Chapter 2: Topological spaces and continuous functions

Section 12: Topological spaces

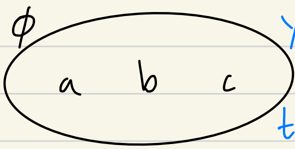
Many concepts in analysis (continuity, convergence, compactness) only require knowledge of the open sets.

Def A *topology* on a set X is a collection τ of subsets, called *open sets*, satisfying

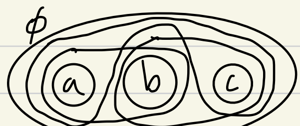
- $\phi, X \in \tau$.
- Arbitrary unions of open sets are open:
  $U_\alpha \in \tau \quad \forall \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \tau$.
- Finite intersections of open sets are open:
  $U_1, \dots, U_n \in \tau \implies U_1 \cap \dots \cap U_n \in \tau$.

We denote this *topological space* by (X, τ) or X.

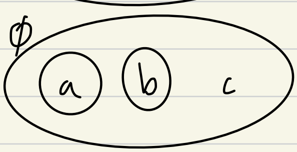
Ex Which of the following are topologies on $X = \{a, b, c\}$?



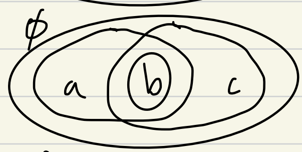
Yes, indiscrete or trivial topology



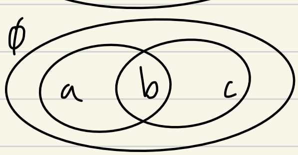
Yes, $\tau = \mathcal{P}(X)$ is the discrete topology



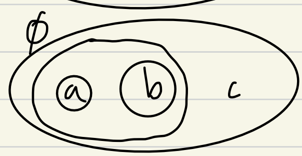
No



Yes



No



Yes

Ex Every metric space is a topological space.
The open sets are unions of (open) balls.

Ex X is a set.
$\tau = \{ U \subset X \mid U = \emptyset \text{ or } X - U \text{ is finite} \}$.
Called the finite complement topology.

For example, if $X = \mathbb{R}$, then a nonempty open set is $\mathbb{R}$ with at most a finite number of points removed:

Pf
- $\emptyset, X \in \tau$
- Let $\emptyset \neq U_\alpha \in \tau \ \forall \alpha \in I$. So $X - U_\alpha$ is finite.
  Note $X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X - U_\alpha)$ is finite.
  So $\bigcup_{\alpha \in I} U_\alpha \in \tau$.
- Let $\emptyset \neq U_1, \dots, U_n \in \tau$. So $X - U_i$ is finite.
  Note $X - (U_1 \cap \dots \cap U_n) = (X - U_1) \cup \dots \cup (X - U_n)$ is finite.
  So $U_1 \cap \dots \cap U_n \in \tau$.

Rmk There is also a countable complement topology.

Def If τ and τ' are two topologies on X with $\tau \subset \tau'$, then we say τ is coarser and τ' is finer.

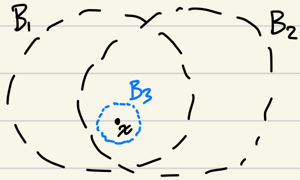
Section 13: Basis for a topology

Instead of specifying all open sets in a topology τ, it is often convenient to specify a nice subset that generates τ.

<u>Def</u> A <u>basis</u> for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

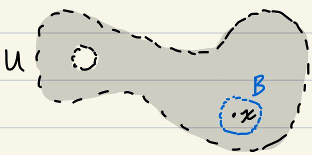
(1) $\forall x \in X \quad \exists B \in \mathcal{B}$ with $x \in B$.

(2) If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

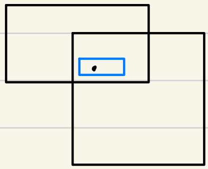


The <u>topology τ generated by $\mathcal{B}$</u> is:
$U \subset X$ is open ($U \in \tau$) if $\forall x \in U$, $\exists B \in \mathcal{B}$ with $x \in B \subset U$.
Equivalently, $U \subset X$ is open if U is a union of sets in $\mathcal{B}$.

<u>Ex</u> X a metric space,
$\mathcal{B} = \{\text{open balls}\}$



<u>Ex</u> $X = \mathbb{R}^2$ $\quad \mathcal{B} = \{\text{open balls}\}$
or $\quad \mathcal{B}' = \{\text{axis-aligned open rectangles}\}$



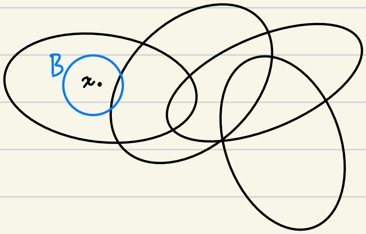
<u>Ex</u> X $\quad \mathcal{B} = \{\text{one point sets}\}$ is a basis for the discrete topology.

<u>Ex</u> Choosing $\mathcal{B} = \tau$ always gives a basis, but it is more valuable to find bases $\mathcal{B} \subsetneq \tau$.

Prop The topology τ generated by a basis $\mathcal{B}$ is indeed a topology.

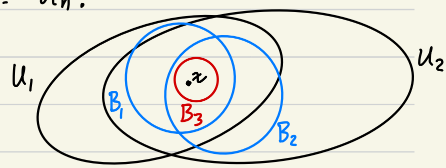
Pf • $\emptyset \in \tau$ since condition is vacuously true,
$X = \bigcup_{B \in \mathcal{B}} B$ by (1), so $X \in \tau$.

• $\{U_\alpha\}_{\alpha \in I}$ with $U_\alpha \in \tau$.



If $x \in \bigcup_{\alpha \in I} U_\alpha$,
then $\exists \alpha \in I$ with $x \in U_\alpha$,
so $\exists B \in \mathcal{B}$ with $x \in B \subset U_\alpha \subset \bigcup_{\alpha \in I} U_\alpha$.

• $U_1, \dots, U_n \in \tau$. Let $x \in U_1 \cap \dots \cap U_n$.
Claim: $\exists B \in \mathcal{B}$ with $x \in B \subset U_1 \cap \dots \cap U_n$.
Use induction on n.



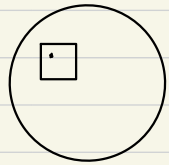
Base case $n=1$ is clear.
For $n=2$, use (2): $x \in U_1 \cap U_2$
$\Rightarrow \exists B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$
$\Rightarrow \exists B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$.
The general inductive step actually quickly follows from the $n=2$ case!

Lemma 13.2 Let (X, τ) be a topological space.
Let $\mathcal{C}$ be a collection of open sets such that if $x \in U$ for $U \in \tau$, then $\exists C \in \mathcal{C}$ with $x \in C \subset U$.
Then $\mathcal{C}$ is a basis for τ.

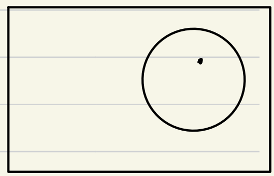
Pf (1) Since $X \in \tau$, $\forall x \in X$ $\exists C \in \mathcal{C}$ with $x \in C \subset X$. ✓
(2) If $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C} \subset \tau$, then $C_1 \cap C_2 \in \tau$,
so $\exists C_3 \in \mathcal{C}$ with $x \in C_3 \subset C_1 \cap C_2$. ✓
So $\mathcal{C}$ is a basis. Also, it is not hard to see that $\mathcal{C}$ generates the topology τ.

