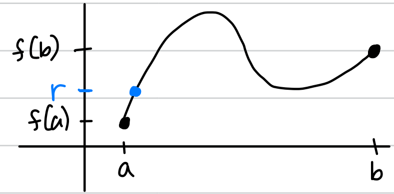
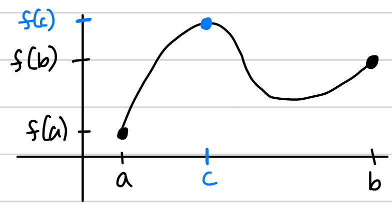
Chapter 3: Connectedness and compactness

Let $f:[a,b]\to\mathbb{R}$ be continuous.



Intermediate value theorem (IVT)

$\forall\ f(a)\le r\le f(b)\quad \exists\ c\in[a,b]$ with $f(c)=r$.



Maximum value theorem

$\exists c\in[a,b]$ s.t. $f(x)\le f(c)\quad \forall x\in[a,b]$.

These properties are really due to the fact that $[a,b]$ is connected and compact, respectively.

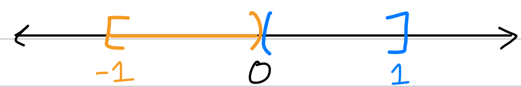
Section 23: Connected spaces

Def X a topological space. A separation of X is a pair U, V of nonempty disjoint open sets whose union is X.
X is connected if it has no separation.

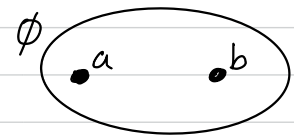
Rmk Topological property. If $X \cong Y$, then X connected $\iff$ Y connected.

Rmk Equivalently, X is connected if its only clopen subsets are $\emptyset, X$.

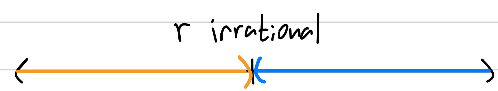
Ex $[-1,0) \cup (0,1]$ has a separation and is not connected



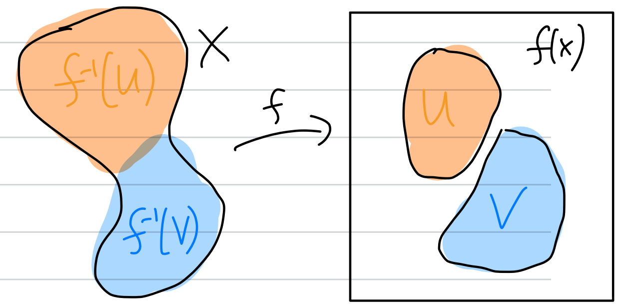
Ex $\{a,b\}$ with the indiscrete topology is connected



Ex $\mathbb{Q}$ is not connected: for r irrational, $(-\infty,r)\cap\mathbb{Q}$, $(r,\infty)\cap\mathbb{Q}$ gives a separation.

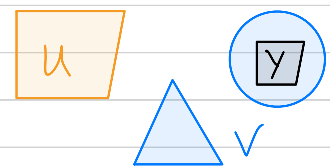


Thm If $f: X \to Y$ is continuous and X is connected, then $f(X)$ is connected.



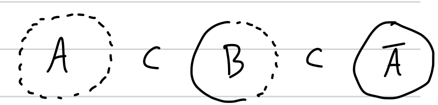
Pf If U, V is a separation of $f(X)$, then $f^{-1}(U), f^{-1}(V)$ is a separation of X (open, nonempty, disjoint, union is X), contradicting the fact X is connected.

Lemma If X has separation U, V and $Y \subset X$ is connected, then $Y \subset U$ or $Y \subset V$.



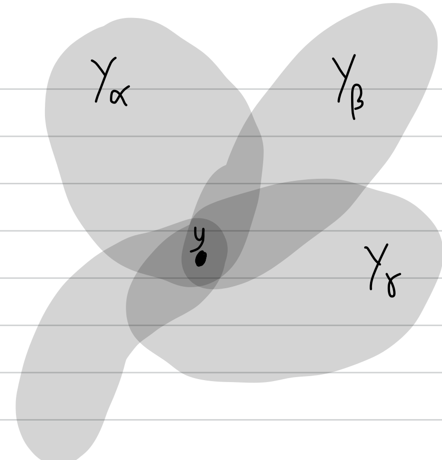
Pf If both $Y \cap U$ and $Y \cap V$ were nonempty, then these open sets in Y would form a separation of Y.

Thm If $A \subset X$ is connected and $A \subset B \subset \overline{A}$, then B is connected.



Rmk Adding in a subset of limit points preserves connectedness.

<u>Thm</u> Unions of connected subspaces with a point in common are connected.



<u>Pf</u> Let $Y = \bigcup_\alpha Y_\alpha$ with $y \in Y_\alpha \underset{\text{connected}}{\subset} X \quad \forall \alpha$.

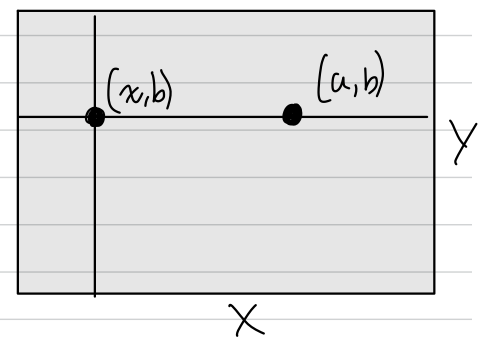
Assume Y has a separation U, V. Suppose $y \in U$.
Then $Y_\alpha \subset U \ \forall \alpha$. So $Y \subset U$ and V is empty, a contradiction.

<u>Thm</u> A finite product of connected spaces is connected

<u>Pf sketch</u> $n=2$ $\quad X \times Y \quad$ Let $(a,b) \in X \times Y$.

