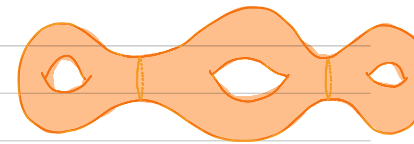
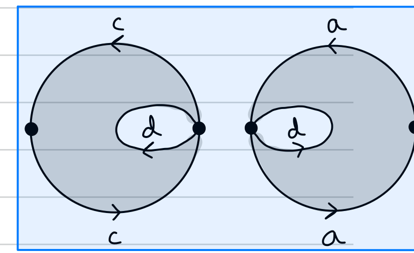
# Chapter 12: Classification of surfaces



$T \# T \# T$

**Thm** Any compact surface is homeomorphic to either

- the 2-sphere $S^2$,
- the n-fold connected sum of tori $\underbrace{T \# \dots \# T}_{n \text{ times}}$, or
- the m-fold connected sum of projective planes $\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}$.




$\mathbb{RP}^2 \# \mathbb{RP}^2$

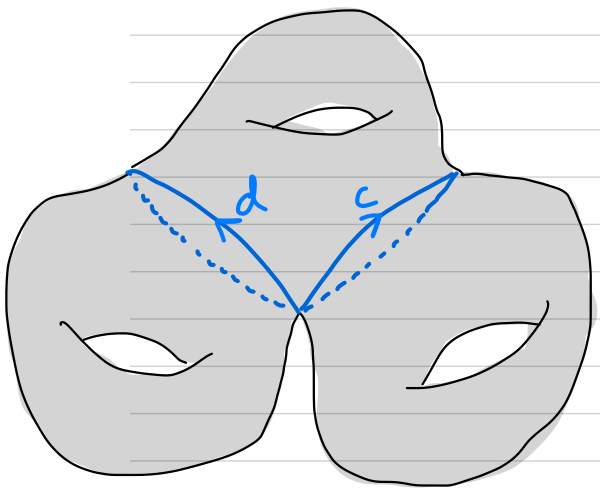
Using SVK, we can compute the fundamental groups of these surfaces (and their abelianizations) to show no two surfaces on this list are homeomorphic.

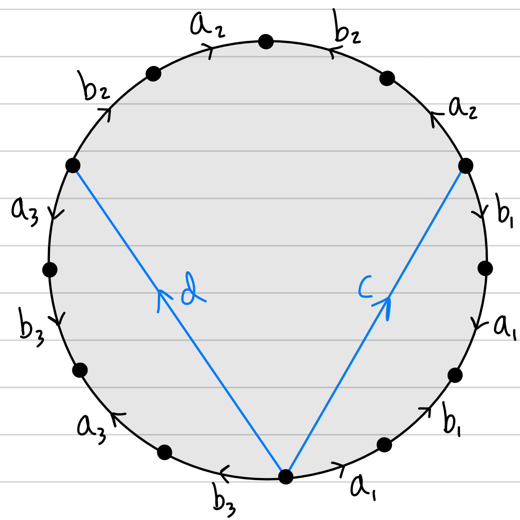
$\pi_1(\underbrace{T \# \dots \# T}_{n \text{ times}}) \cong \langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$ with abelianization $\bigoplus_{i=1}^{2n} \mathbb{Z}$.

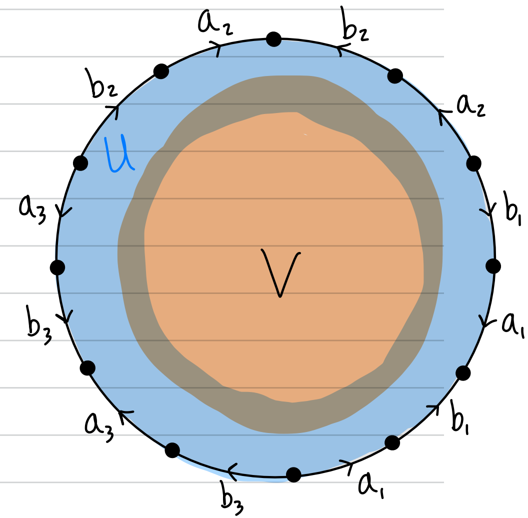
$\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \langle a_1, \dots, a_m \mid a_1^2 a_2^2 \cdots a_m^2 \rangle$ with abelianization $(\bigoplus_{i=1}^{m-1} \mathbb{Z}) \oplus \mathbb{Z}/2$.

