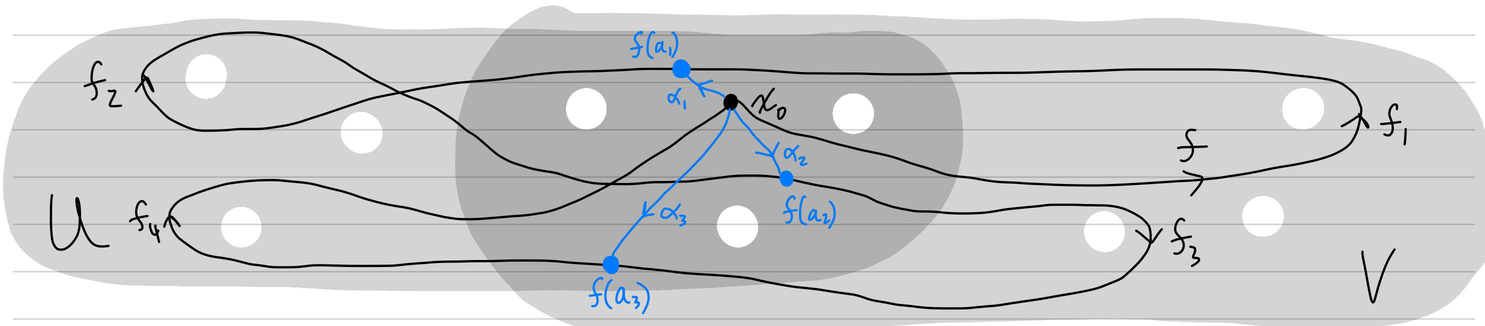
Chapter II: The Seifert-van Kampen theorem

In this chapter, we show how to understand $\pi_1(U \cup V)$ in terms of U, V, and $U \cap V$.



First, we will cover three sections of algebraic preliminaries.

Algebra terminology (from Section 52)

Let G, G' be groups.

A homomorphism $f: G \to G'$ satisfies $f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in G$.

Its kernel is $f^{-1}(e')$, where e' is the identity in G'.

A homomorphism is an isomorphism if it is bijective.

A subgroup H of G is normal if $xhx^{-1} \in H \quad \forall x \in G$ and $\forall h \in H$, or equivalently, if $xH = Hx \quad \forall x \in G$.

If so, the quotient group G/H has elements the cosets $xH \quad \forall x \in G$, with group operation $(xH) \cdot (yH) = (x \cdot y) H$.

Note
$$\begin{aligned} f: G &\to G/H \\ x &\mapsto xH \end{aligned}$$
is a surjective homomorphism with kernel H.

Conversely, if homomorphism $f: G \to G'$ is surjective, then its kernel N is normal in G, and the induced map
$$\begin{aligned} G/N &\to G' \\ xN &\mapsto f(x) \end{aligned}$$
is an isomorphism.

Section 67: Direct sums of abelian groups
Section 68: Free products of groups
Section 69: Free groups

Let $\{G_\alpha\}_{\alpha \in J}$ be a family of (abelian?) groups.

Moral: In the category $\underline{Ab}$ of abelian groups,
the categorical product is the direct product $\prod_\alpha G_\alpha$,
and the categorical coproduct is the direct sum $\bigoplus_\alpha G_\alpha$
In the category $\underline{Gp}$ of groups,
the categorical product is the direct product $\prod_\alpha G_\alpha$,
and the categorical coproduct is the free product $*_\alpha G_\alpha$

Schedule:

Direct products of (abelian?) groups	$\prod_\alpha G_\alpha$	
Direct sums of abelian groups	$\bigoplus_\alpha G_\alpha$	(Section 67)
Free products of groups	$*_\alpha G_\alpha$	(Section 68)
Free abelian groups	$\bigoplus_\alpha \mathbb{Z}$	(Section 67)
Free groups	$*_\alpha \mathbb{Z}$	(Section 69)

Section 67: Direct sums of abelian groups

Let $\{G_\alpha\}_{\alpha \in J}$ be a family of (abelian?) groups.

The <u>direct product</u> $\prod_{\alpha\in J} G_\alpha$ is a group with operation given by

$$(x_\alpha)_{\alpha\in J} \cdot (y_\alpha)_{\alpha\in J} = (x_\alpha \cdot y_\alpha)_{\alpha\in J}.$$

$$(x_\alpha)_{\alpha\in J} + (y_\alpha)_{\alpha\in J} = (x_\alpha + y_\alpha)_{\alpha\in J}.$$

This is only a difference in notation. Additive notation for Abelian groups.

<u>Ex</u> In $\mathbb{Z} \times \mathbb{Z}/4$ we have $(2,1)+(3,3) = (5,0)$, and we have $-(2,1) = (-2,-1) = (-2,3)$.

Universal property (categorical product) in <u>Gp</u> and <u>Ab</u>:

$$\begin{tikzcd} H \arrow[r, dashed, "\exists! f"'] \arrow[dr, "f_\beta"'] & \prod_{\alpha\in J} G_\alpha \arrow[d, "\pi_\beta"] & \ni & (x_\alpha)_{\alpha\in J} \arrow[d, mapsto] \\ & G_\beta & \ni & x_\beta \end{tikzcd}$$

Given any (abelian?) group H and family of homomorphisms $f_\beta : H \to G_\beta$ $\forall \beta \in J$, there exists a unique homomorphism $f : H \to \prod_{\alpha\in J} G_\alpha$ with $\pi_\beta \circ f = f_\beta$ $\forall \beta \in J$.

(Indeed, let $f(h) = (f_\alpha(h))_{\alpha\in J}$ $\forall h \in H$.)

Restrict attention to abelian groups $\{G_\alpha\}_{\alpha\in J}$ (where a universal property holds)

The direct sum $\bigoplus_{\alpha\in J} G_\alpha$ is the subgroup of $\prod_{\alpha\in J} G_\alpha$ consisting of those tuples $(x_\alpha)_{\alpha\in J}$ with $x_\alpha = id_{G_\alpha}$ for all but finitely many α.

For J finite, $\bigoplus_{\alpha\in J} G_\alpha = \prod_{\alpha\in J} G_\alpha$.

Ex In $\mathbb{Z}\oplus\mathbb{Z}/4$ we have $(2,1)+(3,3)=(5,0)$, and we have $-(2,1)=(-2,-1)=(-2,3)$.