<u>Lemma</u> Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies τ, τ' on X. Then τ' is finer than τ ($\tau \subset \tau'$ allowing equality)
$\iff \forall B \in \mathcal{B}$ and $x \in B$, $\exists B' \in \mathcal{B}'$ with $x \in B' \subset B$.

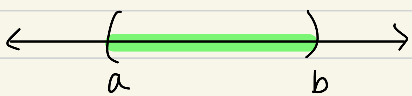
<u>Pf</u> See book

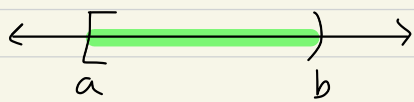


<u>Ex</u> $X = \mathbb{R}^2$ $\quad \mathcal{B} = \{\text{open balls}\}$
and $\quad \mathcal{B}' = \{\text{axis-aligned open rectangles}\}$
generate the same topology.



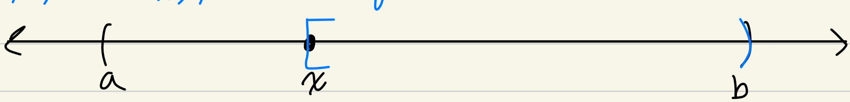
<u>Def</u> $X = \mathbb{R}$
$\mathcal{B} = \{(a,b) \mid a < b\}$ generates the standard topology
$\mathcal{B}' = \{[a,b) \mid a < b\}$ generates the <u>lower limit topology</u>





<u>Fact</u> $\quad \tau \subsetneq \tau'$ $\quad \left(\begin{array}{c}\text{I.e., } \tau' \text{ is finer than } \tau, \\ \text{and not vice-versa}\end{array}\right)$

<u>Pf sketch</u> Apply the prior lemma. Consider $(a,b) \in \mathcal{B}$ and $x \in (a,b)$. Note $[x,b) \in \mathcal{B}'$ satisfies $x \in [x,b) \subset (a,b)$, as required.



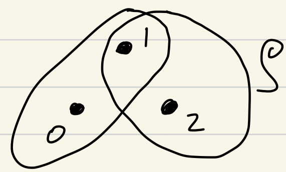
An imperfect analogy

Vector spaces	Topological spaces
$\mathbb{R}^n$	open sets in $\mathbb{R}^n$
vector spaces	topological spaces
basis	basis
Any vector is a sum of basis elements.	Any open set is a union of basis elements.
This description is unique.	Nope.
A vector space has many bases.	A topological space has many bases.
All bases have the same size.	Nope.

Consider the definition of a topology.
Can I start with a collection of sets,
which I declare to be open,
along with all unions and finite intersections thereof?

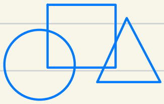
<u>Def</u> A <u>subbasis</u> $\mathcal{S}$ for X is a collection of sets whose union is X. The topology τ generated by subbasis $\mathcal{S}$ is the collection of all unions of finite intersections of elements in $\mathcal{S}$

<u>Ex</u> $\mathcal{S} = \{\{0,1\}, \{0,2\}\}$ is a subbasis but not a basis.
$\tau = \{\phi, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$.
One basis is $\mathcal{B} = \{\{0\}, \{0,1\}, \{0,2\}\}$



<u>Lemma</u> τ is indeed a topology

<u>Pf sketch</u> Show that the collection $\mathcal{B}$ of all finite intersections of elements in $\mathcal{S}$ is a basis.
(1) is easy.
(2) follows since if $B = S_1 \cap \dots \cap S_m$ and $B' = S_1' \cap \dots \cap S_m'$ are two elements of $\mathcal{B}$, then $B \cap B'$ is also an element of $\mathcal{B}$.

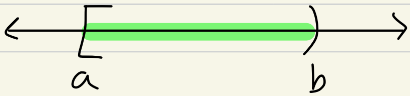


$|\mathcal{S}| = 3$ $|\mathcal{B}| = 6$ if you include ϕ. $|\tau| = 2^5$

Section 14: The order topology

Let X be a set with total order $\leq$.
For $a, b \in X$, define

- $(a,b) = \{x \in X : a < x < b\}$
- $[a,b) = \{x \in X : a \leq x < b\}$
- $(a,b] = \{x \in X : a < x \leq b\}$
- $[a,b] = \{x \in X : a \leq x \leq b\}$.



Def Let $\mathcal{B}$ contain

(1) all intervals (a,b)

(2) all intervals $[a_0, b)$ where a_0 is the smallest element (if any) in X

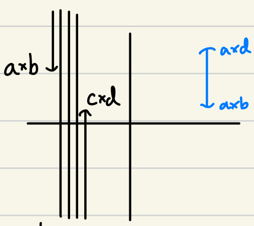
(3) all intervals $(a, b_0]$ where b_0 is the largest element (if any) in X.

The collection $\mathcal{B}$ forms a basis for the <u>order topology</u> on X.

Ex The order topology is the standard topology on $\mathbb{R}$.

Ex $\mathbb{R} \times \mathbb{R}$ with the lexicographic order:

$a \times b < c \times d \iff a < c$ or $a = c,\ b < d$.



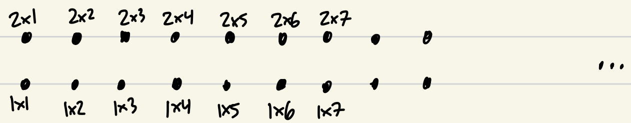
These intervals actually form a basis on their own.

This is not the standard topology on $\mathbb{R}^2$.

Ex The order topology on $\mathbb{Z}_+$ is the <u>discrete topology</u>

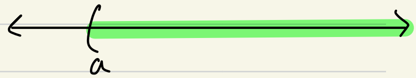
Note $\{n\} = (n-1, n+1)$ for $n > 1$, and $\{1\} = [1, 2)$.

<u>Ex</u> The order topology on $\{1,2\}\times\mathbb{Z}_+$ (lexicographic order) is not the discrete topology, since any basis element containing 2×1 must contain some $1\times n$.



Later: Note $1\times n \xrightarrow[n\to\infty]{} 2\times 1$ is a convergent sequence in this topology!

Ex Let X be an ordered set and $a \in X$.
Let $(a, \infty) = \{x \in X \mid x > a\}$
and $(-\infty, a) = \{x \in X \mid a < x\}$
be the <u>open rays</u>.



Show these are indeed open in the order topology.

Ans If X has a largest element b_0, then
$(a, \infty) = (a, b_0]$ is a basis element, else
$(a, \infty) = \bigcup_{x > a} (a, x)$ is a union of basis elements.

Ex Do the open rays form a basis for $\mathbb{R}$?

Ans No — consider $a < b$. No open ray is contained inside $(-\infty, b) \cap (a, \infty) = (a, b)$.

Ex Do the open rays form a subbasis for the order topology on X?

Ans Yes.
They're open in the order topology, so the topology they generate is contained in the order topology.