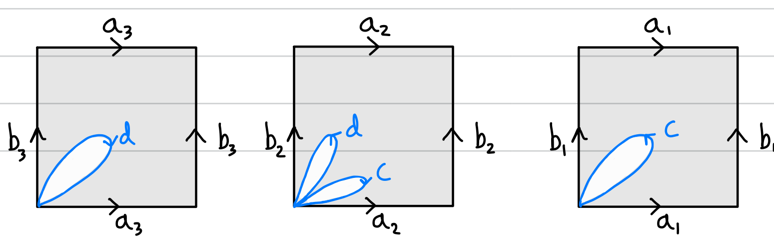
# Section 74: Fundamental groups of surfaces

**Thm** $\pi_1(\underbrace{T\#\cdots\#T}_{n\text{ times}}) \cong \langle a_1,b_1,\dots,a_n,b_n \mid a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}\rangle$

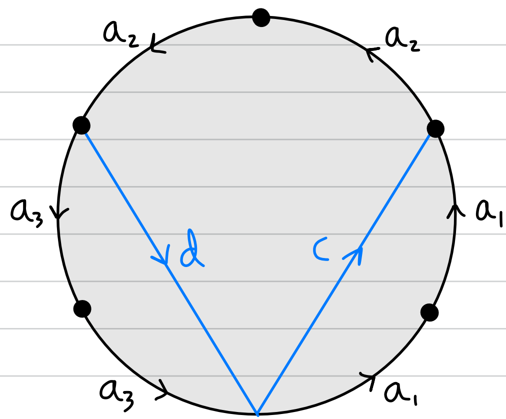


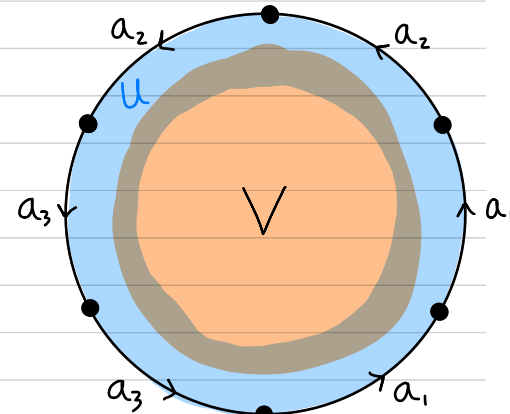



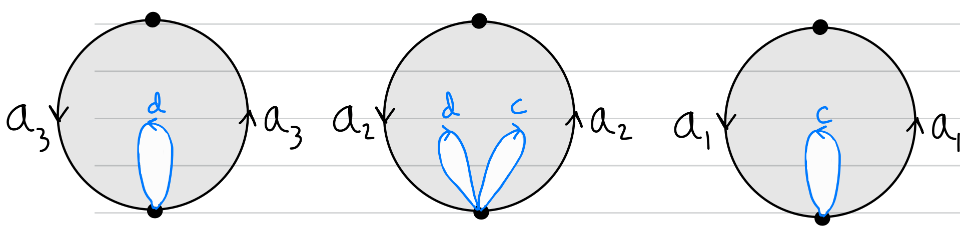



**Thm** $\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \langle a_1, \dots, a_m \mid a_1^2 a_2^2 \cdots a_m^2 \rangle$



Thm $\pi_1(\underbrace{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}_{m\text{ times}}) \cong \langle a_1,\dots,a_m \mid a_1^2a_2^2\cdots a_m^2\rangle$