Universal property (categorical coproduct) in Ab:

$$\begin{tikzcd}
H & \bigoplus_{\alpha\in J} G_\alpha \arrow[l, dashed, "\exists! f"'] & (z_\alpha)_{\alpha\in J} \\
& G_\beta \arrow[ul, "f_\beta"] \arrow[u, "i_\beta"] & x_\beta \arrow[u, mapsto]
\end{tikzcd}$$

$\bigoplus_{\alpha\in J} G_\alpha \ni (z_\alpha)_{\alpha\in J}$, $\quad G_\beta \ni x_\beta$

where $z_\alpha = \begin{cases} x_\beta & \text{if } \alpha=\beta \\ id_{G_\alpha} & \text{otherwise} \end{cases}$

Given any abelian group H and family of homomorphisms $f_\beta : G_\beta \to H$ $\forall \beta\in J$, there exists a unique homomorphism $f : \bigoplus_{\alpha\in J} G_\alpha \to H$ with $f\circ i_\beta = f_\beta$ $\forall\beta\in J$.

(Indeed, let $f\left(\sum_{j=1}^n i_{\beta_j}(x_{\beta_j})\right) = \sum_{j=1}^n f_{\beta_j}(x_{\beta_j})$.)

Ex Let J be infinite and $G_\beta = \mathbb{Z}$ $\forall \beta$.
To see that $\bigoplus_{\alpha \in J} \mathbb{Z}$ (equipped with the projections π_β)
is not the categorical product in **Ab**,
consider $H = \prod_{\alpha \in J} \mathbb{Z}$ with $f_\beta = \pi_\beta$ $\forall \beta$.

Correct:

$$\begin{tikzcd}
H \arrow[r, dashed, "\exists!\, f"'] \arrow[dr, "f_\beta"'] & \prod_{\alpha \in J} G_\alpha \arrow[d, "\pi_\beta"] & \ni & (x_\alpha)_{\alpha \in J} \arrow[d, mapsto] \\
& G_\beta & \ni & x_\beta
\end{tikzcd}$$

Incorrect:

$$\begin{tikzcd}
\prod_{\alpha \in J} \mathbb{Z} \arrow[r, dashed, "\nexists\, f"'] \arrow[dr, "\pi_\beta"'] & \bigoplus_{\alpha \in J} \mathbb{Z} \arrow[d, "\pi_\beta"] & \ni & (x_\alpha)_{\alpha \in J} \arrow[d, mapsto] \\
& \mathbb{Z} & \ni & x_\beta
\end{tikzcd}$$

Section 68 : Free products of groups

Now, let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups.

The free product $\ast_{\alpha \in J} G_\alpha$ will satisfy the universal property making it the coproduct in $\underline{Gp}$:

$$\begin{tikzcd}
H & \ast_{\alpha \in J} G_\alpha \arrow[l, dashed, "\exists ! f"'] & \ni & x_\beta \\
 & G_\beta \arrow[ul, "f_\beta"] \arrow[u, "i_\beta"'] & \ni & x_\beta \arrow[u, mapsto]
\end{tikzcd}$$

Given <u>any</u> group H and family of homomorphisms $f_\beta : G_\beta \to H$ $\forall \beta \in J$, there exists a unique homomorphism $f : \ast_{\alpha \in J} G_\alpha \to H$ with $f \circ i_\beta = f_\beta$ $\forall \beta \in J$.

What is a more explicit definition of the free product $\ast_{\alpha \in J} G_\alpha$?

First, let's do an example.

Let $\mathbb{Z} = \{\dots, g^{-2}, g^{-1}, g^{0}, g^{1}, g^{2}, \dots\} = \langle g \rangle$ and $\mathbb{Z}/4 = \{r, r^2, r^3, r^0\} = \langle r \mid r^4 \rangle$

Though these groups are abelian, their free product $\mathbb{Z} * \mathbb{Z}/4$ is not.

Its elements are finite words such as

$g^7 r^3 g^{-2} r$ and $r^{-1} g^2 r^{-1} g^{-7}$.

Multiplication looks like

$$(g^7 r^3 g^{-2} r)(r^{-1} g^2 r^{-1} g^{-7}) = g^7 r^3 \cancel{g^{-2}}\, \cancel{r}\, \cancel{r^{-1}}\, \cancel{g^{2}}\, r^{-1} g^{-7} = g^7 r^2 g^{-7}$$

and

$$(r^{-1} g^2 r^{-1} g^{-7})(g^7 r^3 g^{-2} r) = r^{-1} g^2 r^{-1} \cancel{g^{-7}}\, \cancel{g^{7}}\, r^3 g^{-2} r = r^{-1} g^2 r^2 g^{-2} r.$$

Inverses look like $(g^7 r^3 g^{-2} r)^{-1} = r^{-1} g^2 r^{-3} g^{-7}$

For another example, let $\mathbb{Z} = \langle g \rangle$, $\mathbb{Z} = \langle h \rangle$, and $\mathbb{Z}/4 = \langle r \mid r^4 \rangle$.

Elements of the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}/4$ are finite words, such as
$gh^2rhrg^{-7}h^{-1}$ and $hr^3h^{-2}g^{-1}$.

Multiplication looks like

$$(gh^2rhrg^{-7}\cancel{h^{-1}})(\cancel{h}r^3h^{-2}g^{-1}) = gh^2rhrg^{-7}r^3h^{-2}g^{-1}$$

and

$$(h\cancel{r^3}\cancel{h^{-2}}\cancel{g^{-1}})(\cancel{g}\cancel{h^2}\cancel{r}hrg^{-7}h^{-1}) = h^2rg^{-7}h^{-1}.$$

$r^4 = \mathrm{id}$ in $\mathbb{Z}/4$

More generally, the elements of $*_{\alpha \in J} G_\alpha$ are <u>reduced words</u>: finite strings of nonidentity elements in the G_α's such that adjacent letters are from different G_α's.
To multiply, concatenate and then reduce.
The identity is the empty word.

(Checking associativity is hard, but the universal property is related to a weaker extension property that helps.)

Section 68 : Free products of groups

Now, let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups.

The free product $*_{\alpha \in J} G_\alpha$ will satisfy the universal property making it the coproduct in $\underline{Gp}$:

$$\begin{tikzcd}
H & & \underset{\alpha \in J}{*} G_\alpha \arrow[ll, dashed, "\exists ! f"'] & \ni & x_\beta \\
& & G_\beta \arrow[ull, "f_\beta"] \arrow[u, "i_\beta"] & \ni & x_\beta \arrow[u, mapsto]
\end{tikzcd}$$

← a reduced word of length one

Do you see now why the definition of i_β makes sense ?

$f(g_1 g_2 \cdot \ldots \cdot g_n) = f_{\beta_1}(g_1) f_{\beta_2}(g_2) \cdot \ldots \cdot f_{\beta_n}(g_n)$ where $g_i \in \beta_i$ $\forall i$.

Let G be a group.
Elements $x, y \in G$ are <u>conjugate</u> if $y = cxc^{-1}$ for some $c \in G$.
A <u>normal</u> subgroup of G is one that contains all conjugates of its elements.

For S a subset of G, let N be the intersection of all normal subgroups of G containing S.
Can you see why N is a normal subgroup of G?
It is called the <u>least normal subgroup</u> containing S.