Also, every basis element for the order topology is a finite intersection of open rays:

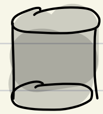
$(a,b) = (-\infty, b) \cap (a, \infty)$

$(a, b_0] = (a, \infty)$ for b_0 largest

$[a_0, b) = (-\infty, b)$ for a_0 smallest

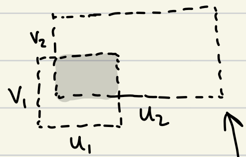
So the reverse containment of topologies is also true.

Section 15: The product topology on $X \times Y$



Def For X and Y topological spaces, the product topology on $X \times Y$ is the topology generated by the basis $\mathcal{B}$ with all sets of the form $U \times V$, with U open in X and V open in Y.

Check Is this a basis?



Note $X \times Y \in \mathcal{B}$.

Also, for $U_1 \times V_1,\ U_2 \times V_2 \in \mathcal{B}$,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

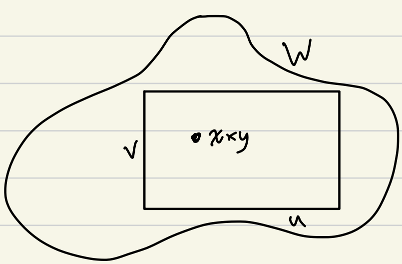
Question Is $\mathcal{B}$ a topology?

No, the union above is not in $\mathcal{B}$

Smaller bases are possible:

Thm If $\mathcal{B}$ is a basis for X and $\mathcal{C}$ is a basis for Y, then $\mathcal{D} = \{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$ is a basis for $X \times Y$.

Pf Sketch



W open in $X \times Y$

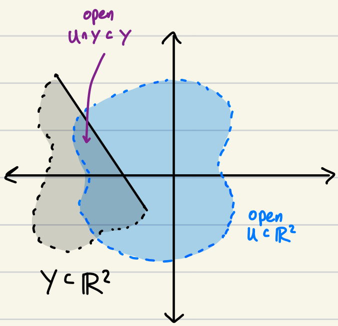
$x \times y \in W$

By definition of product topology and definition of bases $\mathcal{B}$, $\mathcal{C}$:

$\exists B \in \mathcal{B}$ with $x \in B$ and $\exists C \in \mathcal{C}$ with $y \in C$ satisfying $x \times y \in B \times C \subset W$.

By Lemma 13.2, this shows $\mathcal{D}$ is a basis generating the product topology on $X \times Y$.

Section 16: The subspace topology



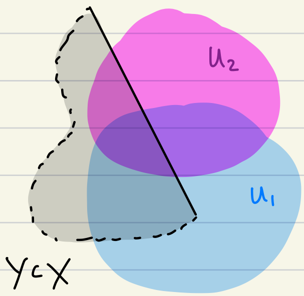
Def Let (X,τ) be a topological space. For $Y\subseteq X$, the collection

$$\tau_Y = \{U\cap Y \mid U\in\tau\}$$

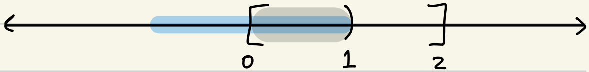
is the <u>subspace topology</u> on Y.

Check it is a topology:

- $\emptyset = \emptyset\cap Y$, $\quad Y = X\cap Y$ ✓
- Arbitrary unions:
  $\bigcup_{\alpha\in J}(U_\alpha\cap Y) = (\bigcup_{\alpha\in J} U_\alpha)\cap Y$ ✓
- Finite intersections:
  $(U_1\cap Y)\cap\dots\cap(U_n\cap Y) = (U_1\cap\dots\cap U_n)\cap Y$ ✓



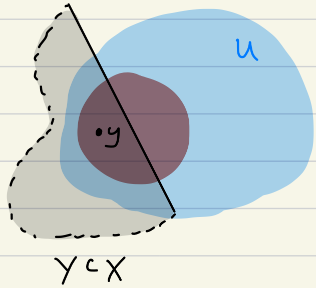
Ex Though $[0,1)$ is not open in $\mathbb{R}$, it is open in the subspace topology on $[0,2]\subset\mathbb{R}$.



Lemma Let $Y\subseteq X$.
If U is open in Y $(U\in\tau_Y)$
and Y is open in X $(Y\in\tau)$,
then U is open in X $(U\in\tau)$.

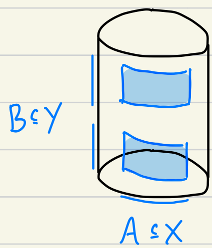
Pf U open in Y
$\Rightarrow \exists\ V\in\tau$ with $U = V\cap Y$
$\Rightarrow U$ is the intersection of two sets in τ
$\Rightarrow U\in\tau$.

Lemma If $\mathcal{B}$ is a basis for the topology on X, then $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for the topology on Y.



Pf Given $U \cap Y$ open in Y (with U open in X) and $y \in U \cap Y$, we can find $B \in \mathcal{B}$ with $y \in B \subset U$.
Note $y \in B \cap Y \subset U \cap Y$.
It follows from Lemma 13.2 that $\mathcal{B}_Y$ is a basis for the topology on Y.

Thm If $A \subseteq X$ and $B \subseteq Y$, then the product topology on $A \times B$ the same as the subspace topology on $A \times B \subset X \times Y$.



Pf Consider first the product topology on the larger space $X \times Y$, which has as a basis all $U \times V$, U open in X, V open in Y.

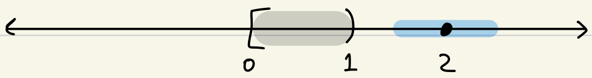
So the subspace topology on $A \times B$ has as a basis all
$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap Y),$$
which is a basis for the product topology on $A \times B$.

These topologies are the same since they have a common basis.

Rmk The order and subspace topologies are not compatible in general.

For example, let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$.

In the subspace topology, $\{2\}$ is open in Y.



But in the order topology, any basis element containing 2 is of the form
$$(a,2] := \{ y \in Y \mid a < y \leq 2 \} \quad \text{for some } a \in Y$$
and it follows that $\{2\}$ is not open.

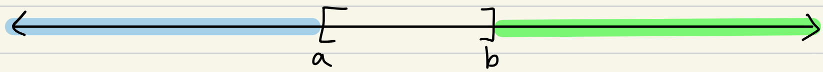
Def If X is totally ordered, a subset $Y \subset X$ is convex if $\forall a,b \in Y$ with $a<b$, the interval $(a,b) = \{ x \in X \mid a < x < b \}$ is contained in Y.

Thm If X is an ordered set with the order topology and $Y \subset X$ is convex, then the order and subspace topologies on Y agree.

Section 17: Closed sets and limit points

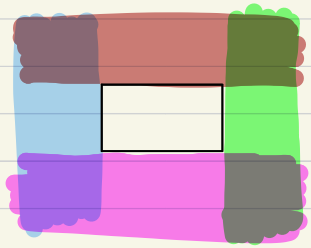
Def A subset A of a topological space X is closed if $X-A$ is open.

Ex $[a,b]$ is closed in $\mathbb{R}$ since $\mathbb{R}-[a,b] = (-\infty,a) \cup (b,\infty)$ is open.