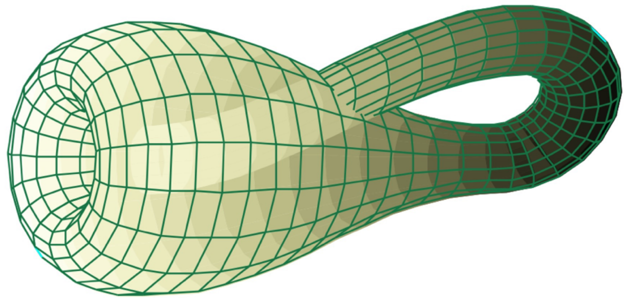
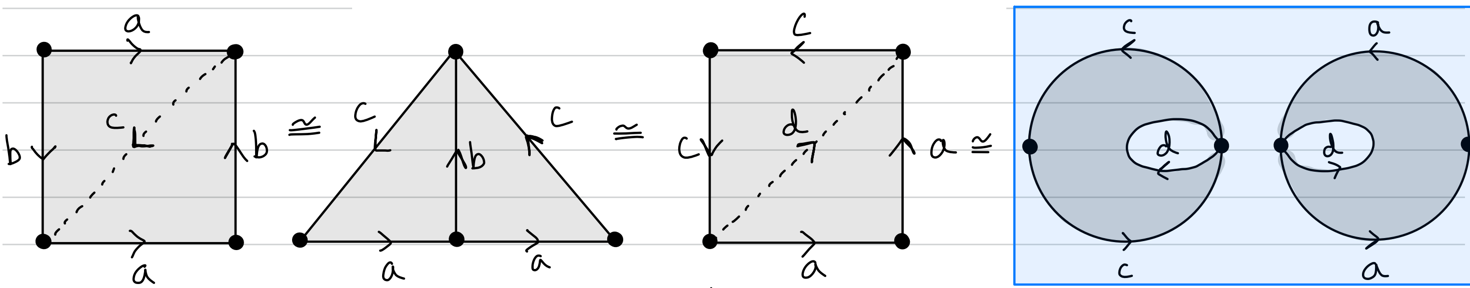


Image from Wikipedia




Klein bottle K

"standard form"

$\mathbb{RP}^2\#\mathbb{RP}^2$

$\pi_1(K)\cong\langle a,b\mid aba^{-1}b\rangle$

$\pi_1(K)\cong\langle a,c\mid a^2c^2\rangle$

# Section 75: Homology of surfaces

Let $G$ be a group.
The commutator of $x,y \in G$ is $[x,y] = xyx^{-1}y^{-1}$.

Def The commutator subgroup of $G$, denoted $[G,G]$, is generated by all commutators. It is a normal subgroup.

Def The abelianization of $G$ is the quotient group $G/[G,G]$. It is abelian.

In the finitely presented case, the abelianization of $\langle g_1,\dots,g_n \mid r_1,\dots,r_m \rangle$ is isomorphic to $\langle g_1,\dots,g_n \mid r_1,\dots,r_m, \underbrace{g_i g_j g_i^{-1} g_j^{-1} \ \forall i<j}_{\binom{n}{2} \text{ more relations}} \rangle$

Def Let $X$ be a path-connected space.
The first homology group of $X$ is the abelianization of the fundamental group:
$H_1(X) = \pi_1(X) / [\pi_1(X), \pi_1(X)]$.

Remark This is not the typical definition.
For $n \geq 2$, the $n$-th homology group $H_n(X)$ is not the abelianization of the $n$-th homotopy group $\pi_n(X)$ (which is already abelian).

Remark Interestingly,
$\pi_n(X)$ is easy to define but hard to compute, whereas
$H_n(X)$ is harder to define but much easier to compute.

Thm $H_1(\underbrace{T \# \cdots \# T}_{n \text{ times}}) \cong \bigoplus_{i=1}^{2n} \mathbb{Z}$.

Pf The abelianization of

$\pi_1(\underbrace{T \# \cdots \# T}_{n \text{ times}}) \cong \langle a_1, b_1, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$

is isomorphic to

redundant

$$\langle a_1, b_1, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1},\ a_i a_j a_i^{-1} a_j^{-1},\ b_i b_j b_i^{-1} b_j^{-1},\ a_i b_j a_i^{-1} b_j^{-1}\ \forall i,j \rangle$$

$$\cong \langle g_1, \ldots, g_{2n} \mid g_i g_j g_i^{-1} g_j^{-1}\ \forall i<j \rangle$$

$$\cong \bigoplus_{i=1}^{2n} \mathbb{Z}.$$

**Thm** $H_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \left(\bigoplus_{i=1}^{m-1} \mathbb{Z}\right) \oplus \mathbb{Z}/2.$

**Pf** The abelianization of
$\pi_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \langle a_1, \dots, a_m \mid a_1^2 a_2^2 \cdots a_m^2 \rangle$
is isomorphic to
$\langle a_1, \dots, a_m \mid a_1^2 a_2^2 \cdots a_m^2,\ a_i a_j a_i^{-1} a_j^{-1} \ \forall i<j \rangle.$

Switching to additive notation (and using Corollary 75.2), this is

- the quotient of the free abelian group generated by $a_1, a_2, \dots, a_{m-1}, a_m$ by the subgroup generated by $2a_1 + 2a_2 + \dots + 2a_m$, i.e.
- the quotient of the free abelian group generated by $a_1, a_2, \dots, a_{m-1}, a_1 + a_2 + \dots + a_m$ by the subgroup generated by $2a_1 + 2a_2 + \dots + 2a_m$.