<u>Lemma</u> The least normal subgroup N of G containing S is generated by all conjugates of elements of S.

<u>Pf</u> Let N' be the subgroup of G generated by all conjugates of elements in S.
Clearly $N' \subset N$ since N is normal.

To show $N \subset N'$, it suffices to show N' is normal in G. So, let $x \in N'$ and $c \in G$.
Then $x = x_1 x_2 \cdot \ldots \cdot x_n$ with $x_i = c_i s_i c_i^{-1}$ for some $c_i \in G$ and s_i satisfying $s_i \in S$ or $s_i^{-1} \in S$.
So $cxc^{-1} = (cx_1c^{-1})(cx_2c^{-1}) \ldots (cx_nc^{-1}) = \big((cc_1) s_1 (cc_1)^{-1}\big)\big((cc_2) s_2 (cc_2)^{-1}\big) \cdot \ldots \cdot \big((cc_n) s_n (cc_n)^{-1}\big)$,
giving $cxc^{-1} \in N'$, as desired.

Thm Let $G = G_1 * G_2$, N_i normal in G_i,
N the least normal subgroup of G containing N_1 and N_2.
Then $G/N \cong (G_1/N_1) * (G_2/N_2)$.

Proof uses the universal property

Corollary If N is the least normal subgroup of G containing G_1, then $(G_1 * G_2)/N \cong G_2$.

Section 69 Free groups

A *free group* is isomorphic to $*_{\alpha \in J} \mathbb{Z}$.
Possible notation is $*_{\alpha \in J} \langle g_\alpha \rangle$, where $\langle g_\alpha \rangle = \{\ldots, g_\alpha^{-2}, g_\alpha^{-1}, g_\alpha^{0}, g_\alpha^{1}, g_\alpha^{2}, \ldots\}$ is infinite cyclic.

A *free abelian group* is isomorphic to $\bigoplus_{\alpha \in J} \mathbb{Z}$.

There are universal properties characterizing these as the "free objects" in the categories *Gp* and *Ab* of groups and abelian groups, respectively.

Any subgroup of $\bigoplus_{i=1}^{m} \mathbb{Z}$ isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}$ for $n \leq m$.

Similarly, any subgroup of a free group is free.
(One beautiful proof uses covering spaces.)
But, surprisingly, $\mathbb{Z} * \mathbb{Z} = *_{i=1}^{2} \mathbb{Z}$ has a subgroup isomorphic to $*_{i=1}^{n} \mathbb{Z}$ for any integer n!

Indeed, the subgroup of $\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ generated by $b, aba^{-1}, a^2ba^{-2}, \ldots, a^{n-1}ba^{-(n-1)}$ is isomorphic to $*_{i=1}^{n} \mathbb{Z}$.

Ex The element $ababa^{-2}b^3a^2b^{-1}ab^2a^{-1}$ is in the subgroup H of $\mathbb{Z} * \mathbb{Z}$ generated by b, aba^{-1}, a^2ba^{-2}.

Aside: $(a^2ba^{-2})^{-1} = a^2b^{-1}a^{-2}$

Check that

$$ababa^{-2}b^3a^2b^{-1}ab^2a^{-1} = (aba^{-1})(a^2ba^{-2})(b)(b)(b)(a^2b^{-1}a^{-2})(aba^{-1})(aba^{-1})$$

is the unique way to write this element as a product of the terms b, aba^{-1}, a^2ba^{-2} and their inverses.

One can show there is an isomorphism

$$\begin{aligned} H &\xleftrightarrow{\cong} \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle g_0, g_1, g_2 \rangle \\ b &\longleftrightarrow g_0 \\ aba^{-1} &\longleftrightarrow g_1 \\ a^2ba^{-2} &\longleftrightarrow g_2 \end{aligned}$$

Group presentations

Let $Z = \{g_\alpha\}_{\alpha\in J}$ be a set (of generators)
and $R = \{r_\beta\}_{\beta\in K}$ be a set (of relations)
with $r_\beta \in \ast_{\alpha\in J}\langle g_\alpha\rangle$ $\forall\beta$.

Recall $\langle g_\alpha\rangle = \{\dots, g_\alpha^{-2}, g_\alpha^{-1}, g_\alpha^{0}, g_\alpha^{1}, g_\alpha^{2}, \dots\}$ is the infinite cyclic group with generator g_α.

Then $\langle Z \mid R\rangle$ is the group $\left(\ast_{\alpha\in J}\langle g_\alpha\rangle\right)/N$,
where N is the least normal subgroup containing R.

This presentation is...
- finitely generated if Z finite,
- finitely presented if Z, R finite: $\langle g_1, \dots, g_n \mid r_1, \dots, r_m\rangle$.

Ex $\langle a, b\rangle \cong \mathbb{Z} \ast \mathbb{Z}$

Ex $\langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$

Ex $\langle r, s \mid r^n, s^2, \underbrace{srsr}_{srs=r^{-1}}\rangle$ is the dihedral group (symmetries of regular n-gon)

Ex $\langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid \alpha_3^{-1}\beta_3\rangle \cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2, g\rangle$

The group isomorphism problem (deciding if two finitely presented groups are isomorphic) is undecidable.
In some finitely presented groups the word problem (deciding if a word is the identity) is undecidable.

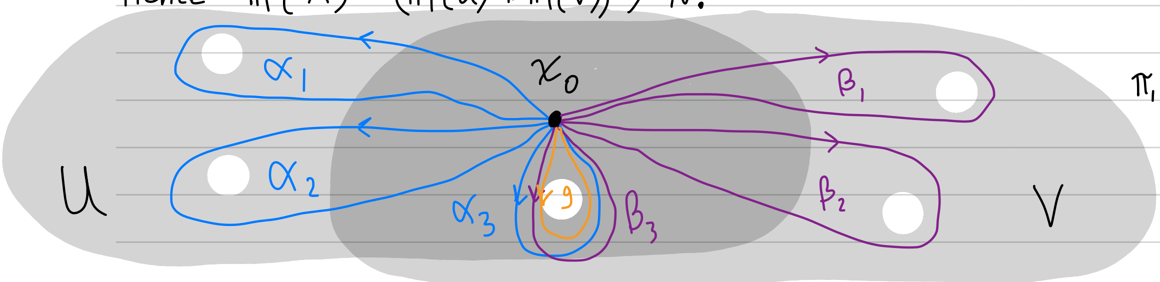
Section 70 The Seifert-van Kampen theorem

Thm (Seifert-van Kampen) Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_U(g)^{-1} i_V(g)$ for $g \in \pi_1(U \cap V, x_0)$.

Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.