For $x \in X$, let $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
Note each T_x is connected by prior lemma.

Hence $X \times Y = \bigcup_{x \in X} T_x$ is connected by prior lemma with (a,b) as common point.

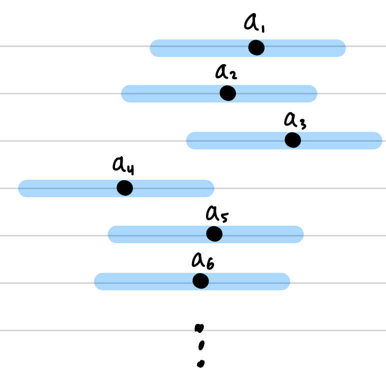


The general case of an n-fold product then follows by induction.

Cor $\mathbb{R}^n$ is connected (assuming $\mathbb{R}$ is).

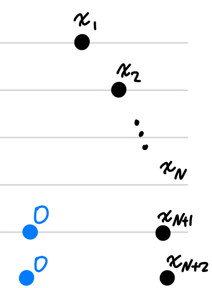
Ex $\mathbb{R}^\omega$ with box topology is not connected.

Pf Let $U = \{\text{bounded sequences}\}$ and $V = \{\text{unbounded sequences}\}$.
They're nonempty, disjoint, and $U \cup V = \mathbb{R}^\omega$. Open in box topology?
Given $a \in \mathbb{R}^\omega$, note $(a_1-1, a_1+1) \times (a_2-1, a_2+1) \times (a_3-1, a_3+1) \times \dots$
is an open neighborhood of a that is contained in
U (resp. V) if a is bounded (resp. unbounded).
Hence U and V are open in box topology,
and $(\mathbb{R}^\omega, \text{box topology})$ is not connected.



Ex $\mathbb{R}^\omega$ (with the product topology) is connected.

Pf Let $\widetilde{\mathbb{R}^n} = \{x \in \mathbb{R}^\omega \mid x_i = 0 \text{ for } i > n\}$. Connected since $\widetilde{\mathbb{R}^n} \cong \mathbb{R}^n$.
Let $\mathbb{R}^\infty = \bigcup_{n \geq 1} \widetilde{\mathbb{R}^n}$. Connected since union of connected sets containing $(0,0,0,\dots)$.
We claim $\mathbb{R}^\infty \subset \mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$, giving that $\mathbb{R}^\omega$ is connected.
Indeed, for $x \in \mathbb{R}^\omega$ and $x \in \prod_i U_i$ a basic open set,
$\exists N$ with $U_i = \mathbb{R}$ $\forall i > N$, meaning $(x_1, x_2, \dots, x_N, 0, 0, \dots) \in \mathbb{R}^\infty \cap \prod_i U_i$.

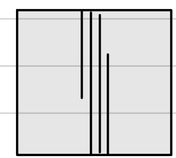


Section 24: Connected subspaces of $\mathbb{R}$

$\mathbb{R}$ is connected, and so are intervals and rays in $\mathbb{R}$.
Rmk: This doesn't rely on the algebraic structure of $\mathbb{R}$, only on its order properties. Indeed, see the theorem below.

Def A simply ordered set L with more than one element is a linear continuum if
(1) L has the least upper bound property
(2) If $x<y$, then $\exists z$ with $x<z<y$.

Ex $\mathbb{R}$, $[0,1]\times[0,1]$ with dictionary order.

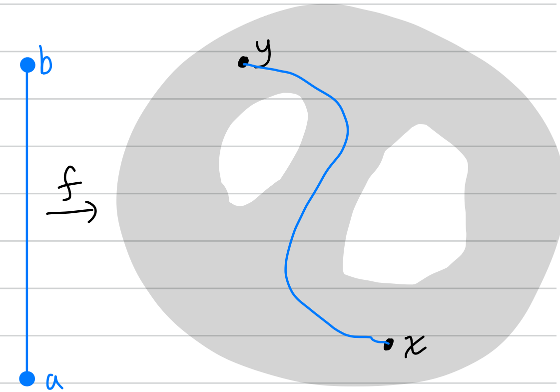


Def $Y \subset L$ is convex if $a,b \in Y$ with $a<b \Rightarrow [a,b] \subset Y$.

Ex L, intervals in L, rays in L.

Thm If L is a linear continuum with the order topology and $Y \subset L$ is convex, then Y is connected.

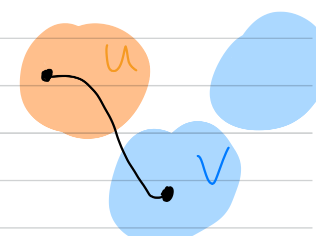
That intervals in $\mathbb{R}$ are connected gives a sufficient condition for showing a space is connected:



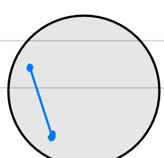
<u>Def</u> A space X is <u>path connected</u> if every $x, y \in X$ can be joined by a path, i.e. a continuous map $f: [a,b] \to X$ with $f(a) = x$ and $f(b) = y$.

<u>Lemma</u> A path connected space X is connected.

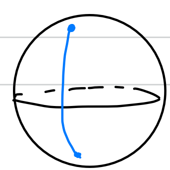
<u>Pf</u> Suppose U, V separate X.
Let $f: [a,b] \longrightarrow X$ be continuous with $f(a) \in U$, $f(b) \in V$.
But the continuous image of the connected set $[a,b]$ is connected, meaning $f([a,b])$ must be contained in U or in V.