Ex $[a,b]\times[c,d]$ is closed in $\mathbb{R}^2$.

(Complement is union of four basic open sets.)



Ex In the finite complement topology on a set X, the closed sets are X, $\emptyset$, and all finite subsets of X.

Ex In the discrete topology, every set is closed.

Rmk closed $\not\equiv$ not open

Ex $[0,2)$ is neither open nor closed in $\mathbb{R}$.

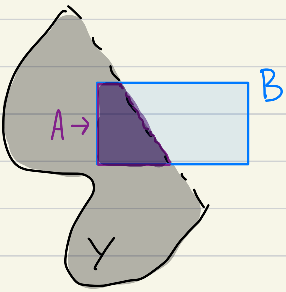
Ex Let $Y = [0,2) \cup \{4\} \subset \mathbb{R}$ have the subspace topology.
Is $[0,2)$ open in Y Yes.
Is $\{4\}$ open in Y? Yes.
Is $[0,2)$ closed in Y? Yes.
Is $\{4\}$ closed in Y? Yes.

Thm For X a topological space,
- ϕ and X are closed
- arbitrary intersections of closed sets are closed
- finite unions of closed sets are closed.

Pf See book. $\left(X - \bigcap_{\alpha \in J} C_\alpha = \bigcup_{\alpha \in J} (X - C_\alpha) \right)$

Rmk Topological spaces could have instead been defined via closed sets.

Thm For $Y \subset X$ with the subspace topology, a set $A \subset Y$ is closed in Y $\iff$ $A = B \cap Y$ for some closed set B in X



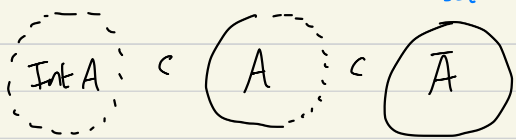
$X = \mathbb{R}^2$

Pf See book.

Def For X a topological space and $A \subset X$,

- the <u>interior of A</u>, denoted $\operatorname{Int} A$, is the union of all open sets contained in A
- the <u>closure of A</u>, denoted $\operatorname{Cl} A$ or $\overline{A}$, is the intersection of all closed sets containing A.

$$\underset{\substack{\uparrow \\ \text{open set}}}{\operatorname{Int} A} \subset A \subset \underset{\substack{\uparrow \\ \text{closed set}}}{\overline{A}}$$

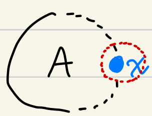


Ex For $X = \mathbb{R}$ and $A = [0,2)$,
$\operatorname{Int} A = (0,2)$ and $\overline{A} = [0,2]$.

Thm X topological space with basis $\mathcal{B}$, $A \subset X$.

(a) $x \in \overline{A} \iff$ every open set containing x intersects A.

(b) $x \in \overline{A} \iff$ every $B \in \mathcal{B}$ containing x intersects A.



Rmk An open set containg x is called a <u>neighborhood</u> of x.

Pf (a) ($\Rightarrow$) U a nbhd of x that doesn't intersect A
$\Rightarrow X - U$ is a closed set containing A
$\Rightarrow \overline{A} \subset X - U \Rightarrow x \notin \overline{A}$

($\Leftarrow$) $x \notin \overline{A}$ means $X - \overline{A}$ is a nbhd of x not intersecting A.

(b) ($\Rightarrow$) Basis elements are open

($\Leftarrow$) A nbhd containing x contains a basis element containing x.

Ex $A = [0,2) \subset \mathbb{R}$, $\overline{A} = [0,2]$, $\overline{A}$ is set of limit points.

Ex $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$, $\overline{B} = B \cup \{0\}$, 0 is only limit point.

Ex $\mathbb{Q} \subset \mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$, all points in $\mathbb{R}$ are limit points.

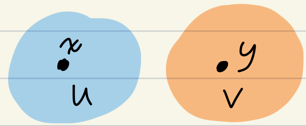
<u>Def</u> X topological space, $A \subset X$. A point $x \in X$ is a <u>limit point</u> of A if every nbhd of x contains a point in A other than x.

(x may or may not be in A)

<u>Thm</u> X topological space, $A \subset X$.
Let A' be the set of limit points of A.
Then $\overline{A} = A \cup A'$.

<u>Cor</u> A subset of a topological space is closed $\Leftrightarrow$ it contains all its limit points.

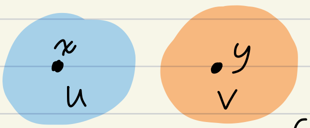
Def A topological space X is a <u>Hausdorff space</u> if $\forall$ distinct $x, y \in X$, $\exists$ (open) neighborhoods U of x and V of y with $U \cap V = \emptyset$



Thm In a Hausdorff space X, finite sets are closed.

Pf It suffices to show that $\{x\}$ is closed $\forall x \in X$, since finite unions of closed sets are closed.

So, let $y \neq x$ in X. By the Hausdorff assumption, $\exists$ (open) neighborhood $V \ni y$ with $x \notin V$.

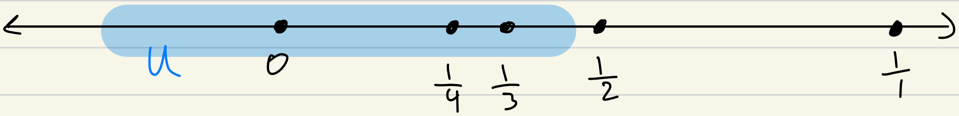


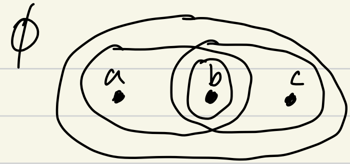
So $y \notin \overline{\{x\}}$ $\forall$ $y \neq x$ in X.

So $\overline{\{x\}} = \{x\}$, meaning $\{x\}$ is closed.

Def A sequence $x_1, x_2, x_3, \ldots$ <u>converges</u> to a point $x \in X$ if $\forall$ (open) neighborhoods U of x, $\exists N \in \mathbb{Z}_+$ such that $x_n \in U$ $\forall n \geq N$.

Ex $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \longrightarrow 0$ in $\mathbb{R}$

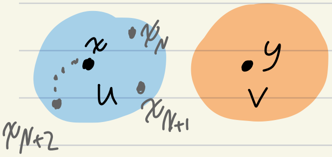


Ex In the topological space 

note that $\{b\}$ is not closed,
and note that the sequence $b, b, b, b, b, \ldots$
converges not only to b, but also to a or to c!

Thm In a Hausdorff space, sequences converge to at most one point.

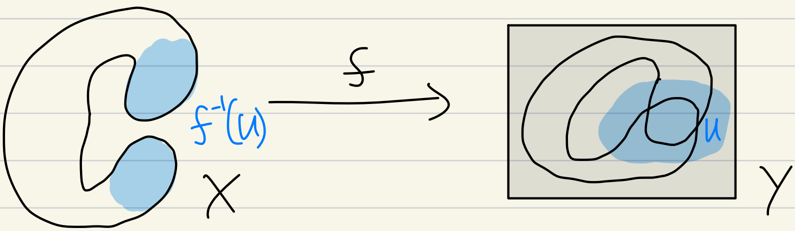
Pf If $x_n \to x$ and $y \neq x$, then let $U \ni x$ and $V \ni y$ be disjoint nbhds. Note U contains all but finitely many elements of the sequence, and hence V cannot.