$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "j_U"] & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "j_V"'] &
\end{tikzcd}$$

$$\Phi(\alpha_1 \beta_2^2 \alpha_2^{-1}) = j_U(\alpha_1)\, j_V(\beta_2)^2\, j_U(\alpha_2)^{-1}$$

Section 70 The Seifert-van Kampen theorem

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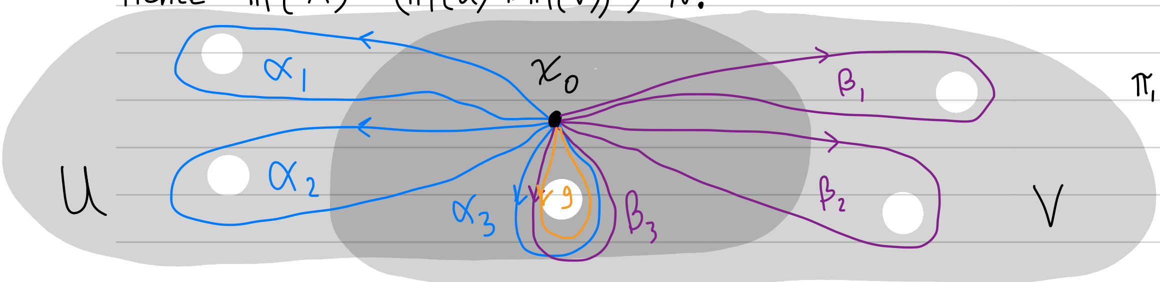
Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.



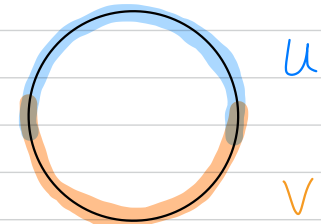
$$\begin{tikzcd}
 & \pi_1(U) \ni \alpha_3 \arrow[dr, "j_U"] & \\
\pi_1(U \cap V) \ni g \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\
 & \pi_1(V) \ni \beta_3 \arrow[ur, "j_V"'] &
\end{tikzcd}$$

$\pi_1(U) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ $\quad \pi_1(V) = \langle \beta_1, \beta_2, \beta_3 \rangle$ $\quad \pi_1(U \cap V) = \langle g \rangle$

$$\begin{aligned}
\pi_1(X) &\cong \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid i_U(g)^{-1} i_V(g) \rangle \\
&= \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid \alpha_3^{-1} \beta_3 \rangle \\
&\cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2, g \rangle
\end{aligned}$$

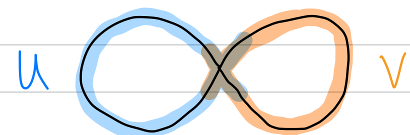
Note $i_U(g) = \alpha_3$ and $i_V(g) = \beta_3$

To see that the path-connected assumption is necessary, note that if $X = S^1$ with U, V as drawn, then Φ is not even surjective.



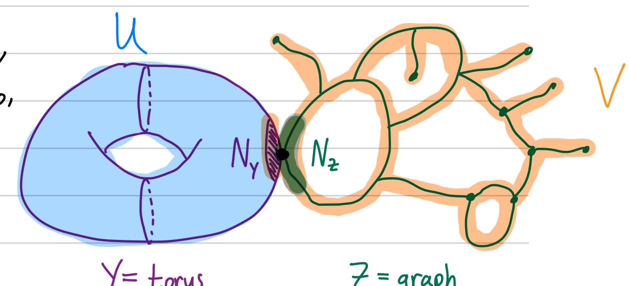
<u>Corollary</u> If $U \cap V$ is simply connected, then $\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$ is an isomorphism.

<u>Ex</u> $\pi_1(\text{figure eight}) = \pi_1(S^1 \vee S^1) \cong \langle a, b \rangle$.



More generally, if (Y, y_0) and (Z, z_0) are two pointed spaces, then their <u>wedge sum</u> $Y \vee Z$ is the quotient space $(Y \amalg Z) / (y_0 \sim z_0)$.

True if X, Y are "CW complexes" } If path-connected Y has a contractible nbhd N_Y about y_0, and path-connected Z has a contractible nbhd N_Z about z_0, then applying the corollary (with $U = Y \vee N_Z$, $V = N_Y \vee Z$, $U \cap V = N_Y \vee N_Z \simeq *$, $U \cup V = Y \vee Z$) gives $\pi_1(Y \vee Z) \cong \pi_1(Y) * \pi_1(Z)$.



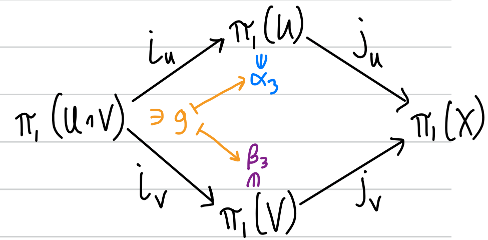
Thm (Seifert-van Kampen)

Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_U(g)^{-1} i_V(g)$ for $g \in \pi_1(U \cap V, x_0)$.

Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.



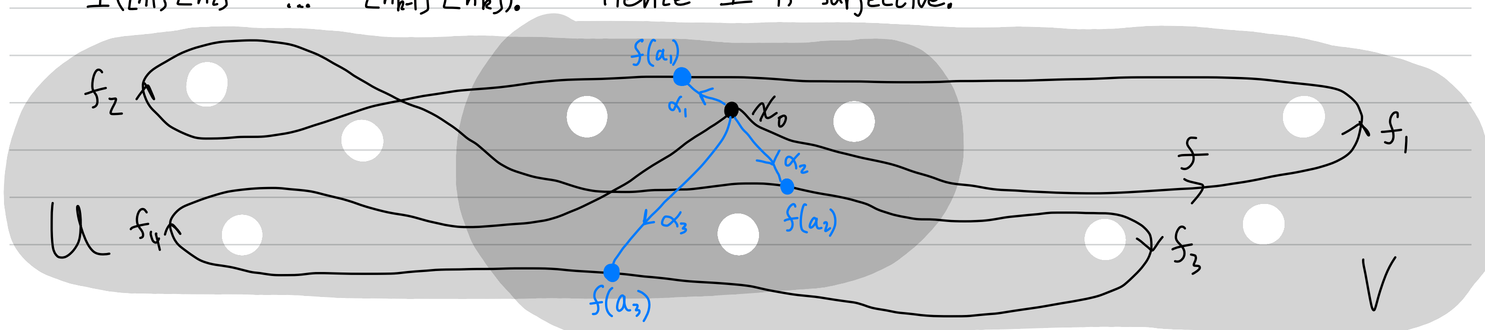
Proof Sketch (following Hatcher Thm 1.20)

Notation: A *factorization* of $[f] \in \pi_1(X)$ is a product $[f] = \Phi([h_1] \cdots [h_k])$ with each $[h_i]$ in $\pi_1(U)$ or $\pi_1(V)$.