Hence $H_1(\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m \text{ times}}) \cong \left(\bigoplus_{i=1}^{m-1} \mathbb{Z}\right) \oplus \mathbb{Z}/2.$

For completeness (but without emphasizing it), here is Corollary 75.2 that we used:

**Theorem 75.1.** *Let $F$ be a group; let $N$ be a normal subgroup of $F$; let $q : F \to F/N$ be the projection. The projection homomorphism*

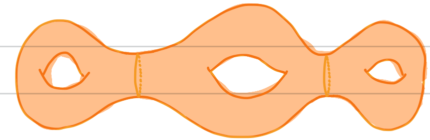
$$p : F \to F/[F, F]$$

*induces an isomorphism*

$$\phi : q(F)/[q(F), q(F)] \to p(F)/p(N).$$

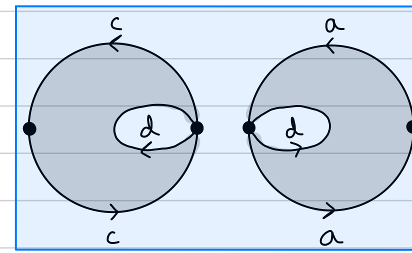
**Corollary 75.2.** *Let $F$ be a free group with free generators $\alpha_1, \ldots, \alpha_n$; let $N$ be the least normal subgroup of $F$ containing the element $x$ of $F$; let $G = F/N$. Let $p : F \to F/[F, F]$ be projection. Then $G/[G, G]$ is isomorphic to the quotient of $F/[F, F]$, which is free abelian with basis $p(\alpha_1), \ldots, p(\alpha_n)$, by the subgroup generated by $p(x)$.*

Section 76: Cutting and pasting
Section 77: The classification theorem



$T \# T \# T$

Thm Any compact surface is homeomorphic to $S^2$, $\underbrace{T \# \ldots \# T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{m \text{ times}}$.




$\mathbb{RP}^2 \# \mathbb{RP}^2$

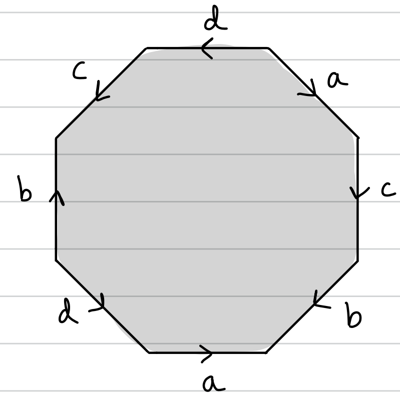
Our proof follows Justin Huang's REU paper "Classification of Surfaces".

# CLASSIFICATION OF SURFACES

JUSTIN HUANG

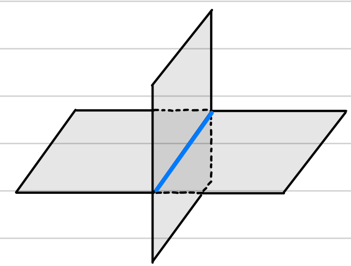
Abstract. We will classify compact, connected surfaces into three classes: the sphere, the connected sum of tori, and the connected sum of projective planes.

**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T \# \ldots \# T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{m \text{ times}}$.