Thm A subspace of a Hausdorff space is Hausdorff.
The product of two Hausdorff spaces is Hausdorff.

Section 18: Continuous functions

Def X, Y topological spaces. A function $f: X \to Y$ is continuous if $\forall$ open U in Y, $f^{-1}(U)$ is open in X.



Rmk It suffices to check this condition on basis elements of Y:

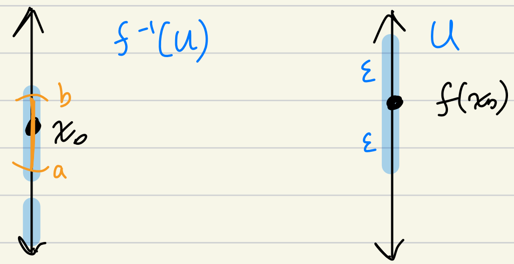
$$U = \bigcup_{\alpha \in J} B_\alpha \qquad f^{-1}(U) = f^{-1}\Big(\bigcup_{\alpha \in J} B_\alpha\Big) = \underbrace{\bigcup_{\alpha \in J} \underbrace{f^{-1}(B_\alpha)}_{\text{open}}}_{\text{open}}$$

Rmk It suffices to check this condition on subbasis elements of Y:

$$B = S_1 \cap \ldots \cap S_n \qquad f^{-1}(B) = f^{-1}(S_1 \cap \ldots \cap S_n) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n)$$

Ex $f : \mathbb{R} \to \mathbb{R}$ cont. as defined above
$\iff$ f is cont. with the ε-δ condition.

Pf $(\Rightarrow)$ Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$.
Note $U = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open
$\Rightarrow$ $f^{-1}(U)$ is open.
So $\exists$ a basic open set $x_0 \in (a, b) \subset f^{-1}(U)$.
Let $\delta = \min(x_0 - a, b - x_0)$.
Then x within δ of x_0 $\Rightarrow$ $f(x)$ within ε of $f(x_0)$.



$(\Leftarrow)$ See book

Ex $Id : \mathbb{R}_\ell \to \mathbb{R}$ (defined by $Id(x) = x$ $\forall x \in \mathbb{R}_\ell$)
is continuous since $Id^{-1}((a,b)) = (a,b)$ is open in $\mathbb{R}_\ell$.

$Id : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous since $Id^{-1}([a,b)) = [a,b)$
is not open in $\mathbb{R}$.

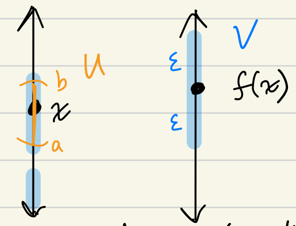
Thm Let X and Y be topological spaces, and let $f: X \to Y$. The following are equivalent:

(1) f is continuous

(2) $\forall$ closed sets B in Y, $f^{-1}(B)$ is closed in X.

(3) $\forall$ $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$

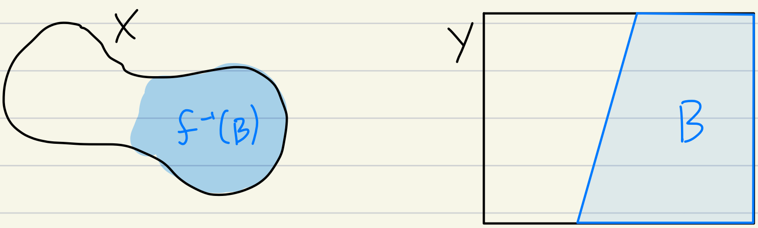
(4) $\forall$ $x \in X$ and nbhds V of $f(x)$, $\exists$ nbhd U of x with $f(U) \subset V$.



Rmk If (4) holds at $x_0 \in X$ but not necessarily at all points in X, then we say <u>f is continuous at x_0</u>

Pf See book.

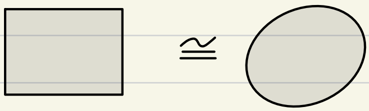
Picture of $(1) \Longleftrightarrow (2)$:

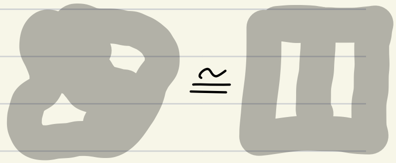


Note $X = f^{-1}(X - B) \coprod f^{-1}(B)$

Def A <u>homeomorphism</u> is a continuous bijection $f: X \to Y$ such that $f^{-1}: Y \to X$ is also continuous.

We say "X is homeomorphic to Y" and write $X \cong Y$.

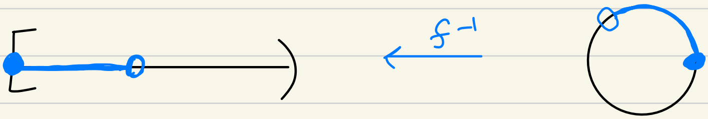
Ex 

Ex 

Ex $f: (-1,1) \to \mathbb{R}$ defined by $f(x) = \frac{x}{1-x^2}$ is a homeomorphism with inverse $f^{-1}: \mathbb{R} \longrightarrow (-1,1)$ defined by
$$f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}.$$

So homeomorphisms need not preserve boundedness.

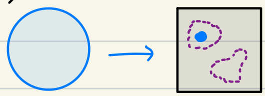
Non-Ex $f: [0,2\pi) \to S^1$ defined by $f(t) = (\cos t, \sin t)$ is a continuous bijection that is not a homeomorphism.



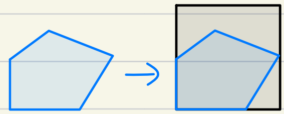
Rmk A homeomorphism gives a bijection b/w the open sets of X and Y. So it preserves all topological properties.

Thm (Constructing continuous functions)

(a) Constant functions are cont.



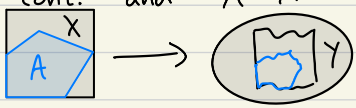
(b) The inclusion of a subspace is cont.



(c) Compositions are continuous: If $f: X \to Y$, $g: Y \to Z$ are cont., then so is $g \circ f: X \to Z$.

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & Z \end{tikzcd}$$

(d) $f: X \to Y$ cont. and $A \subset X \Rightarrow f|_A$ cont.

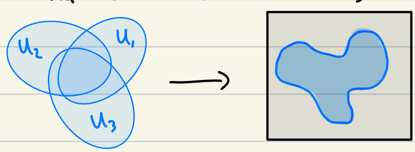


(e) $f: X \to Y$ cont. $\Rightarrow$ $f: X \to Z$ cont. for $Y \subset Z$

$\Rightarrow$ $f: X \to W$ cont. for $f(X) \subset W$

(f) $f: X \to Y$, $X = \bigcup_\alpha U_\alpha$, $f|_{U_\alpha}$ cont $\forall \alpha \Rightarrow f$ cont.