Recall from the proof of Munkres Thm 59.1 that $[f]$ has a factorization

$$\begin{aligned}[f] &= [f_1 \bar{\alpha}_1][\alpha_1 f_2 \bar{\alpha}_2] \cdot \ldots \cdot [\alpha_{k-2} f_{k-1} \bar{\alpha}_{k-1}][\alpha_{k-1} f_k] \\ &= \Phi([h_1][h_2] \cdot \ldots \cdot [h_{k-1}][h_k]).\end{aligned}$$

Hence Φ is surjective.



It is clear $N \subset \ker(\Phi)$ since $\forall\, g \in \pi_1(U \cap V)$,

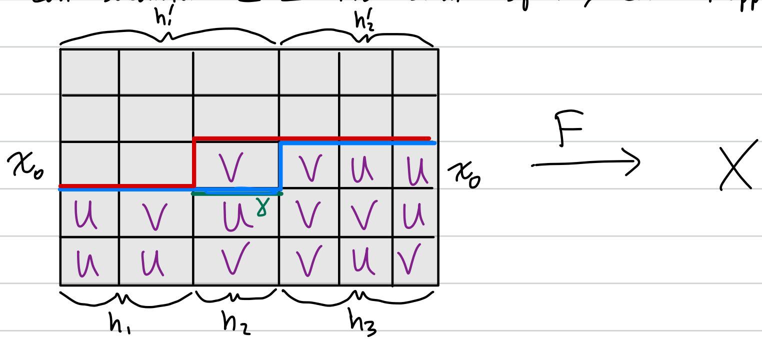
$$\begin{aligned}
\Phi(i_U(g)^{-1} i_V(g)) &= \Phi(i_U(g)^{-1})\, \Phi(i_V(g)) \\
&= \Phi(i_U(g))^{-1}\, \Phi(i_V(g)) \\
&= j_U(i_U(g))^{-1}\, j_V(i_V(g)) \\
&= \text{identity}
\end{aligned}$$

since
$$\begin{tikzcd}
 & U \arrow[dr, hook] & \\
U \cap V \arrow[ur, hook] \arrow[dr, hook] & & X \\
 & V \arrow[ur, hook] &
\end{tikzcd}$$
commutes.

<u>Aside</u> Similary for conjugates:
Given any $c \in \pi_1(U) * \pi_1(V)$,
we have
$$\begin{aligned}
\Phi(c\, i_U(g)^{-1} i_V(g)\, c^{-1}) &= \Phi(c)\, \Phi(i_U(g)^{-1} i_V(g))\, \Phi(c^{-1}) \\
&= \Phi(c)\, \Phi(c)^{-1} \\
&= \text{identity}.
\end{aligned}$$

It remains to show $\ker(\Phi) \subset N$, i.e. if two factorizations have the same image
$\Phi([h_1] \cdots [h_k]) = \Phi([h_1'] \cdots [h_\ell'])$,
then one can be obtained from the other by regarding $[h_i] \in \pi_1(U \cap V)$ as lying in $\pi_1(U)$ or $\pi_1(V)$ and reducing, i.e. by applying relations in N.

- Let $F: I \times I \to X$ be a homotopy from $h_1 \cdots h_k$ to $h_1' \cdots h_\ell'$.
- By compactness of $I \times I$ (and the Lebesgue number lemma) we can subdivide $I \times I$ into small squares, each mapped into U or V.



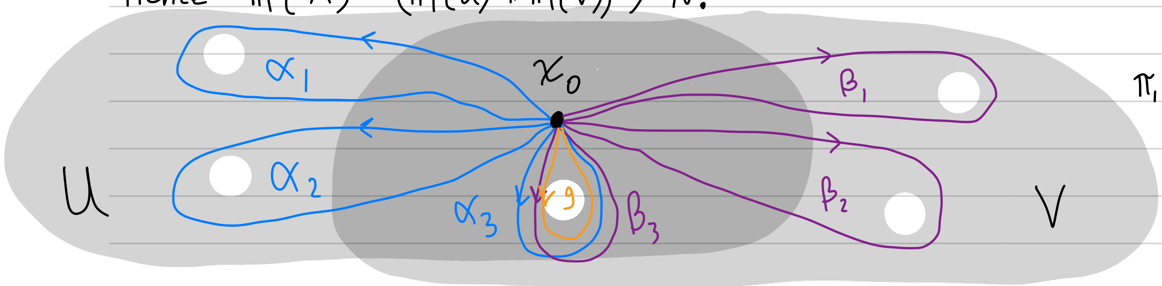
- At each vertex of the subdivision, choose a path in U, in V, or in $U \cap V$ to x_0.
- Obtain the blue factorization from the red one by regarding the green loop γ (including the paths to x_0) as lying in V insead of in U, using a relation in N. Then homotope across the small square.
- Continue until we obtain the factorization $[h_1'] \cdots [h_\ell']$ from $[h_1] \cdots [h_k]$ after applying relations in N.

Thm (Seifert–van Kampen) Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_U(g)^{-1} i_V(g)$ for $g \in \pi_1(U \cap V, x_0)$.

Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.



$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "j_U"] & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "j_V"'] &
\end{tikzcd}$$

Ex In the finitely presented case, if

$$\pi_1(U, x_0) = \langle \alpha_1, \ldots, \alpha_k \mid r_1, \ldots, r_\ell \rangle$$
$$\pi_1(V, x_0) = \langle \beta_1, \ldots, \beta_m \mid s_1, \ldots, s_n \rangle$$
$$\pi_1(U \cap V, x_0) = \langle g_1, \ldots, g_p \mid t_1, \ldots, t_q \rangle$$

then $\pi_1(X, x_0) \cong \langle \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \mid r_1, \ldots, r_\ell, s_1, \ldots, s_n, i_U(g_1)^{-1} i_V(g_1), \ldots, i_U(g_p)^{-1} i_V(g_p) \rangle$.

Aside $i_U(g_1 g_2)^{-1} i_V(g_1 g_2) = i_U(g_2)^{-1} i_U(g_1)^{-1} i_V(g_1) i_V(g_2) = i_U(g_2)^{-1}\, i_U(g_1)^{-1} i_V(g_1)\, i_U(g_2)\, i_U(g_2)^{-1} i_V(g_2)$.

Thm (Seifert-van Kampen) Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_U(g)^{-1} i_V(g)$ for $g \in \pi_1(U \cap V, x_0)$.

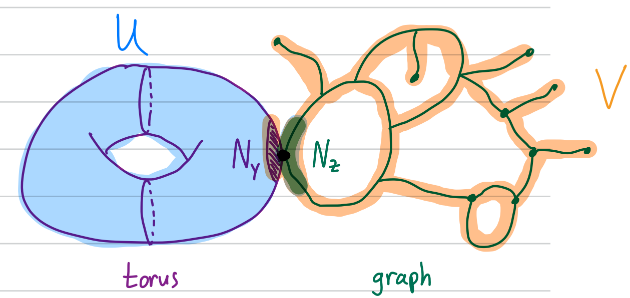
Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "j_U"] & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "j_V"'] &
\end{tikzcd}$$

Corollary Given the hypotheses of the theorem, if $U \cap V$ is simply connected, then

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism.