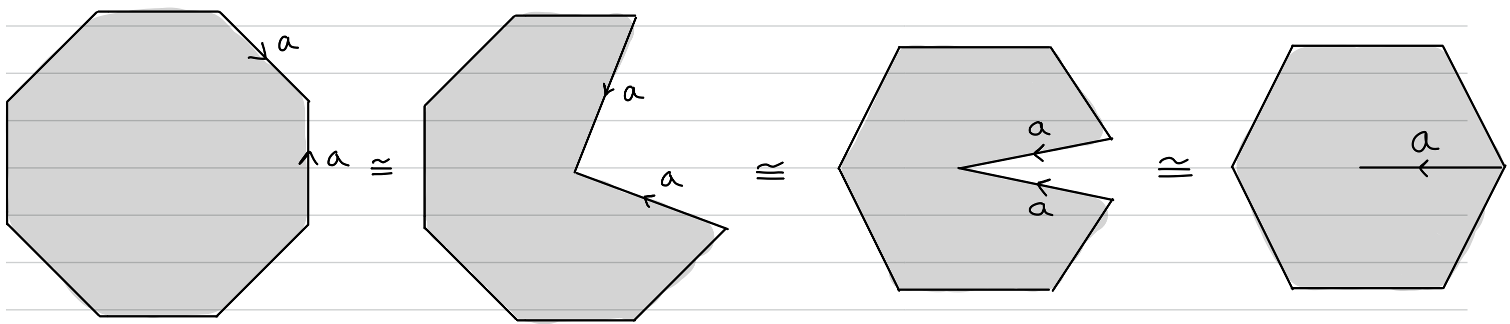

**Step 1/6:** Any compact surface can be obtained from a planar polygonal region by identifying its edges in pairs.

The strategy for steps 2-6 will be to cut and paste to put the planar polygon in a recognizable standard form, such as $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$ or $a_1^2 a_2^2 a_3^2$.

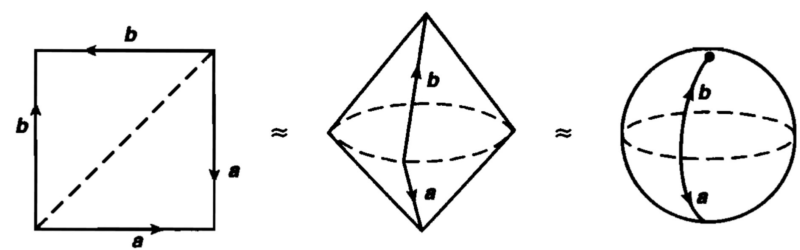


**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T \# \ldots \# T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{m \text{ times}}$.

**Step 2/6:** Remove adjacent opposing edges $aa^{-1}$.

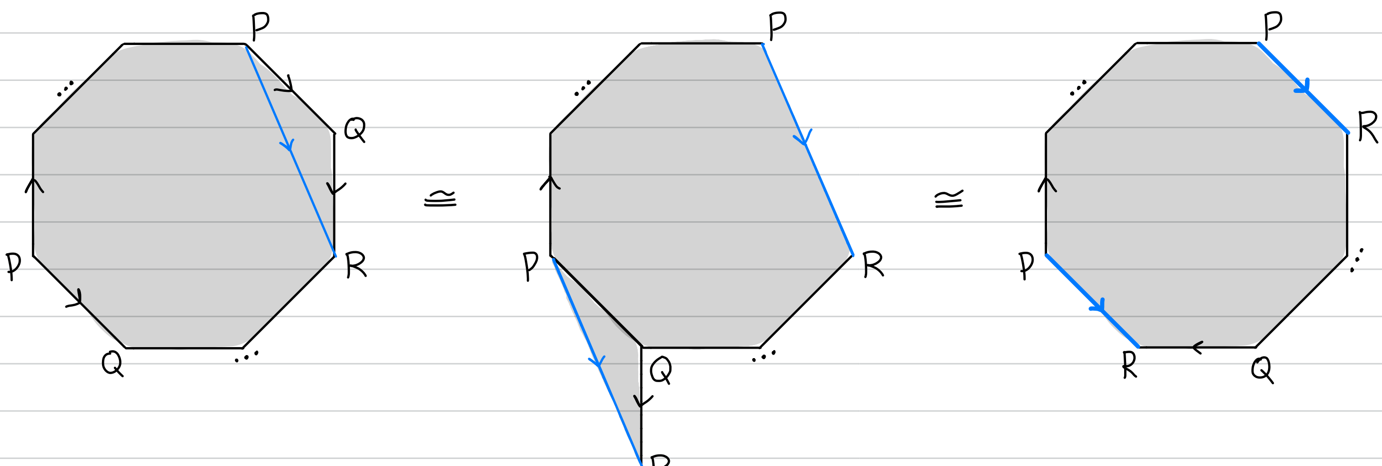



If all edges are of this form, then the surface is a 2-sphere $S^2$.



**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T \# \ldots \# T}_{n \text{ times}}$, or $\underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{m \text{ times}}$.

