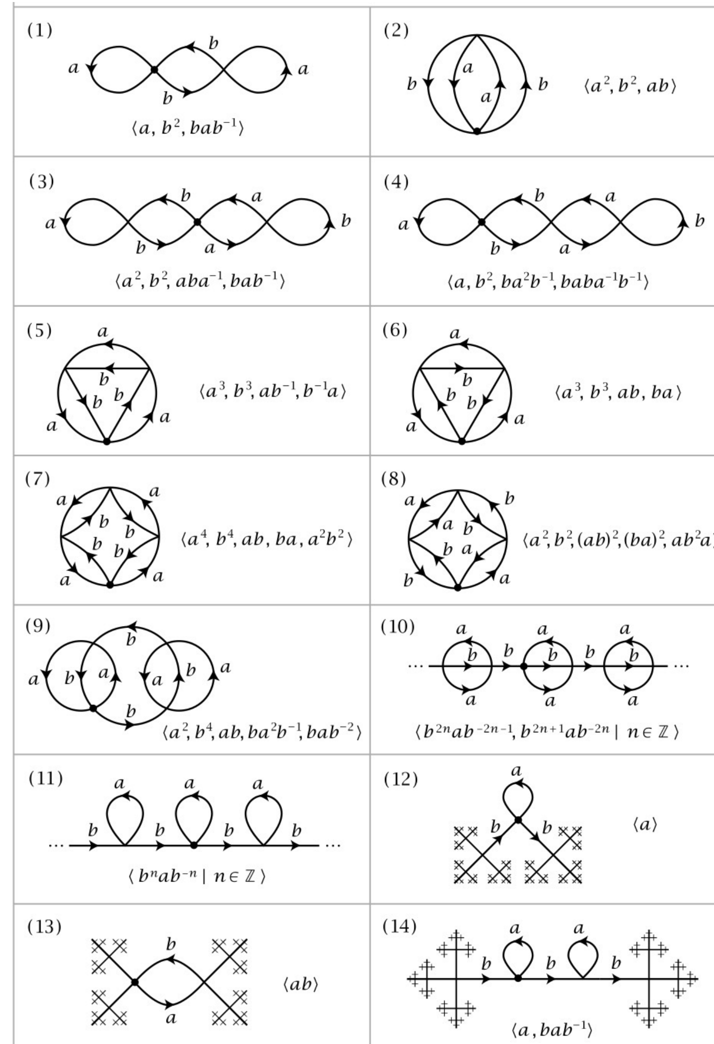
equipped with composition.

Ex (7) $G(\tilde{X}) \cong \mathbb{Z}/4$ (8) $G(\tilde{X}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

A covering space is normal if $\forall x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, $\exists\, h \in G(\tilde{X})$ with $h(\tilde{x}) = h(\tilde{x}')$.

(Maximal symmetry)

Ex (1), (2), (5)-(8), (11) normal.



Deck transformations and group actions

Prop 1.39 Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a p.c. covering space of the p.c., l.p.c. space X. Let $H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

(a) $p:\tilde{X}\to X$ normal $\iff$ H normal in $\pi_1(X,x_0)$.

(b) $G(\tilde{X})\cong N(H)/H$, where the normalizer of H is $N(H)=\{g\in\pi_1(X,x_0)\mid g^{-1}Hg=H\}$.

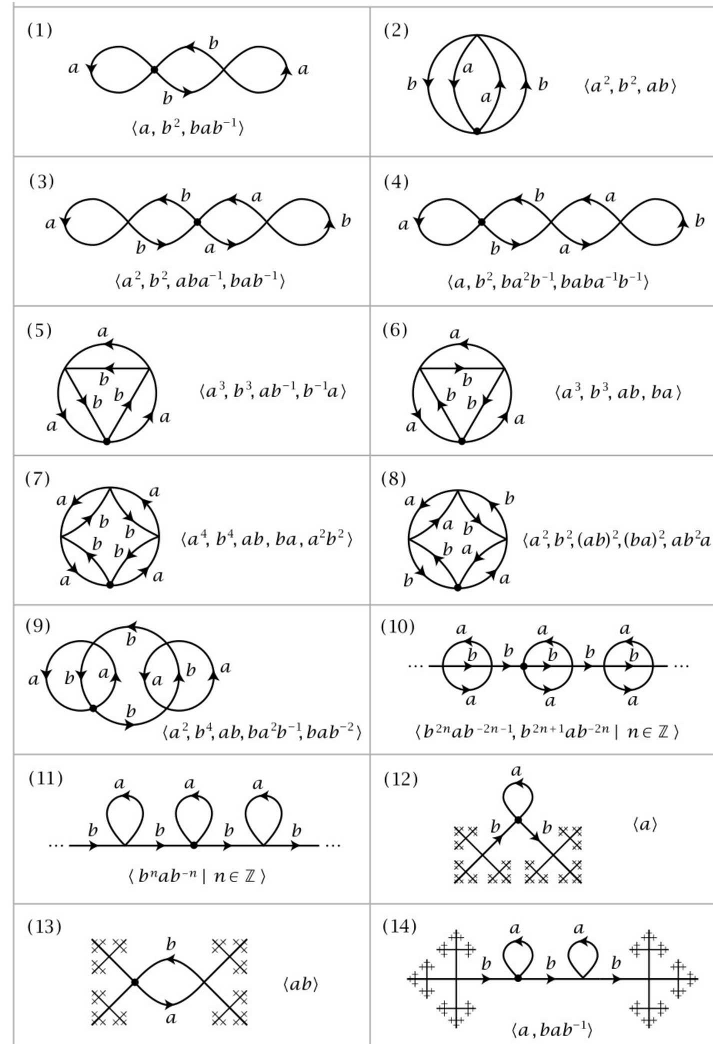
- $G(\tilde{X})\cong\pi_1(X,x_0)/H$ if $p:\tilde{X}\to X$ normal.
- $G(\tilde{X})\cong\pi_1(X,x_0)$ for $\tilde{X}$ the universal cover.