<u>Ex</u> Ball $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is connected
Sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is connected for $n \geq 1$



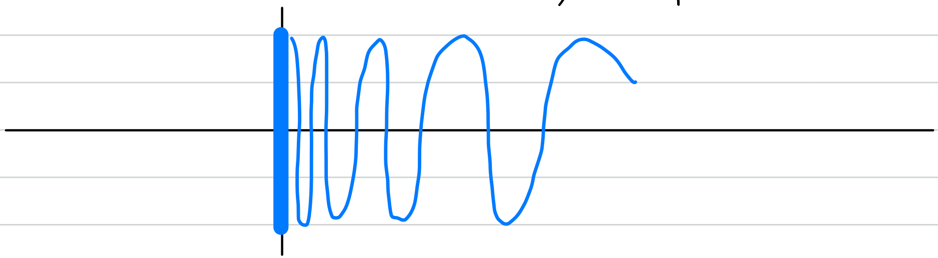
B^2



S^2

Ex Topologist's sine curve is $\overline{S} = \{(x, \sin(1/x)) \mid 0 < x \leq 1\} \cup \{(0,y) \mid -1 \leq y \leq 1\}$.

Here $S = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$ is path-connected $\Longrightarrow$ S connected.



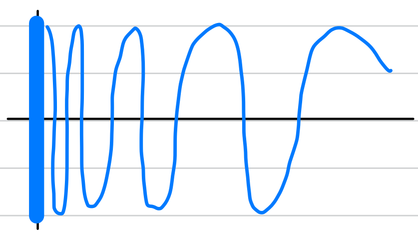
Note S connected $\Longrightarrow$ $\overline{S}$ connected.

But $\overline{S}$ is not path connected.

Section 25: Components and local connectedness

Def X topological space.
Declaring $x \sim y$ when $\exists$ connected subspace containing x, y gives equivalence relation; equivalence classes called components.

Declaring $x \sim y$ when $\exists$ a path in X from x to y gives equivalence relation; equivalence classes called path components.

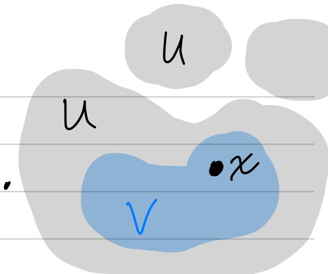
Ex 

One component
Two path components

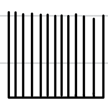
Each path component is connected, hence contained in a single component.

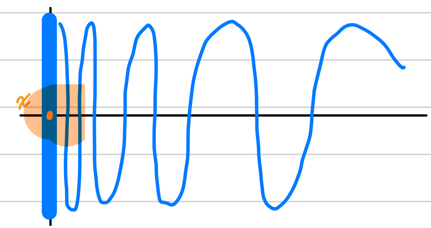
When do components and path components coincide?
X being "locally path connected" suffices.

Def A topological space X is locally (path) connected if $\forall x \in X$ and nbhds $U \ni x$, $\exists$ a (path) connected nbhd $x \in V \subset U$.



Locally path connected $\Longrightarrow$ locally connected, since path connected $\Longrightarrow$ connected.

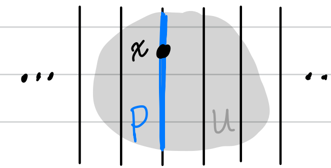
Ex	Connected	Not connected
Locally connected	Intervals in $\mathbb{R}$	$[-1,0) \cup (0,1]$
Not locally connected	Topologist's sine curve or  $[0,1] \times \{0\} \cup \mathbb{Q} \times [0,1]$	$\mathbb{Q}$



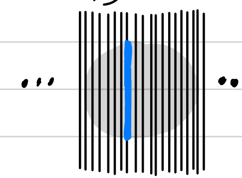
Thm A space X is locally (path) connected
$\iff \forall$ open $U \subset X$, each (path) component P of U is open.

Pf Let's prove the path connected version.
($\Rightarrow$) Open $U \subset X$ has path component P.
For $x \in P$, let nbhd $x \in V \subset U$ be path connected; hence $V \subset P$.
So P is open.
($\Leftarrow$) Given nbhd $U \ni x$, let P be the (open) path component of U containing x. Note $x \in P \subset U$.
So X is locally path connected.

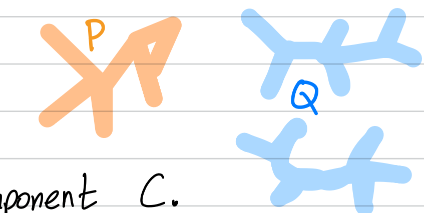
Ex $\mathbb{Z} \times [0,1]$



Non-Ex $\mathbb{Q} \times [0,1]$



Thm If X is locally path connected, then the components and path components coincide.



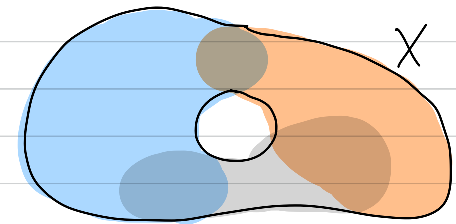
Pf Let P be a path component contained in a component C.
If $P \subsetneq C$, then let Q be the union of all other path components in C.
X locally connected, so prior theorem (connected version with $U = X$) says C open.
Prior theorem (path connected version with $U = C$) then says P and Q are open.
Then $C = P \cup Q$ gives a separation of C. So it must be that $P = C$.

Section 26: Compact spaces

Analogy: (Sets, functions) $\rightsquigarrow$ (Topological spaces, continuous functions)

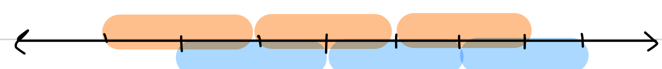
$\cup$ Finite sets $\qquad$ $\cup$ Compact spaces

A cover $\mathcal{U}$ of a topological space X is a collection of subsets whose union is X. If these sets are open, then $\mathcal{U}$ is an <u>open cover</u>.