Thm (Seifert–van Kampen) Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

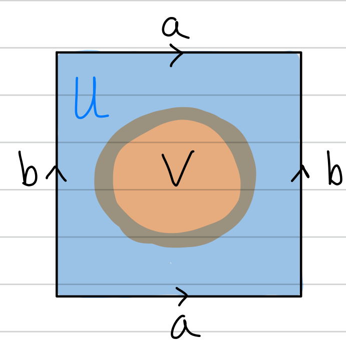
is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_u(g)^{-1} i_v(g)$ for $g \in \pi_1(U \cap V, x_0)$.
Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "j_u"] & \\ \pi_1(U \cap V) \arrow[ur, "i_u"] \arrow[dr, "i_v"'] & & \pi_1(X) \\ & \pi_1(V) \arrow[ur, "j_v"'] & \end{tikzcd}$$

Corollary Given the hypotheses of the theorem, if V is simply connected, then Φ induces an isomorphism

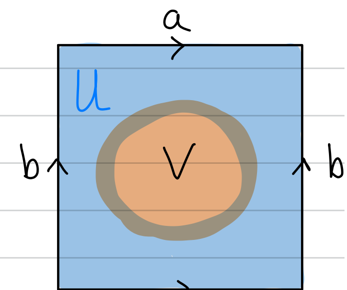
$$\pi_1(U, x_0)/N \longrightarrow \pi_1(X, x_0)$$

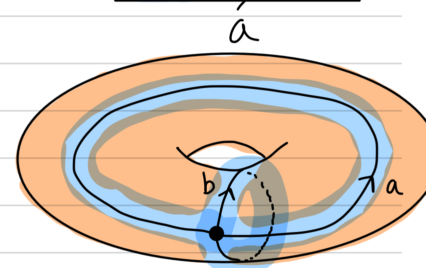
where N is the least normal subgroup of $\pi_1(U, x_0)$ containing all words of the form $i_u(g)$ for $g \in \pi_1(U \cap V, x_0)$.



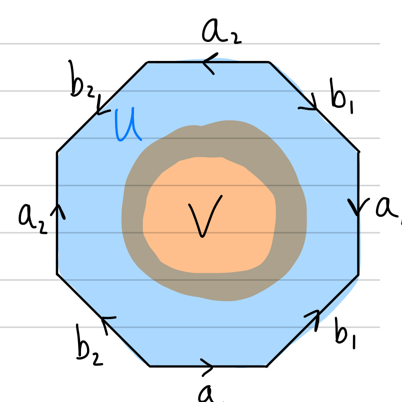
Ex Let $T = S^1 \times S^1$ be the torus.
We already saw $\pi_1(T) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

Alternatively, apply SVK with $\pi_1(U) = \langle a, b \rangle$,
$\pi_1(V) = \{id\}$, $\pi_1(U \cap V) = \langle g \rangle$, $i_U(g) = a\,b\,a^{-1}b^{-1}$
to get
$\pi_1(T) \cong \langle a, b \mid a\,b\,a^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$.



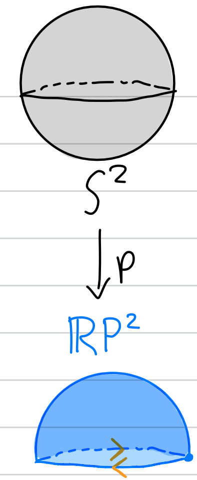


Ex Let $T \# T$ be the double torus.
Apply SVK with $\pi_1(U) = \langle a_1, b_1, a_2, b_2 \rangle$,
$\pi_1(V) = \{id\}$, $\pi_1(U \cap V) = \langle g \rangle$, $i_U(g) = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$
to get
$\pi_1(T \# T) \cong \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle$.



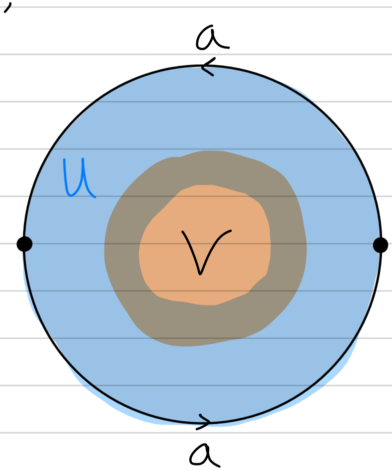
Aside: In homework, you determine $\pi_1(T \# T)$ using SVK on a union using different pieces.

<u>Ex</u> Let $\mathbb{R}P^2$ be real projective space:
$\mathbb{R}P^2 = S^2/\sim$, where $x \sim -x \ \forall x \in S^2$.

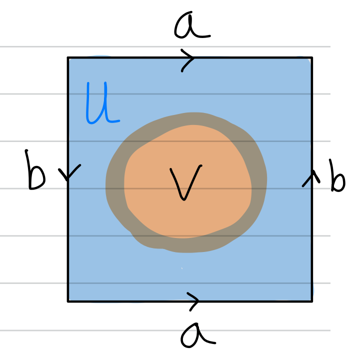


We already saw $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ since $\mathbb{R}P^2$ has a simply connected 2-fold covering space (S^2).

Alternatively, apply SVK with $\pi_1(U) = \langle a \rangle$, $\pi_1(V) = \{id\}$, $\pi_1(U \cap V) = \langle g \rangle$, $i_U(g) = a^2$ to get
$$\pi_1(\mathbb{R}P^2) \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}/2.$$



Ex Let K be the Klein bottle.



Apply SVK with $\pi_1(U) = \langle a, b \rangle$,
$\pi_1(V) = \{id\}$, $\pi_1(U \cap V) = \langle g \rangle$, $i_U(g) = aba^{-1}b$
to get
$\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle$.