**Step 3/6:** Reduce to one vertex.



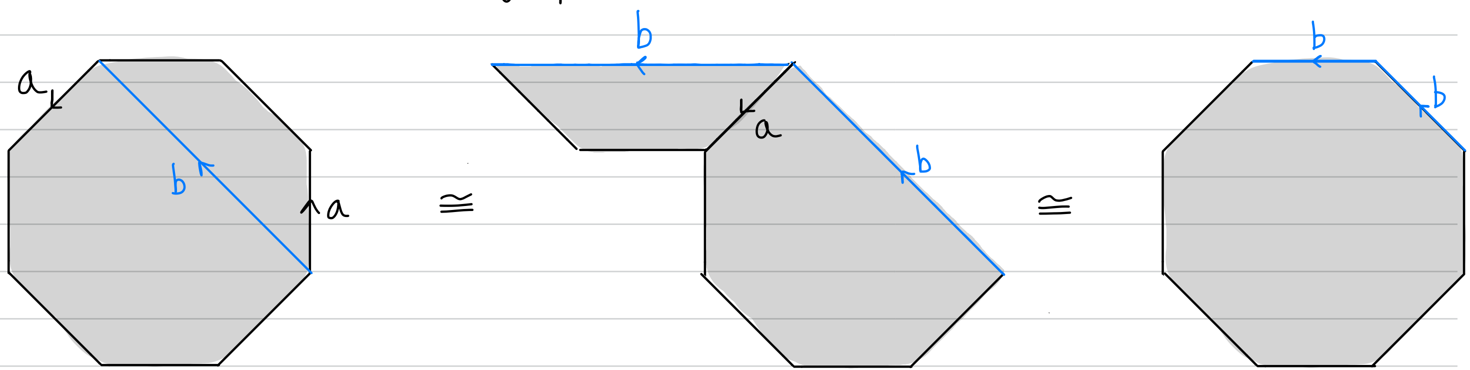

Note we reduced the # of Q labels.

Once only a single Q label remains, remove it as in Step 2.

Continuing as such, we can reduce to a single vertex.

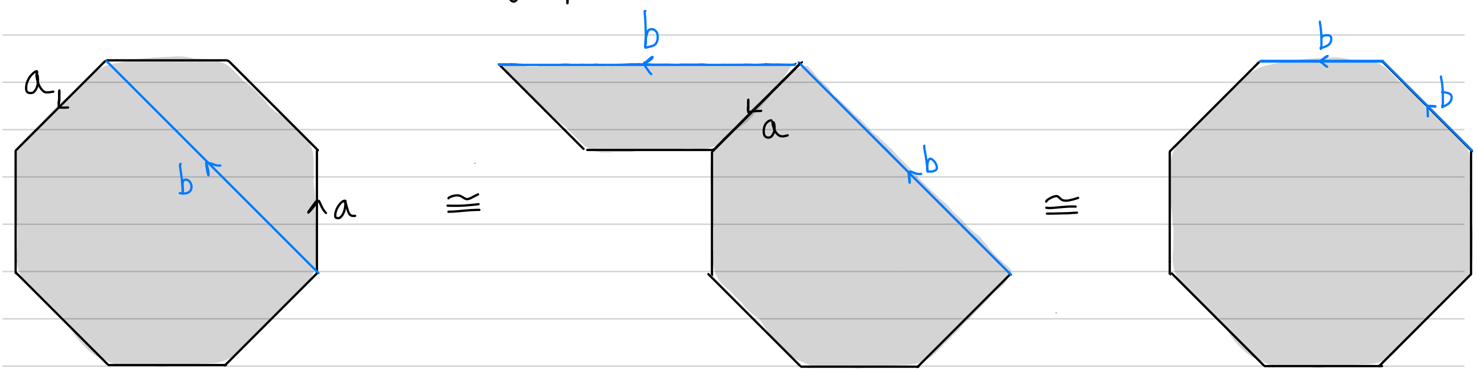
**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\dots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{m \text{ times}}$.

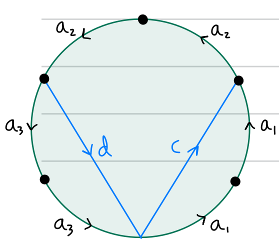
**Step 4/6:** Collect twisted edge pairs $\dots a \dots a \dots \rightsquigarrow \dots b^2 \dots$



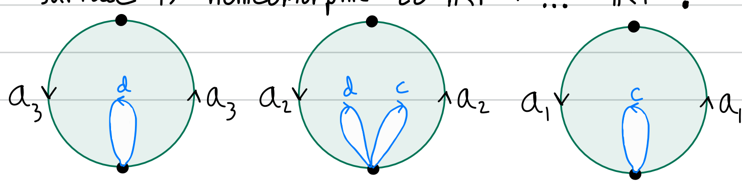
**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\ldots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{m \text{ times}}$.