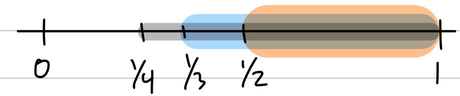
<u>Def</u> A topological space X is <u>compact</u> if every open cover $\mathcal{U}$ has a finite subcover, i.e. a subcollection $\{U_1, \dots, U_n\} \subset \mathcal{U}$ with $X = U_1 \cup \dots \cup U_n$.

<u>Ex</u> $\mathbb{R}$ not compact: $\{(n-1, n+1)\}_{n \in \mathbb{Z}}$ is an open cover with no finite subcover.

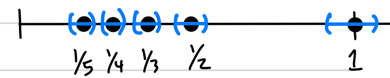


<u>Ex</u> A finite set X is compact (regardless of the topology).
(Given an open cover, for each $x \in X$, choose an open set containing x.)

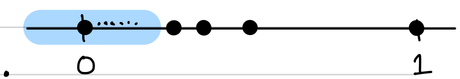
Ex $[0,1]$ not compact: the open cover $\{[1/n, 1]\}_{n\in\mathbb{Z}_+}$ has no finite subcover.



Ex $\{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$ is not compact: the open cover by singletons has no finite subcover.

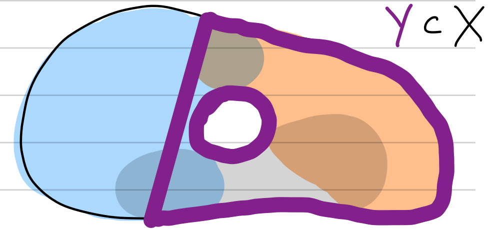


But $\{0\}\cup\{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$ is compact: Given an open cover, note an open set containing 0 contains all but finitely many elements of $\{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$.



Ex We'll see: $X\subset\mathbb{R}^n$ compact $\iff$ X closed and bounded.

We say a collection of sets $\{U_\alpha\}_{\alpha \in J}$ in X covers $Y \subset X$ if $Y \subset \bigcup_\alpha U_\alpha$.

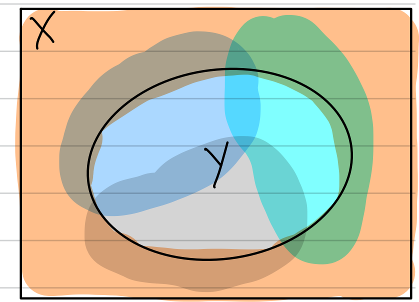


<u>Lemma</u> $Y \subset X$ is compact $\iff$ every cover of Y by open sets in X has a finite subcover

<u>Pf</u> Y compact $\iff$ every cover $\{\underbrace{U_\alpha}_{\text{open in } X} \cap Y\}$ has a finite subcover

$\iff$ every cover $\{\underbrace{U_\alpha}_{\text{open in } X}\}$ has a finite subcover

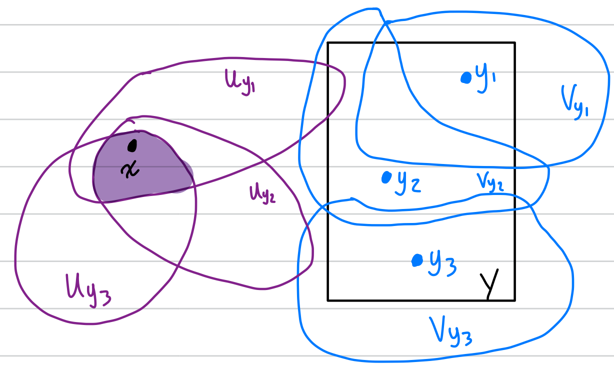
<u>Thm</u> A closed subset Y of a compact space X is compact.



<u>Pf</u> Let $\mathcal{U}$ be a cover of Y by open sets in X.
Then $\mathcal{U} \cup \{X - Y\}$ is an open cover of X.
X compact $\Rightarrow \exists$ finite subcover $\{U_1, \ldots, U_n, X - Y\}$ of X.
So $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ is a finite subcover of Y.

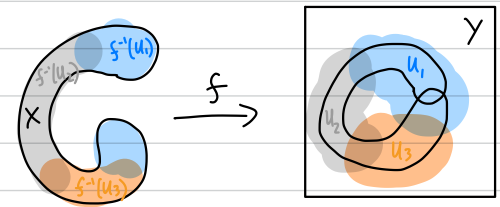
Thm Every compact subspace Y of a Hausdorff space X is closed.

Pf We'll show $X-Y$ is open.
Let $x \in X-Y$. For each $y \in Y$ $\exists$ disjoint opens $V_y \ni Y$, $U_y \ni x$.
Y compact $\Rightarrow$ Y has finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$.
Note $Y \subset V_{y_1} \cup \dots \cup V_{y_n}$ is disjoint from the open set $U_{y_1} \cap \dots \cap U_{y_n} \ni x$.
Hence $X-Y$ is open.

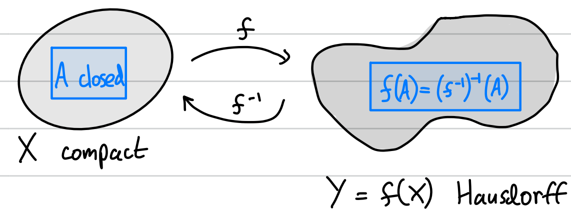


Thm If $f: X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Pf Let $\mathcal{U}$ be a cover of $f(X)$ by open sets in Y.
Then $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of X.
X compact $\Rightarrow$ $\exists$ finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$ of X.
So $U_1, \dots, U_n$ is a finite subcover of $f(X)$.



Thm If $f: X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.