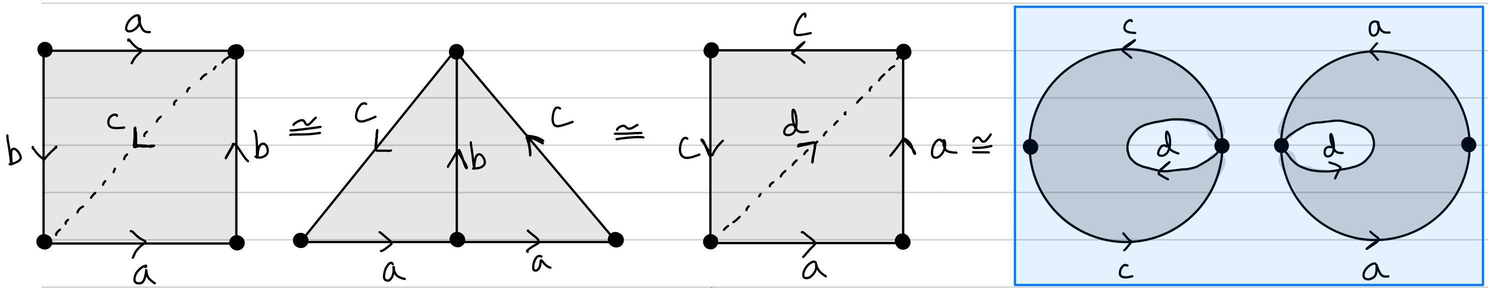
Later, we will see that any compact surface is homeomorphic to either the 2-sphere S^2, the n-fold connected sum of tori $\underbrace{T \# \dots \# T}_{n \text{ times}}$, or the m-fold connected sum of projective planes $\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}$.

Using SVK, we can compute the fundamental groups of these surfaces (and their abelianizations) to show no two surfaces on this list are homeomorphic.

$\pi_1(\underbrace{T \# \dots \# T}_{n \text{ times}}) \cong \langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$ with abelianization $\bigoplus_{i=1}^{2n} \mathbb{Z}$.

$\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \langle a_1, \dots, a_m \mid a_1^2 a_2^2 \cdots a_m^2 \rangle$ with abelianization $(\bigoplus_{i=1}^{m-1} \mathbb{Z}) \oplus \mathbb{Z}/2$.

Note the Klein bottle K is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$:



Klein bottle K "standard form" $\mathbb{RP}^2 \# \mathbb{RP}^2$

$\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle$ $\pi_1(K) \cong \langle a, c \mid a^2 c^2 \rangle$

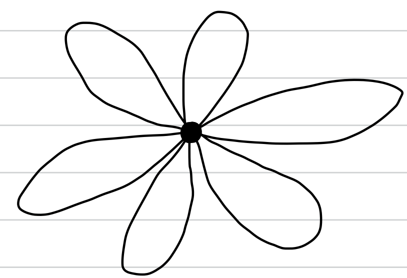
Section 71: The fundamental group of a wedge of circles

Let $\{(S^1_\alpha, p_\alpha)\}_{\alpha\in J}$ be a collection of circles, each with a chosen basepoint $p_\alpha \in S^1_\alpha$.

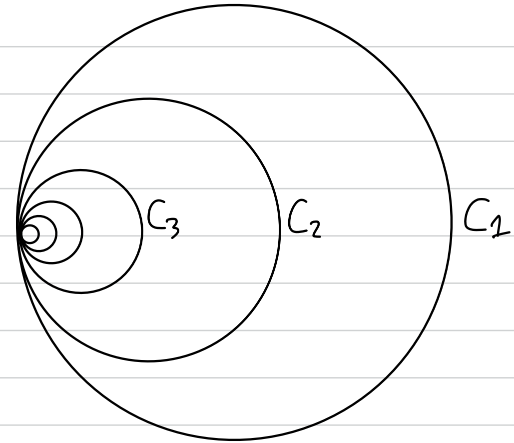
Their wedge sum is $\bigvee_{\alpha\in J} S^1_\alpha = \left(\coprod_{\alpha\in J} S^1_\alpha\right) \big/ (p_\alpha \sim p_\beta \ \forall\, \alpha,\beta \in J)$.

(The topology is such that a set U is open in the wedge sum $\iff$ each intersection $U \cap S^1_\alpha$ is open in S^1_α.)

Thm $\pi_1\left(\bigvee_{\alpha\in J} S^1_\alpha\right) \cong \mathop{\ast}_{\alpha\in J} \langle f_\alpha \rangle$



Be careful with the topology!



Ex The Hawaiian earrings space is $X = \bigcup_{n \geq 1} C_n$,
where $C_n = \{(x,y) \in \mathbb{R}^2 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$,
with the subspace topology from $\mathbb{R}^2$.
(This is not the wedge sum topology.)

The loop $g : [0,1] \to X$ that wraps around C_n over $[\frac{1}{n+1}, \frac{1}{n}]$ is continuous.
Note $[g]$ does not belong to the subgroup of $\pi_1(X)$ generated by $[f_1], [f_2], \ldots, [f_n]$ for any n, where f_i wraps around C_i.

(To see this, for $N > n$ consider the map $h : X \to C_N$ which is the identity on C_N and maps C_i to the basepoint for $i \neq N$.
Note $h_*([g]) \neq 0$ but $h_*([f_i]) = 0$ for $i = 1, \ldots, n$.)

Hence $\pi_1(X) \neq *_{i=1}^{\infty} \langle [f_i] \rangle$.

elements are words of <u>finite</u> length (in an infinite alphabet)

Aside: Interestingly, while the abelianization of $\pi_1(\vee_{i=1}^{\infty} S^1) = *_{i=1}^{\infty} \mathbb{Z}$ is $\bigoplus_{i=1}^{\infty} \mathbb{Z}$, the abelianization of $\pi_1(X)$ is

$$\prod_{i=1}^{\infty} \mathbb{Z} \oplus \left(\prod_{i=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{i=1}^{\infty} \mathbb{Z} \right).$$

Section 72: Adjoining a two-cell

If desired, you can be more careful with basepoints.
Let $x_0 \in U \cap V$, let $y_0 \in U$, let γ be a path in U from y_0 to x_0.

Apply SVK with

$\pi_1(U, y_0) = \langle [a], [b] \rangle$

$\pi_1(U, x_0) = \langle \hat{\gamma}[a], \hat{\gamma}[b] \rangle$ Recall $\hat{\gamma}[\alpha] := [\bar{\gamma} * \alpha * \gamma]$

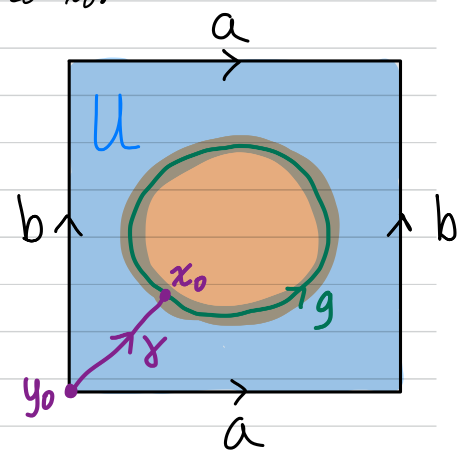
$\pi_1(V, x_0) = \{id\}$

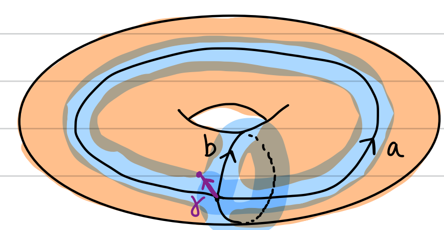
$\pi_1(U \cap V, x_0) = \langle [g] \rangle$