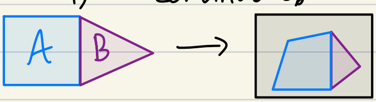
↑ open



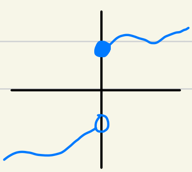
(g) (Pasting lemma) $X = A \cup B$, A, B closed in X.
$f: A \to Y$ and $g: B \to Y$ cont. and $f(x) = g(x)$ $\forall x \in A \cap B$.
Then the function $h: X \to Y$ defined via

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.



Ex Why is $f: \mathbb{R} \to \mathbb{R}$ not continuous?



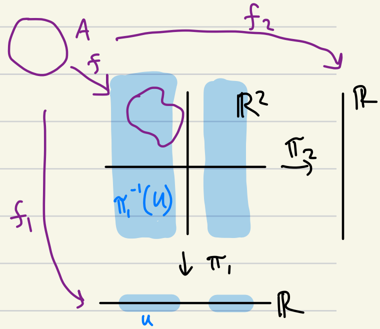
Thm Let $f: A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. Then f is continuous $\iff$ f_1, f_2 are continuous.

Pf Let $\pi_1 : X \times Y \to X$ and
$(x, y) \longmapsto x$

$\pi_2 : X \times Y \longrightarrow Y$.
$(x, y) \longmapsto y$

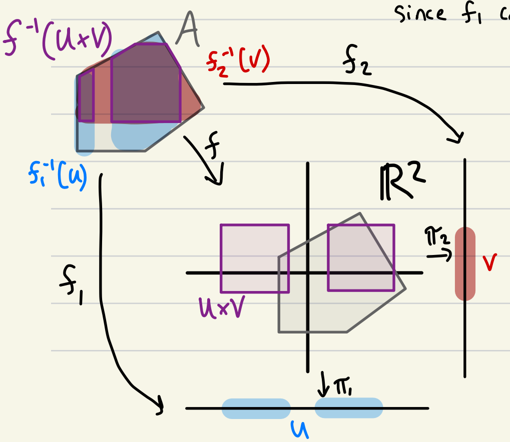
$$\begin{tikzcd} A \arrow[dr, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & X \times Y \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Y \\ & X & \end{tikzcd}$$

Note π_1 is continuous since if U is open in X, then $\pi_1^{-1}(U) = U \times X$ is open in $X \times Y$. And similarly for π_2.



($\Longrightarrow$) f cont. implies
$\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$
are continuous.

($\Longleftarrow$) For $U \times V$ a basic open set in $X \times Y$ (meaning U open in X, V open in Y), note $f^{-1}(U \times V) = \underset{\substack{\uparrow \\ \text{open in } A \\ \text{since } f_1 \text{ cont.}}}{f_1^{-1}(U)} \cap \underset{\substack{\nwarrow \\ \text{open in } A \\ \text{since } f_2 \text{ cont.}}}{f_1^{-1}(V)}$ is open in A



Section 19 Product topology

Def Given $\{X_\alpha\}_{\alpha\in J}$, the <u>cartesian product</u> $\prod_{\alpha\in J} X_\alpha$ is the set of all <u>J-tuples</u> $(x_\alpha)_{\alpha\in J}$ which are maps $x: J \to \bigcup_{\alpha\in J} X_\alpha$ with $x_\alpha := x(\alpha) \in X_\alpha$.

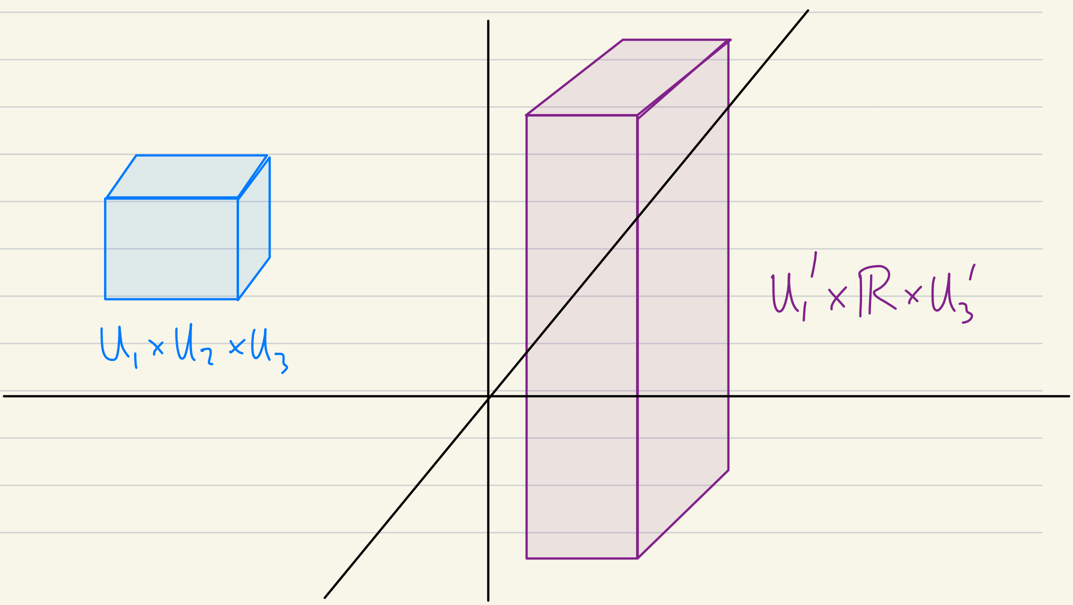
Def The (less-important) <u>box topology</u> on $\prod_{\alpha\in J} X_\alpha$ has as its basis all sets

$$\left\{ \prod_{\alpha\in J} U_\alpha \;\middle|\; U_\alpha \text{ open in } X_\alpha \ \forall \alpha \right\}$$

Def The (more-important) <u>product topology</u> on $\prod_{\alpha\in J} X_\alpha$ has as its basis all sets

$$\left\{ \prod_{\alpha\in J} U_\alpha \;\middle|\; \begin{array}{l} U_\alpha \text{ open in } X_\alpha \\ U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \end{array} \right\}$$

Rmk These topologies agree if J is finite.



Thm Let $f_\alpha : A \to X_\alpha \quad \forall \alpha \in J$

Define $f : A \longrightarrow \prod_{\alpha \in J} X_\alpha$ by $a \longmapsto (f_\alpha(a))_{\alpha \in J}$.

Let $\prod X_\alpha$ have the product topology.

Then f is continuous $\iff$ f_α is continuous $\forall \alpha$.

Pf Note each projection $\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$ is continuous.

$(\Rightarrow)$ f cont. $\Rightarrow$ $f_\alpha = \pi_\alpha \circ f$ cont. $\forall \alpha$

$(\Leftarrow)$ A basis element for the product topology can be written as

$$\prod_{\alpha \in J} U_\alpha = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

where $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \dots \beta_n$. Note

$$f^{-1}\left(\prod_{\alpha \in J} U_\alpha\right) = f^{-1}\left(\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})\right)$$

$$= \bigcap_{i=1}^{n} \underbrace{f_{\beta_i}^{-1}(U_{\beta_i})}_{\text{open in } A \text{ since } f_{\beta_i} \text{ cont.}} \quad \text{is open in } X.$$

Rmk $(\Leftarrow)$ need not be true if $\prod X_\alpha$ has the box topology. Let $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} X_n$ with $X_n = \mathbb{R}$ $\forall n$.