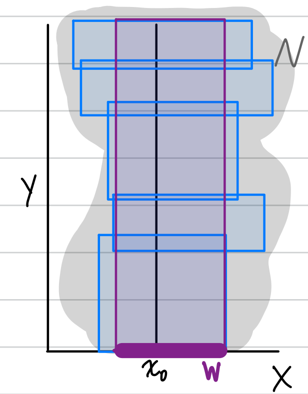
Pf To see that $f^{-1}: Y \to X$ is continuous, note

$$A \text{ closed in } X \underset{\text{since } X \text{ compact}}{\Longrightarrow} A \text{ compact} \Longrightarrow f(A) \text{ compact} \underset{\text{since } Y \text{ Hausdorff}}{\Longrightarrow} f(A) = (f^{-1})^{-1}(A) \text{ closed.}$$

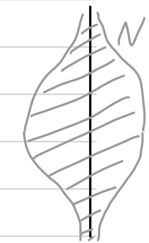
Thm Finite products of compact spaces are compact.

(Tychonoff theorem: Arbitrary products of compact spaces are compact.
One proof in Munkres uses Zorn's lemma, another the well-ordering theorem.)

Tube lemma X space, Y compact. Let $x_0 \times Y \subset N \underset{\text{open}}{\subset} X \times Y$.
Then $\exists\, x_0 \in W \underset{\text{open}}{\subset} X$ with $W \times Y \subset N$.



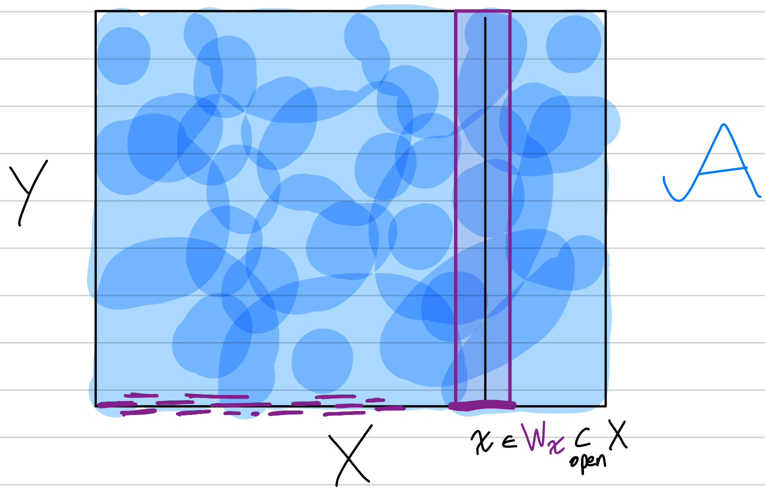
Not:



Pf Cover $\underbrace{x_0 \times Y}_{\cong\, Y \text{ compact}}$ with basic opens $\{U \times V\}$, each $U \times V \subset N$.

$\exists$ finite subcover $U_1 \times V_1, \ldots, U_n \times V_n$ with $x_0 \in U_i$ $\forall i$.
Let $W = U_1 \cap \ldots \cap U_n$.

Pf theorem For X, Y compact, let $\mathcal{A}$ be open cover of $X \times Y$. (General case is by induction.)
For each $x \in X$, compact $x \times Y$ covered by $A_1, \ldots, A_n \in \mathcal{A}$.
Apply lemma with $N = A_1 \cup \ldots \cup A_n$:
Get $x \in W_x \underset{\text{open}}{\subset} X$ s.t. $W_x \times Y$ is covered by finitely many sets in $\mathcal{A}$.
X compact $\Rightarrow$ open cover $\{W_x\}$ of X has finite subcover $W_1, \ldots, W_k$.
So $W_1 \times Y, \ldots, W_k \times Y$ cover $X \times Y$ and are each covered by finitely many sets in $\mathcal{A}$.


Y
A
X
$x \in W_x \underset{\text{open}}{\subset} X$

__Def__ A collection $\mathcal{C}$ of subsets of X has the __finite intersection property__ (f.i.p.) if $\forall \{C_1, \dots, C_n\} \subset \mathcal{C}$, $\quad C_1 \cap \dots \cap C_n \neq \emptyset$.

__Ex__ Nested sequence $\quad C_1 \supset C_2 \supset C_3 \supset \dots$

$C_n = [-1/n, 1/n] \subset [-1,1]$ compact $\qquad\qquad C_n = [n, \infty) \subset \mathbb{R}$ not compact

__Thm__ X topological space. Then X compact $\iff$ every collection of closed sets with f.i.p. has nonempty intersection.

__Pf__ X compact $\iff$

For every collection of open sets, no finite subcover implies not a cover.

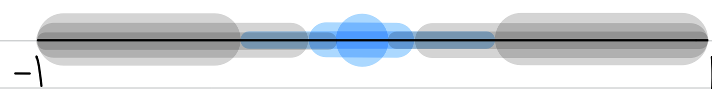
complement $\updownarrow$ $\qquad$ $\updownarrow$ $\qquad$ $\updownarrow$

closed sets $\qquad$ f.i.p. $\qquad$ nonempty intersection

__Picture__ $X = [-1,1]$

Open sets $\mathcal{U} = \{[-1, -1/n) \cup (1/n, 1]\}_{n \in \mathbb{Z}_+}$

Closed sets $\mathcal{C} = \{[-1/n, 1/n]\}_{n \in \mathbb{Z}_+}$



Section 27 Compact subspaces of $\mathbb{R}$

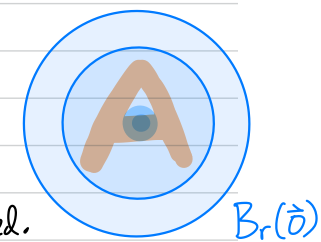
Every closed interval $[a,b]$ in $\mathbb{R}$ is compact. More generally,

Thm Let X be a totally ordered set with the least upper bound property. Then in the order topology, each closed interval $[a,b]$ is compact.