$i_U([g]) = [\bar{\gamma} * a * b * \bar{a} * \bar{b} * \gamma] = \hat{\gamma}[a] * \hat{\gamma}[b] * \hat{\gamma}[a]^{-1} * \hat{\gamma}[b]^{-1}$

to get

$$\pi_1(T, x_0) \cong \langle \hat{\gamma}[a], \hat{\gamma}[b] \mid \hat{\gamma}[a] * \hat{\gamma}[b] * \hat{\gamma}[a]^{-1} * \hat{\gamma}[b]^{-1} \rangle$$
$$\cong \langle a, b \mid a\, b\, a^{-1}\, b^{-1} \rangle.$$

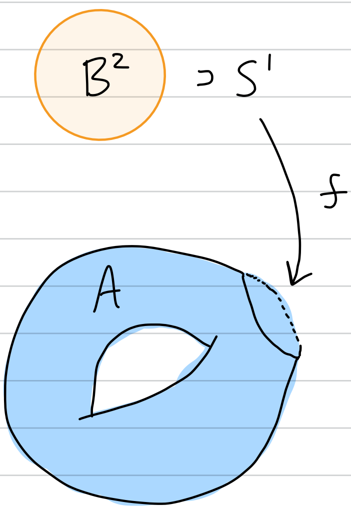


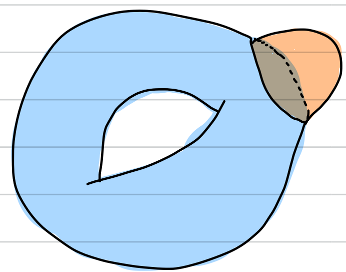


Aside: Compact Hausdorff spaces are normal.

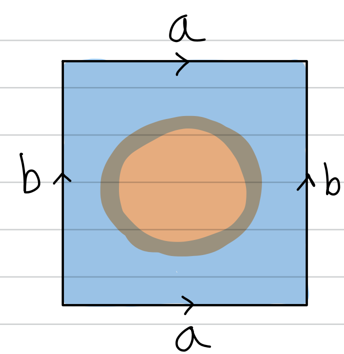
Thm (See Munkres Thm 72.1 and §72 Exercise #2)

Let A be a normal space and $f: S^1 \to A$ be continuous, $S^1 \subset B^2$.
Consider the adjunction space $A \cup_f B^2 := (A \amalg B^2) / (f(x) \sim x \quad \forall x \in S^1)$.
Then $A \to A \cup_f B^2$ induces a surjection $\pi_1(A) \to \pi_1(A \cup_f B^2)$
whose kernel N is the least normal subgroup of $\pi_1(A)$ containing $[f]$.



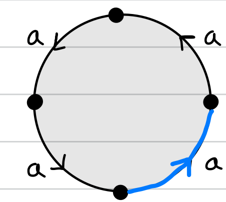


$A \cup_f B^2$



$\pi_1(T) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$

Section 73: The fundamental group of the torus and dunce cap

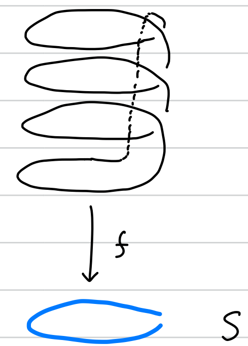


Def The <u>n-fold dunce cap</u> is the quotient space

$$B^2/(x\sim r(x)\ \forall x\in S^1),$$

where $r: S^1 \longrightarrow S^1$ is a rotation by $2\pi/n$.
$e^{i\theta} \mapsto e^{i(\theta+2\pi/n)}$

Equivalently,



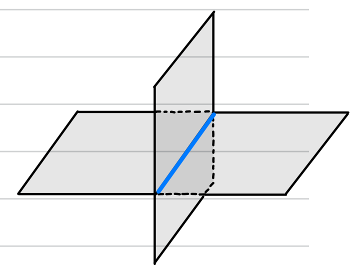
Def The <u>n-fold dunce cap</u> is the adjunction space

$$S^1 \cup_f B^2 := (S^1 \amalg B^2)/(f(x)\sim x),$$

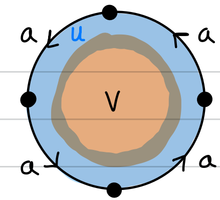
where $f: S^1 \longrightarrow S^1$ via $e^{i\theta} \mapsto e^{in\theta}$.

The n-fold dunce cap is

- a ball B^2 for $n=1$,
- the projective plane $\mathbb{R}P^2$ for $n=2$,
- not a manifold for $n\geq 3$.



Thm $\pi_1(\text{n-fold dunce cap}) \cong \mathbb{Z}/n \cong \langle a \mid a^n \rangle$



Ex Find a space X with $\pi_1(X) \cong \mathbb{Z}/3 \times \mathbb{Z}/9$.

Ans $X = (\text{3-fold dunce cap}) \times (\text{9-fold dunce cap})$.

Ex Find a space X with $\pi_1(X) \cong \mathbb{Z}/3 * \mathbb{Z}/9 * \mathbb{Z}$.

Ans $X = (\text{3-fold dunce cap}) \vee (\text{9-fold dunce cap}) \vee S^1$.

Ex Find a space X with $\pi_1(X) \cong (\mathbb{Z} * \mathbb{Z}) \times (\mathbb{Z} * \mathbb{Z})$.

Ans $X = (S^1 \vee S^1) \times (S^1 \vee S^1)$.

More generally, ...

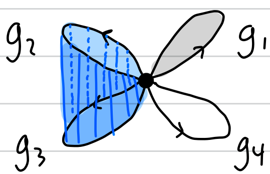
<u>Fact</u> For any finitely presented group $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$ there is a space X with $\pi_1(X) \cong G$.

(See also §73 Exercise #2)

Indeed, let X be the adjunction space obtained from $A = \bigvee_{i=1}^{n} S^1$ (so $\pi_1(A) \cong \langle g_1, \dots, g_n \rangle$) by attaching m balls B^2 along their boundary circles, where the j-th attaching map $S^1 \longrightarrow A$ for $j = 1, \dots, m$ is given by the j-th relation r_j.

<u>Ex</u> $G = \langle g_1, g_2, g_3, g_4 \mid g_1, \; g_2 g_3^{-1} \rangle$

$X =$



<u>Ex</u> $G = \langle g_1, g_2, g_3 \mid g_1^3, \; g_1 g_3 g_2^{-1} g_1^7 g_3, \; g_2 g_3^2 \rangle$

$X =$ hard to visualize, but not hard to understand abstractly!

This is a theme in algebraic topology: many algebraic objects (or morphisms) have topological analogues. ("Eilenberg-MacLane spaces")