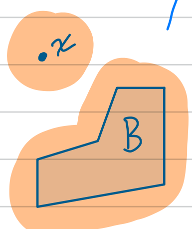
Chapter 4: Countability and separation axioms

e.g. first countable — e.g. Hausdorff

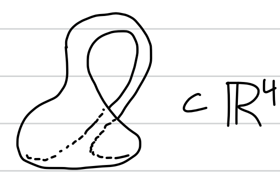
When can a given space be embedded in a metric space or a compact Hausdorff space?

Munkres' goal: Urysohn metrization theorem, which says
a second countable regular space is metrizable.

countable basis

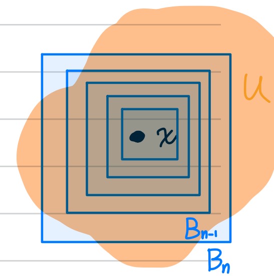


A second goal: A compact manifold can be embedded in some finite-dimensional Euclidean space.



Klein bottle

Section 30: The countability axioms



Def A space X has a countable basis at $x \in X$ if $\exists$ a countable collection of nbhds $\{B_n \ni x\}_{n \in \mathbb{Z}_+}$ such that for each nbhd $U \ni x$, $\exists$ some n with $x \in B_n \subset U$.

Space X is first countable if it has a countable basis at each $x \in X$

Ex A metric space (X, d) is first countable: Consider $\{B_{1/n}(x) \mid n \in \mathbb{Z}_+\}$

We previously saw the following theorem with $(\Leftarrow)$ for metric spaces:

Thm X a topological space.

(a) $A \subset X$. $\exists$ sequence $(a_n) \subset A$ with $a_n \to x \Rightarrow x \in \bar{A}$ (and $(\Leftarrow)$ if X first countable).

(b) $f: X \to Y$. f continuous $\Rightarrow \forall$ sequences $x_n \to x$, $f(x_n) \to f(x)$ (and $(\Leftarrow)$ if X first countable).

Def Space X is <u>second countable</u> if it has a countable basis.

Rmk Second countable spaces are first countable.

Ex $\mathbb{R}$ $\quad \{(a,b) \mid a,b \in \mathbb{Q}, a<b\}$

$\mathbb{R}^n$ $\quad \{(a_1,b_1)\times\dots\times(a_n,b_n) \mid a_i, b_i \in \mathbb{Q}\}$

$\mathbb{R}^\omega$ $\quad$ Similarly

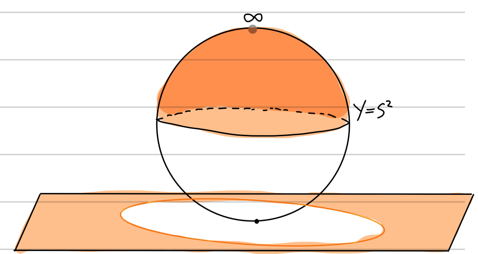
Thm Subspaces and countable products of first/second countable spaces are first/second countable.

Pf Quick.

Def $A \subset X$ is <u>dense</u> if $\overline{A} = X$

Ex $\mathbb{Q} \subset \mathbb{R}$ is dense

$\mathbb{Q}^2 \subset \mathbb{R}^2$ is dense



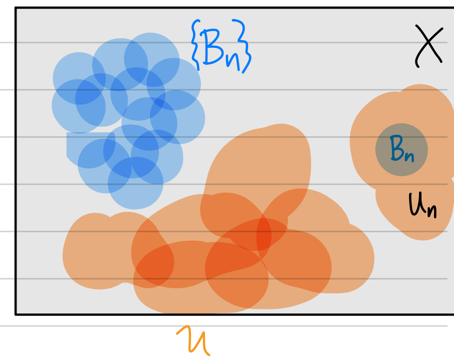
A non-compact (locally compact Hausdorff) space is dense in its one-point compactification.

<u>Thm</u> For X second countable,
(a) Every open cover has a countable subcover (Lindelöf property)
(b) $\exists$ countable dense subset of X (separable property).

<u>Rmk</u> The three are equivalent if X metrizable.

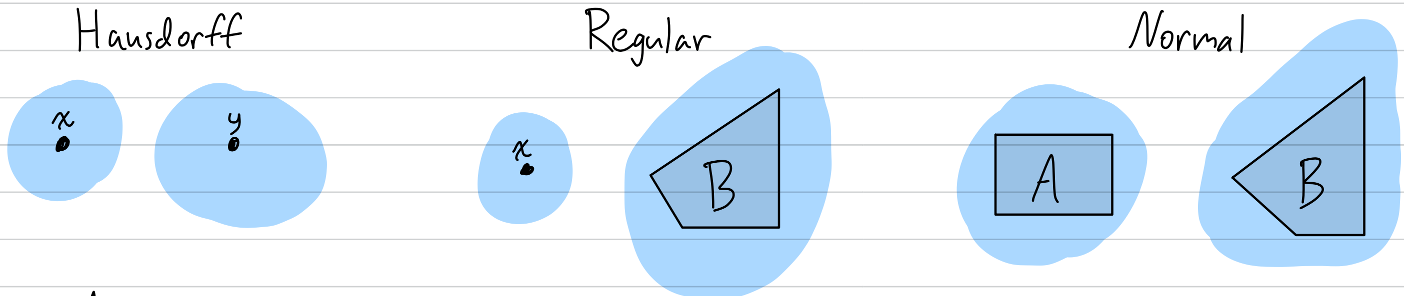
<u>Pf</u> Let $\{B_n\}$ be a countable basis for X.