Define $f : \mathbb{R} \longrightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, t, \dots)$

Each coordinate function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) = t$ is continuous.

But, f is not continuous if $\mathbb{R}^\omega$ has the box topology, since $B = (-1, 1) \times (\frac{-1}{2}, \frac{1}{2}) \times (\frac{-1}{3}, \frac{1}{3}) \times \dots$ is open in the box topology, but $f^{-1}(B) = \{0\}$ is not open in $\mathbb{R}$.

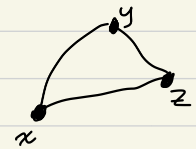
Section 20: The metric topology

Def A metric on a set X is a function $d: X \times X \to \mathbb{R}$ s.t.

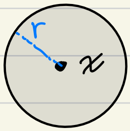
(1) $d(x,y) \geq 0, \quad d(x,y) = 0 \iff x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leq d(x,y) + d(y,z)$ triangle inequality

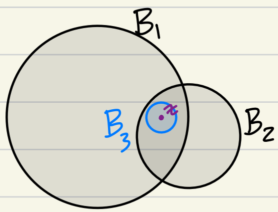


$B_r(x) = \{ y \in X \mid d(x,y) < r \}$ is the r-ball centered at x.



Def Given a metric space (X,d), $\{ B_r(x) \mid x \in X, r > 0 \}$ is a basis for a metric topology on X.

Check its a basis (2)



Rmk U is open in (X,d)

$\iff \forall x \in U \quad \exists \; x \in B_s(y) \subset U$

$\iff \forall x \in U \quad \exists \; x \in B_r(x) \subset U$

Def A topological space X is metrizable if $\exists$ a metric on X that induces the topology on X.

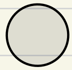
Important Question Is a given topological space metrizable?

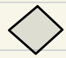
Ex For X a set, defining $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

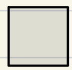
gives a metric inducing the discrete topology.

Ex Metrics on $\mathbb{R}^n$

For $1 \leq p \leq \infty$, let $d_p(x,y) = \|x-y\|_p$, where

where $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$ 

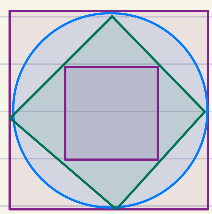
$\|x\|_1 = |x_1| + \dots + |x_n|$  "taxicab" metric

$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$  "sup" metric

$$\|x\|_p = \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p} \text{ for all } 1 \leq p < \infty.$$

Note $B_1^{d_\infty}(\vec{0}) \supset B_1^{d_2}(\vec{0}) \supset B_1^{d_1}(\vec{0}) \supset B_{1/2}^{d_\infty}(\vec{0})$.

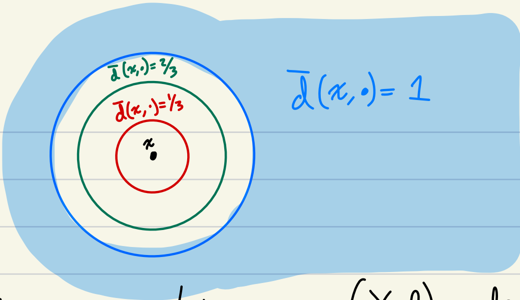
Hence the following lemma shows that all of these metrics induce the same topology on $\mathbb{R}^n$:



(and, moreover, this topology is the product topology)

Lemma Let X have metrics d, d' generating the topologies τ, τ'. Then τ' is finer than τ (i.e. $\tau \subset \tau'$) if $\forall\, B_r^d(x)$, $\exists\, B_s^{d'}(x) \subset B_r^d(x)$.

Pf See book.



<u>Def</u> Given a metric space (X,d), define $\bar{d}(x,y) = \min(d(x,y),1)$. This is the <u>standard bounded metric</u>.

<u>Thm</u> $\bar{d}$ is metric on X and induces the same topology as d.

<u>Pf</u> See book.

<u>Thm</u> The product topology on $\mathbb{R}^\omega$ is induced by the metric

$$D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$$

<u>Pf</u> See book.

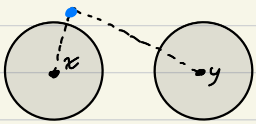
<u>Rmk</u> Metric $\sup\{\bar{d}(x_i,y_i)\}$ gives a topology that is too fine, for example containing $(-1,1)^\omega$ as an open set.

<u>Rmk</u> $\sup\{d(x_i,y_i)\}$ is not a metric (it gives a function from $X\times X$ into $[0,\infty]$, not into $[0,\infty)$).

<u>Rmk</u> More generally, countable products of metric spaces are metrizable.

Section 21: Metric topology (continued)

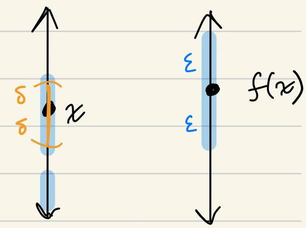
Rmk Metric spaces are Hausdorff: If $x \neq y$, then $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint for $0 < \varepsilon < \frac{1}{2} d(x,y)$ by the triangle inequality.



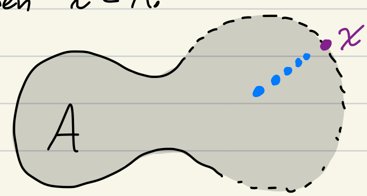
Thm $f: (X, d_X) \to (Y, d_Y)$ continuous

$\iff$

Given $x \in X$, $\varepsilon > 0$, $\exists \delta > 0$ s.t.
$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.



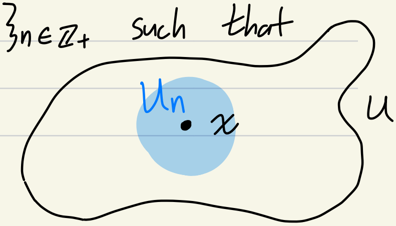
Lemma (Sequence Lemma) X topological space, $A \subset X$.
If a sequence in A converges to x, then $x \in \bar{A}$.
Converse holds if X metrizable.



Pf ($\Rightarrow$) $x_n \to x$ implies every nbhd of x contains a point in A, so $x \in \bar{A}$.

Pf ($\Leftarrow$) Let d be a metric giving the topology on X. $\forall n \in \mathbb{Z}_+$, choose $x_n \in B_{1/n}(x) \cap A$. Note $x_n \to x$.

Rmk For the converse of this (and the next) lemma, assumption "X metrizable" can be relaxed to <u>X first countable</u>, which means:
$\forall x \in X$, $\exists$ countable collection of nbhds $\{U_n\}_{n \in \mathbb{Z}_+}$ such that $\forall$ nbhds $U \ni x$, $\exists n \in \mathbb{Z}_+$ with $x \in U_n \subset U$.