(**Pf** See book.)

Recall $A \subset \mathbb{R}^m$ is bounded if $A \subset B_M(\vec{0}) = \{x \in \mathbb{R}^m \mid \|x\| < M\}$ for some M.

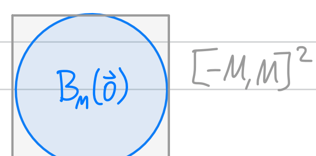
(Equivalently, $d(a,a') \le N$ for some $N \in \mathbb{R}$.)



Heine-Borel Thm $A \subset \mathbb{R}^m$ is compact $\Longleftrightarrow$ it is closed and bounded.

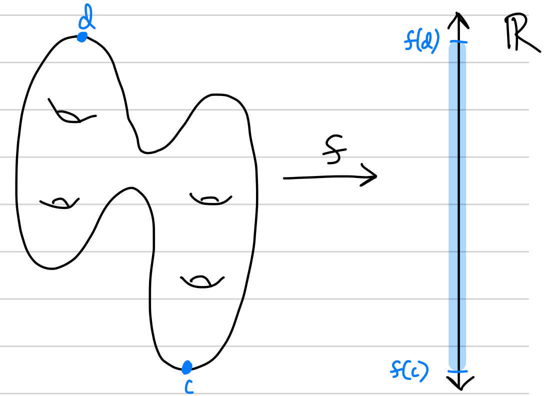
Pf ($\Rightarrow$) A compact means A closed (since $\mathbb{R}^m$ is Hausdorff).
Also, the open cover $\{B_r(\vec{0})\}_{r>0}$ of A has a finite subcover, so A bounded.

($\Leftarrow$) A bounded means $A \subset B_M(\vec{0}) \subset [-M,M]^m$, which is compact as a finite product of the compact space $[-M,M]$. So A is a closed subset of a compact space, hence compact.



Extreme value theorem

If $f: X \to \mathbb{R}$ is continuous and X is compact, then $\exists\ c, d \in X$ with $f(c) \leq f(x) \leq f(d)\ \forall x \in X$.



Pf X compact, f continuous $\Longrightarrow f(X)$ compact
$\Longrightarrow f(X)$ closed and bounded (Heine-Borel).
The least upper bound of $f(X)$ must be in $f(X)$:
if not, it would then be a limit point of $f(X)$, but closed sets contain their limit points.

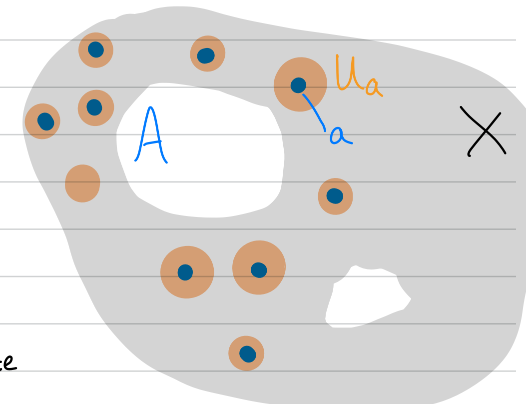
Similarly for the greatest lower bound.

Section 28: Limit point compactness

Def Topological space X is <u>limit point compact</u> if every infinite set has a limit point.
(Also called Bolzano–Weierstrass property.)

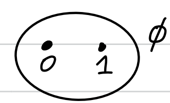
Recall x is a limit point of $A \subset X$ if every nbhd of x intersects A at some point other than x.

Thm X compact $\Rightarrow$ X limit point compact.

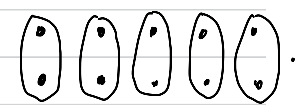


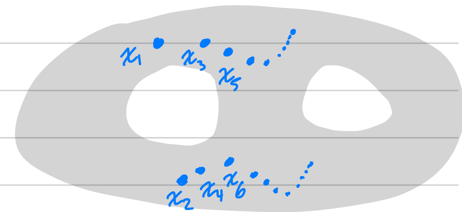
Pf If $A \subset X$ has no limit points, we'll show A finite.
A is closed (contains all its limit points) and hence compact.
$\forall a \in A$, $\exists$ nbhd $U_a \ni a$ s.t. $U_a \cap A = \{a\}$
(otherwise a is a limit point of A).
The open cover $\{U_a\}_{a \in A}$ of A has a finite subcover $\Rightarrow$ A finite

Ex Converse not true.



$X = \mathbb{Z}_+$ (discrete topology). $Y = \{0,1\}$ with indiscrete topology.
$X \times Y$ is limit point compact since every nonempty subset has a limit point.
$X \times Y$ is not compact since the open cover $\{n \times Y\}_{n \in \mathbb{Z}_+}$ has no finite subcover.





<u>Def</u> X is <u>sequentially compact</u> if every sequence of points in X has a convergent subsequence.

<u>Thm</u> If X is metrizable, then TFAE:

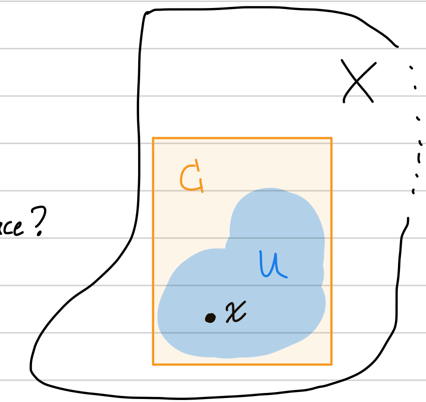
(1) X compact
(2) X limit point compact
(3) X sequentially compact

<u>Pf</u> (1) $\Rightarrow$ (2) above.