(a) $\mathcal{U}$ open cover of X.
For each n, if $\exists\, U \in \mathcal{U}$ with $B_n \subset U$, choose one such U_n.
This gives a countable subcollection $\{U_n\}$ of $\mathcal{U}$.
To see this subcollection covers X, note if $x \in X$ then $x \in U$ for some $U \in \mathcal{U}$. So $x \in B_n \subset U$. $\quad B_n \subset U_n$



(b) $\forall n$ choose $x_n \in B_n$. Let $D = \{x_n \mid n \in \mathbb{Z}_+\}$.
For any $x \in X$, note any basic open B_n intersects D;
hence $x \in \overline{D}$ and $X = \overline{D}$.

Section 31: Separation axioms



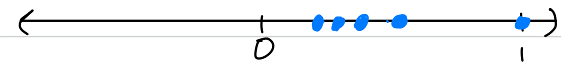
Def A topological space is regular if one point sets are closed and for each $x \in X$ and $B \underset{\text{closed}}{\subseteq} X$ with $x \notin B$, $\exists$ disjoint opens $U \ni x$ and $V \supset B$.

X is normal if one-point sets are closed and for each disjoint $A, B \underset{\text{closed}}{\subseteq} X$, $\exists$ disjoint opens $U \supset A$ and $V \supset B$.

Rmk A regular space is Hausdorff: let $B = \{y\}$.
A normal space is regular: let $A = \{x\}$.

<u>Ex</u> Recall $\mathbb{R}_K$ is $\mathbb{R}$ with basis $\{(a,b), (a,b)-K\}$, where $K = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$.

(We've added enough open sets so that K is closed.)



$\mathbb{R}_K$ is Hausdorff: Use open intervals.

$\mathbb{R}_K$ is not regular: Consider 0 and $0 \notin K \underset{\text{closed}}{\subseteq} \mathbb{R}_K$.

Can show any open sets about 0 and K intersect.

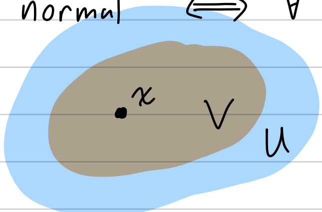
Later, we'll see a space that is regular but not normal.

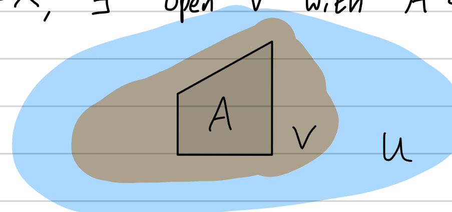
Another way to state these properties is:

<u>Lemma</u> Let X be a topological space with one-point sets closed.

X is regular $\iff$ $\forall$ nbhd $U \ni x$, $\exists$ open V with $x \in V \subset \overline{V} \subset U$.

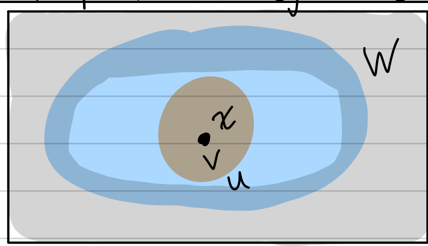
X is normal $\iff$ $\forall$ $A \underset{\text{closed}}{\subset} X$, $A \subset U \underset{\text{open}}{\subset} X$, $\exists$ open V with $A \subset V \subset \overline{V} \subset U$.





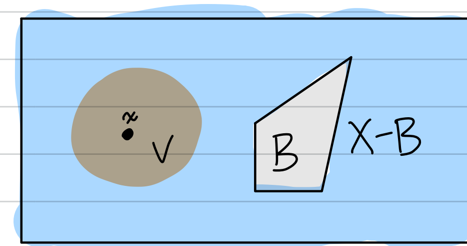
<u>Picture of proof for regular case</u>

$(\Rightarrow)$



Get disjoint opens $V \ni x$, $W \supset X - U$.
W is nbhd of any point in $X-U$ and
W is disjoint from V.
So $(X-U) \cap \overline{V} = \emptyset$, i.e. $\overline{V} \subset U$.

$(\Leftarrow)$

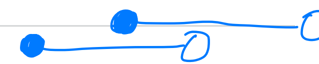


Let $U = X - B$, get $x \in V \subset \overline{V} \subset X - B$.
So $V \ni x$ and $X - \overline{V} \supset B$ are disjoint open sets.