**Step 4/6:** Collect twisted edge pairs $\ldots a \ldots a \ldots \rightsquigarrow \ldots b^2 \ldots$



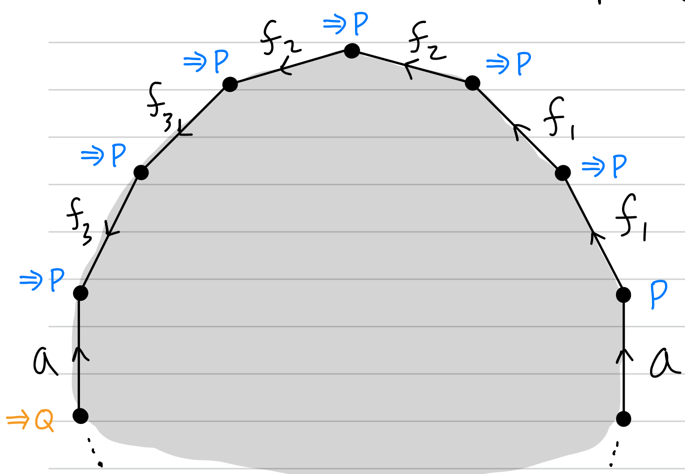

We continue by induction, and if the only remaining edges are twisted pairs, then the surface is homeomorphic to $\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2$.



**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\ldots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{m \text{ times}}$.

**Step 5/6:** Collect pairs of opposing edge pairs $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots \rightsquigarrow \ldots cdc^{-1}d^{-1}\ldots$

First, we argue that we can't have $a f_1^2 f_2^2 \cdots f_n^2 a^{-1}$, since that would contradict Step 2 (one vertex):