Ex (5) a normal covering space.

$H=\langle a^3, b^3, ab^{-1}, b^{-1}a\rangle$.

Note $b^{-1}a^3b \in b^{-1}Hb = H$.

$$b^{-1}a^3b = (b^{-1}a)\,a^3\,(a^{-1}b)$$



Deck transformations and group actions

Prop 1.39 Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a p.c. covering space of the p.c., l.p.c. space X. Let $H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

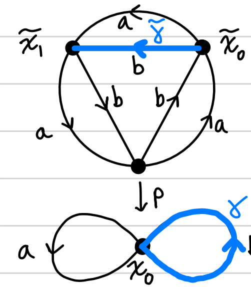
(a) $p:\tilde{X}\to X$ normal $\iff$ H normal in $\pi_1(X,x_0)$.

(b) $G(\tilde{X})\cong N(H)/H$, where the normalizer of H is $N(H)=\{g\in\pi_1(X,x_0)\mid g^{-1}Hg=H\}$.

- $G(\tilde{X})\cong\pi_1(X,x_0)/H$ if $p:\tilde{X}\to X$ normal.
- $G(\tilde{X})\cong\pi_1(X,x_0)$ for $\tilde{X}$ the universal cover.

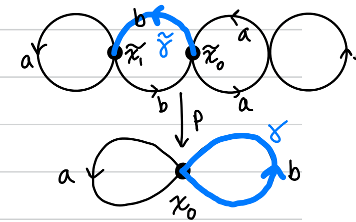
Pf (a) Let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1\in p^{-1}(x_0)$.

Note $p_*(\pi_1(\tilde{X},\tilde{x}_1))=[\gamma]^{-1}H[\gamma]$ for $[\gamma]\in\pi_1(X,x_0)$ with $\gamma=p\circ\tilde{\gamma}$.



$H=\langle a^3,b^3,ab^{-1},b^{-1}a\rangle$

$[\gamma]\in N(H)$



$H=\langle a^2,b^2,aba^{-1},bab^{-1}\rangle$

$[\gamma]\notin N(H)$. $b^{-1}(bab^{-1})b=a\notin H$

$p_*(\pi_1(\tilde{X},\tilde{x}_1))\neq H$.

No deck transformation $\tilde{x}_1\mapsto\tilde{x}_0$.

So $[\gamma]\in N(H)$ iff $p_*(\pi_1(\tilde{X},\tilde{x}_1))=p_*(\pi_1(\tilde{X},\tilde{x}_0))$, which by the lifting criterion is equivalent to deck transformations taking $\tilde{x}_1$ to $\tilde{x}_0$ and vice-versa.

Hence $p:\tilde{X}\to X$ is normal
$\iff$ such deck transformations exist $\forall\tilde{x}_1\in p^{-1}(x_0)$
$\iff$ $N(H)=\pi_1(X,x_0)$
$\iff$ H is normal in $\pi_1(X,x_0)$.

Deck transformations and group actions

A <u>group action</u> on a set Y is a function $G \times Y \to Y$, denoted $(g,y) \mapsto g.y$, satisfying

- $id.\, y = y \quad \forall y \in Y$
- $g'.(g.y) = (g'g).y \quad \forall g, g' \in G, \ \forall y \in Y.$

That is, it is a homomorphism from G to the group of permutations of Y.

A <u>group action</u> on a space Y is a homomorphism from G to the group of homeomorphisms of Y.

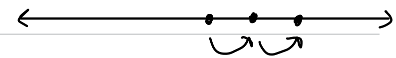
The <u>orbit space</u> Y/G is the quotient space $Y/\sim$, where $y \sim g.y \quad \forall y \in Y$ and $g \in G$.

<u>Ex</u> The group $G(\tilde{X})$ of deck transformations acts on the covering space $\tilde{X}$ by

$$G(\tilde{X}) \times \tilde{X} \to \tilde{X}$$
$$(h, \tilde{x}) \mapsto h(\tilde{x})$$

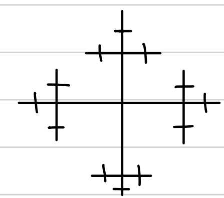
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & \circlearrowleft & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

<u>Ex</u> $\mathbb{Z}$ acts on $\mathbb{R}$



<u>Ex</u> $\mathbb{Z}^n$ acts on $\mathbb{R}^n$

<u>Ex</u> $\langle a, b \rangle$ acts on



So does $\mathbb{Z}/4$, via rotations, but not freely.

For a normal covering space $\tilde{X} \to X$, the orbit space $\tilde{X}/G(\tilde{X})$ is homeomorphic to X.

Deck transformations and group actions

Proposition 1.40. *If an action of a group G on a space Y satisfies $(*)$, then:*

(a) *The quotient map $p : Y \to Y/G$, $p(y) = Gy$, is a normal covering space.*

(b) *G is the group of deck transformations of this covering space $Y \to Y/G$ if Y is path-connected.*

(c) *G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$ if Y is path-connected and locally path-connected.*

$(*)$ Each $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.