(2) $\Rightarrow$ (3). Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in X.
Let $A = \{x_1, x_2, x_3, \dots\}$.
If A finite, then $\exists$ constant (hence convergent) subsequence.
If A infinite, then $\exists$ limit point $a \in A$.
Hence $\forall n \in \mathbb{Z}_+$, $B_{1/n}(a)$ intersects A in infinitely many points. (since X metrizable; see Thm 17.9)
Hence we can choose a convergent subsequence (x_{n_k}) with $x_{n_k} \in B_{1/n_k}(a)$ $\forall k$.

(3) $\Rightarrow$ (1) hardest part; see book.

Section 29: Local compactness (and one-point compactification)

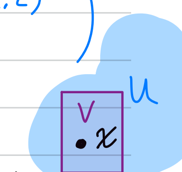


Particularly nice topological spaces include
- metric spaces
- compact Hausdorff spaces.

Any subspace of a metric space is a metric space.
When is a topological space a subspace of a compact Hausdorff space?

Def A topological space X is locally compact if $\forall x \in X$, $\exists$ nbhd $U \ni x$ and compact C with $U \subset C \subset X$.

(Rmk If X is Hausdorff, then this definition looks more familiar (see Thm 29.2):
X is locally compact $\iff$ $\forall x \in X$ and nbhd $U \ni x$, $\exists$ nbhd $x \in V \in U$ with $\overline{V}$ compact.)



Ex $\mathbb{R}$ is locally compact. $x \in \mathbb{R}$ $U = (x-1, x+1) \subset [x-1, x+1] = C$

Ex $\mathbb{R}^n$ is locally compact. $x \in \mathbb{R}^n$ $U = (x_1-1, x_1+1) \times \ldots \times (x_n-1, x_n+1) \subset [x_1-1, x_1+1] \times \ldots \times [x_n-1, x_n+1] = C$

Non-Ex $\mathbb{R}^\omega$ not locally compact — basic open sets are not contained in compact sets.

Thm Let X be a topological space.

X is locally compact Hausdorff

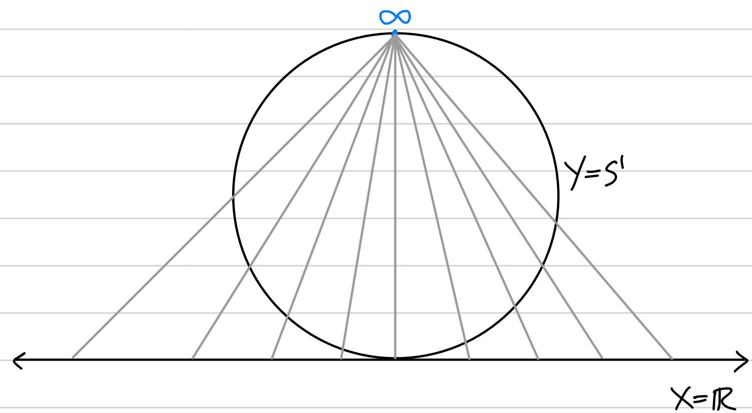
$\iff \exists$ a topological space Y s.t.

(1) $X \subset Y$

(2) $Y - X$ is a single point

(3) Y is compact Hausdorff.

Furthermore, for any two such spaces Y, Y', $\exists$ homeomorphism $h: Y \to Y'$ with $h|_X = id_X$.

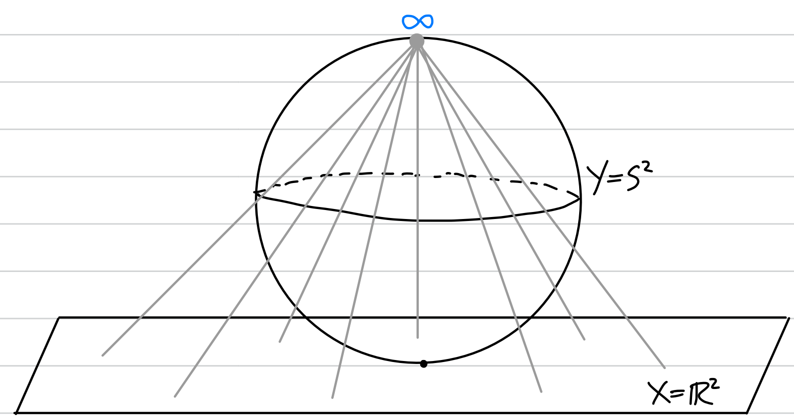


Ex $X = \mathbb{R} \cong (0,1)$

$Y = S^1 \cong [0,1]/_{0\sim 1}$

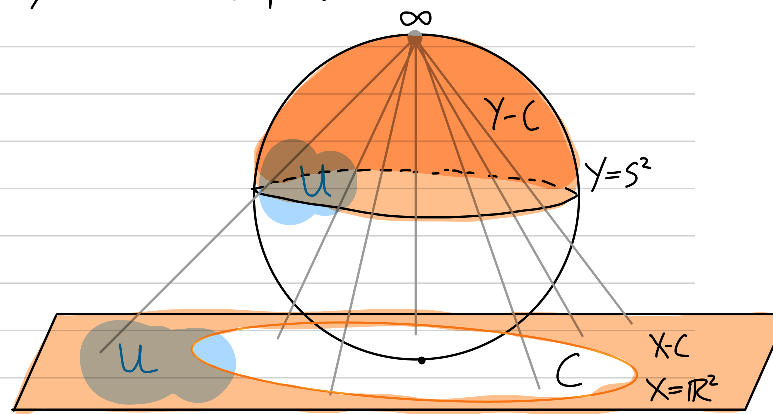
Ex $X = \mathbb{R}^n \qquad Y = S^n$

Special case $X = \mathbb{C}$, $Y = S^2$ Riemann sphere



Recall A closed subspace of a compact space is compact
A compact subspace of a Hausdorff space is closed
Thus for a subspace of a compact Hausdorff space, closed $\Leftrightarrow$ compact.