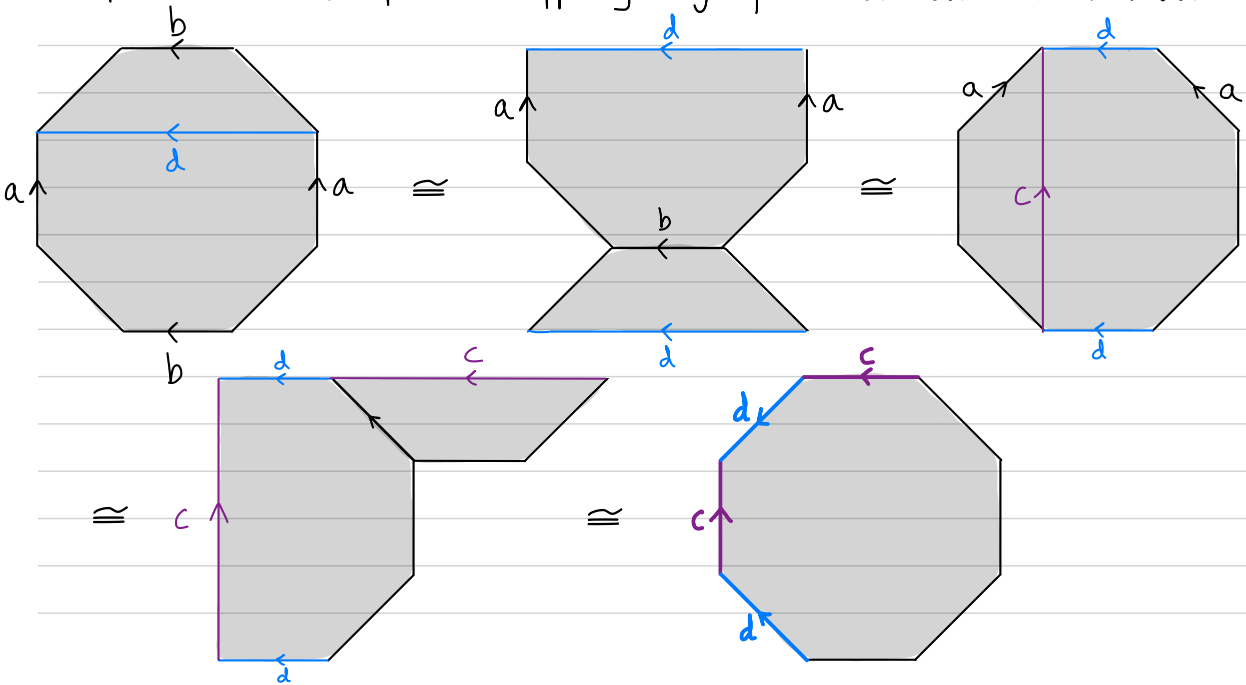

Indeed, if the head of edge $a$ was labeled $P$, then each $f_i$ would have tail and head labeled $P$.

The vertex labels are induced from edge pairings, meaning the tail of edge $a$ would have label $Q$.

So, we have interleaved opposing edge pairs $\ldots a \ldots b \ldots a^{-1} \ldots b \ldots$, which we collect as follows:

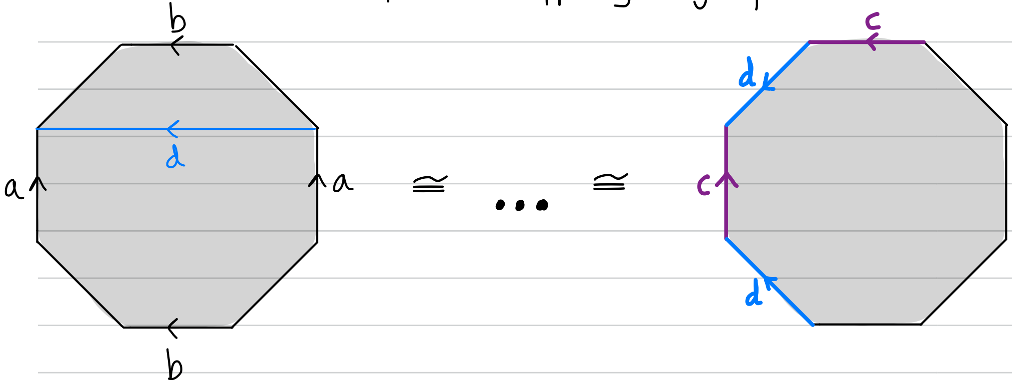
**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\dots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{m \text{ times}}$.

**Step 5/6:** Collect pairs of opposing edge pairs $\dots a \dots b \dots a^{-1} \dots b^{-1} \dots \rightsquigarrow \dots c d c^{-1} d^{-1} \dots$



**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\dots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{m \text{ times}}$.

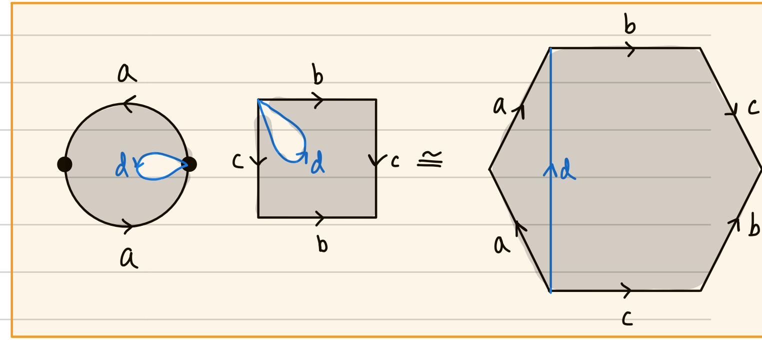
**Step 5/6:** Collect pairs of opposing edge pairs $\dots a\dots b\dots a^{-1}\dots b^{-1}\dots \rightsquigarrow \dots cdc^{-1}d^{-1}\dots$




By induction, what remains are
(i) twisted edge pairs $\dots f^2\dots$, and
(ii) pairs of opposing edge pairs $\dots cdc^{-1}d^{-1}\dots$.

If only (i), the surface is $\mathbb{RP}^2\#\dots\#\mathbb{RP}^2$.
If only (ii), the surface is $T\#\dots\#T$.
If both, the surface is $\mathbb{RP}^2\#\dots\#\mathbb{RP}^2$, as we see from Step 6:

**Thm** Any compact surface is homeomorphic to $S^2$, $\underbrace{T\#\dots\#T}_{n \text{ times}}$, or $\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{m \text{ times}}$.




**Step 6/6:**
Show $T\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$.