Rmk $\mathbb{R}^J$ not metrizable for J uncountable. Indeed, let

$$A = \{ x = (x_\alpha) \in \mathbb{R}^J \mid x_\alpha = 1 \text{ for all but finitely many } \alpha \in J \}.$$

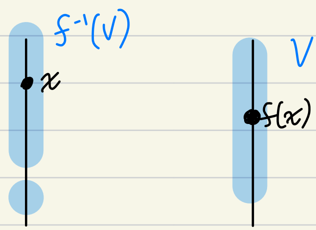
Define $\vec{0} \in \mathbb{R}^J$ to be the point x with $x_\alpha = 0 \quad \forall \alpha \in J$.

Then $\vec{0} \in \overline{A}$ since any basic open set about $\vec{0}$ is $\mathbb{R}$ in all but finitely many coordinates, hence intersects A.

But for any sequence $x^1, x^2, x^3, \ldots \in A$, $\exists$ some $\beta \in J$ with $x^n_\beta = 1 \quad \forall n$ (since a countable union of finite sets is countable), hence $\pi_\beta^{-1}((-1/2, 1/2))$ is a nbhd about $\vec{0}$ containing no x^n, so no sequence in A can converge to $\vec{0}$.

Thm X, Y topological spaces, $f: X \to Y$. If f is continuous, then $\forall\, x_n \to x$, we have $f(x_n) \to f(x)$.
Converse holds if X is a metric space.

Pf ($\Longrightarrow$) Given nbhd $V \ni f(x)$, note $f^{-1}(V)$ is a nbhd of x, so x_n eventually in $f^{-1}(V)$ implies that $f(x_n)$ is eventually in V.



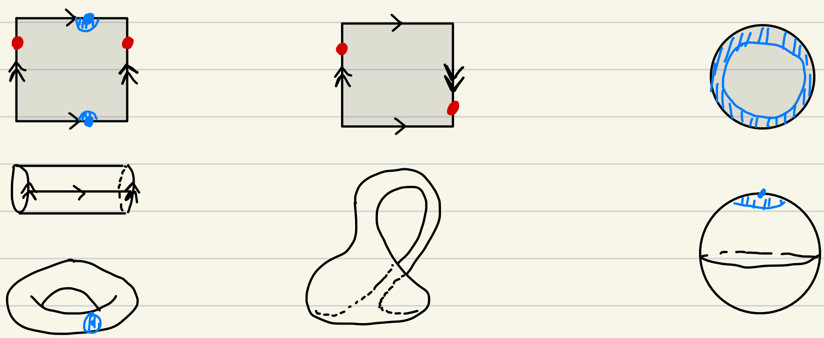
($\Longleftarrow$) Suffices to show $f(\overline{A}) \subset \overline{f(A)}$ for any $A \subset X$.
If $x \in \overline{A}$, then by prior lemma (since X metrizable), $\exists\, x_n \in A$ with $x_n \to x$. By assumption, $f(x_n) \to f(x)$.
Since $f(x_n) \in f(A) \; \forall n$, the prior lemma gives $f(x) \in \overline{f(A)}$.
Hence $f(\overline{A}) \subset \overline{f(A)}$ as desired.

Section 22: The quotient topology

Let X be a topological space, and let X^* be a partition of X, namely a collection of disjoint subsets whose union is X.

(In other words, suppose we have an equivalence relation on X.)

Ex $[0,1]\times[0,1]/\sim$ $\qquad$ $[0,1]\times[0,1]/\sim'$ $\qquad$ D^2/S^1



torus $\qquad$ Klein bottle $\qquad$ sphere

From the topology on X, how do we get a topology on X^*?
Give X^* the finest topology such that $p: X \longrightarrow X^*$ is continuous.

$$x \longmapsto [x]$$

U open in X^* $\iff$ $p^{-1}(U)$ open in X. $\qquad$ (Coarsest such topology would give only the open sets $\emptyset, X^*$)

Def Let X be a topological space, Y be a set, $p: X \to Y$ be surjective. In the quotient topology on Y,

U open in Y $\iff$ $p^{-1}(U)$ open in X.

<u>Check</u> This is a topology.

$p^{-1}(Y) = X$ open in $X \Rightarrow Y$ open in Y.

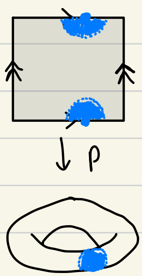
$p^{-1}(\emptyset) = \emptyset$ open in $X \Rightarrow \emptyset$ open in Y.

$p^{-1}(\bigcup_\alpha \underbrace{U_\alpha}_{\text{open in } Y}) = \bigcup_\alpha \underbrace{p^{-1}(U_\alpha)}_{\text{open in } X}$ open in $X \Rightarrow \bigcup_\alpha U_\alpha$ open in Y.

open in Y $\Rightarrow$ open in X

$p^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} p^{-1}(U_i)$ open in $X \Rightarrow \bigcap_{i=1}^{n} U_i$ open in Y.

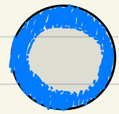
<u>Ex</u>



X

$\downarrow p$

torus Y



X

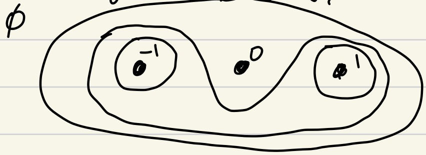
$\downarrow p$

sphere

Y

<u>Ex</u> $p: \mathbb{R} \to \{-1, 0, 1\}$ by $p(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

The induced quotient topology on $\{-1, 0, 1\}$ is



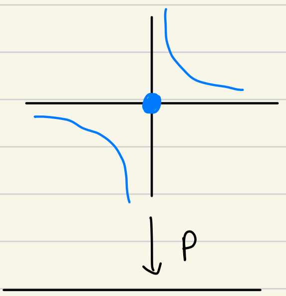
Let $p: X \to Y$ be surjective, X a topological space.
What if Y already has a topology?

<u>Def</u> For X, Y topological spaces and $p: X \to Y$ surjective,
p is a <u>quotient map</u> if

$$U \text{ open in } Y \iff p^{-1}(U) \text{ open in } X.$$

<u>Ex</u> All the examples above, where Y has quotient topology.

<u>Non-Ex</u>



$$X = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\} \cup \{(0,0)\}$$

$$Y = \mathbb{R}$$

$$p((x,y)) = x$$

Note $p: X \to Y$ is continuous and surjective, but not a quotient map since $p^{-1}(\{0\}) = \{(0,0)\}$ is open in X, but $\{0\}$ is not open in Y.

Thm (Continuous maps out of quotient spaces)

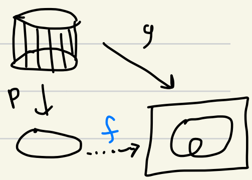
X, Y, Z topological spaces, $p: X \to Y$ a quotient map.

Let $g: X \to Z$ be constant on each $p^{-1}(\{y\})$,

hence inducing a function $f: Y \to Z$ with $f \circ p = g$.

Then f continuous $\Longleftrightarrow$ g continuous.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$



Pf ($\Rightarrow$) f cont. implies $f \circ p = g$ cont.

($\Leftarrow$) Given V open in Z, $\underset{p^{-1}(f^{-1}(V))}{\underset{\parallel}{g^{-1}(V)}}$ open in X since g is continuous.

Now, p a quotient map implies $f^{-1}(V)$ open in Y, so f is continuous.