Pf of thm ($\Rightarrow$) Let $Y = X \cup \{\infty\}$.
Let topology τ for Y consist of:
(a) U, open in X
(b) $Y - C$, C compact in X.



To see that τ is a topology, note:

- $\emptyset$ open in X, $Y = Y - \emptyset$ with $\emptyset$ compact in X
- $U_1 \cap U_2$ open in X

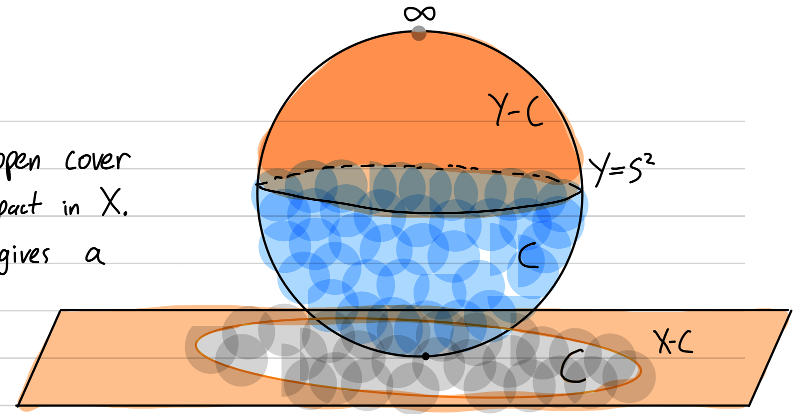
  $(Y - C_1) \cap (Y - C_2) = Y - \underline{(C_1 \cup C_2)}$ finite union of compact sets is compact

  $U \cap (Y - C) = U \cap \underline{(X - C)}$ open in X closed in X since X Hausdorff
- $\bigcup_\alpha U_\alpha =: U$ open in X

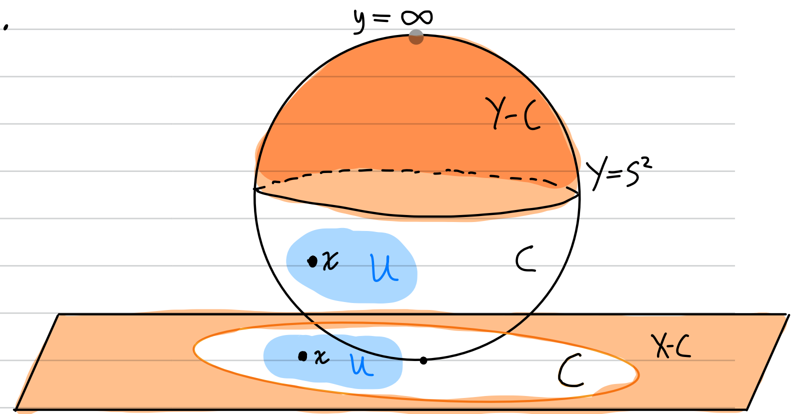
  $\bigcup_\beta (Y - C_\beta) = Y - \underbrace{\bigcap_\beta C_\beta}_{=:C}$ arbitrary intersection of compact spaces of Hausdorff space is compact

$\bigcup_\alpha U_\alpha \cup \bigcup_\beta (Y - C_\beta) = U \cup (Y - C) = Y - \underline{(C - U)}$ closed subspace of compact C is compact

To see that Y is compact, note if $\mathcal{U}$ is an open cover of Y, then $\mathcal{U}$ has an element $Y-C \ni x$, C compact in X. $\exists$ finite subcover of C, and adding in $Y-C$ gives a finite subcover of Y.



To see that Y is Hausdorff,
note for $x, x' \in X$, can use Hausdorff property of X.
Remaining case is $x \in X$, $y = \infty$.
Since X is locally compact,
choose compact set C containing a nbhd $U \ni x$.
Then $\underset{\substack{\Psi \\ x}}{U}$, $\underset{\substack{\Psi \\ y=\infty}}{X-C}$ are disjoint opens.



($\Leftarrow$) $X \subset Y = X \cup \{\infty\}$

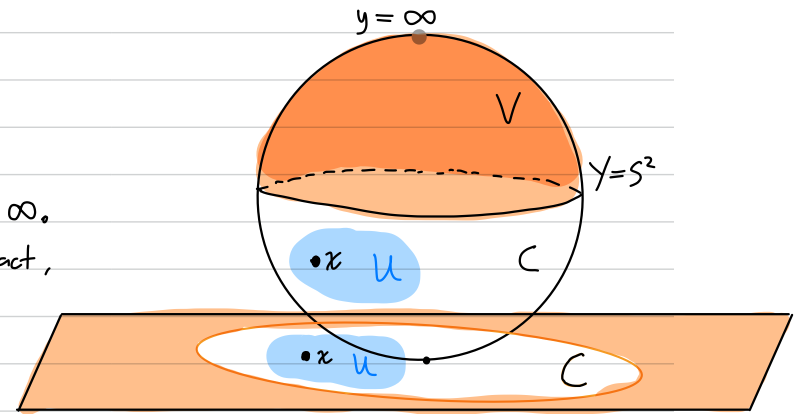
Note Y Hausdorff $\Rightarrow$ X Hausdorff.

Remains to show X locally compact.

For $x \in X$, choose disjoint opens $U \ni x$, $V \ni \infty$.

Let $C = Y - V$. C closed in $Y \Rightarrow C$ compact,

with $U \subset C \subset X$.



It remains to show that Y is unique up to $\cong$.

Define $h: Y \to Y'$ by $h(\infty) = \infty'$, $h(x) = x$ $\forall x \in X$.

One can show h is a homeomorphism (see book).