<u>Thm</u> Subspaces and products of Hausdorff spaces are Hausdorff.
Subspaces and products of regular spaces are regular.
(The same is not true for normal spaces.)

<u>Ex</u> $\mathbb{R}_\ell$ (lower limit topology) has basis: $\{(a,b), [a,b)\}$.

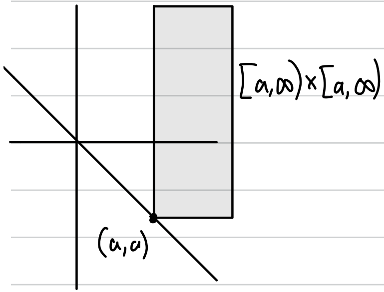
$\mathbb{R}_\ell$ is normal. Indeed, let $A, B \underset{\text{closed}}{\subset} \mathbb{R}_\ell$ be disjoint.
$\forall a \in A$, since $a \notin B = \overline{B}$, $\exists$ open $[a, a+\varepsilon_a)$ disjoint from B.
$\forall b \in B$, $\exists$ open $[b, b+\varepsilon_b)$ disjoint from A.



Then $\bigcup_{a \in A} [a, a+\varepsilon_a) \supset A$ and $\bigcup_{b \in B} [b, b+\varepsilon_b) \supset B$ are disjoint opens.

Hence $\mathbb{R}_\ell$ is regular.
By the above theorem, the Sorgenfrey plane $(\mathbb{R}_\ell)^2$ is regular.
But $(\mathbb{R}_\ell)^2$ is not normal.

Indeed, $L = \{(x,-x) : x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$, hence closed in $(\mathbb{R}_\ell)^2$.



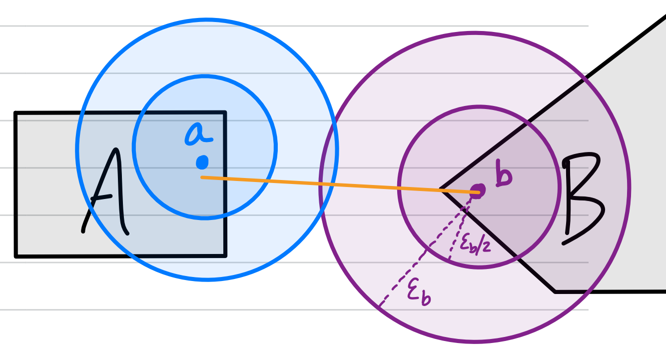
Note $\{(a,-a)\}$ is open in L,
so L has the discrete subspace topology.
I.e., all subsets of L are closed in L and hence in $(\mathbb{R}_\ell)^2$.

One can show $A = \{(x,-x) \mid x \in \mathbb{Q}\}$ and $B = L - A$ are closed sets in $(\mathbb{R}_\ell)^2$ not contained in disjoint opens.
So $(\mathbb{R}_\ell)^2$ is not normal.

Section 32: Normal spaces

Thm Every metrizable space (X,d) is normal.

Pf Metric space X Hausdorff $\Rightarrow$ one-point sets are closed.

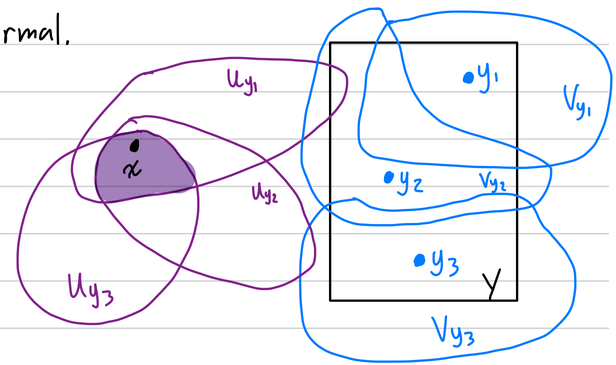


Let $A, B \subset X$ be disjoint closed subsets.
$\forall a \in A \ \exists \ B_{\varepsilon_a}(a)$ disjoint from B
(else a is a limit point of B and hence in B)
$\forall b \in B \ \exists \ B_{\varepsilon_b}(b)$ disjoint from A.

Let $U = \bigcup_{a \in A} B_{\varepsilon_a/2}(a)$ and $V = \bigcup_{b \in B} B_{\varepsilon_b/2}(b)$

These open sets containing A, B are disjoint since if $z \in U \cap V$,
then $\exists \, a \in A$ and $b \in B$ with $z \in B_{\varepsilon_a/2}(a) \cap B_{\varepsilon_b/2}(b)$.
WLOG let $\varepsilon_a \leq \varepsilon_b$.
We'd have $d(a,b) \leq d(a,z) + d(z,b) \leq \varepsilon_a/2 + \varepsilon_b/2 \leq \varepsilon_b$, a contradiction.

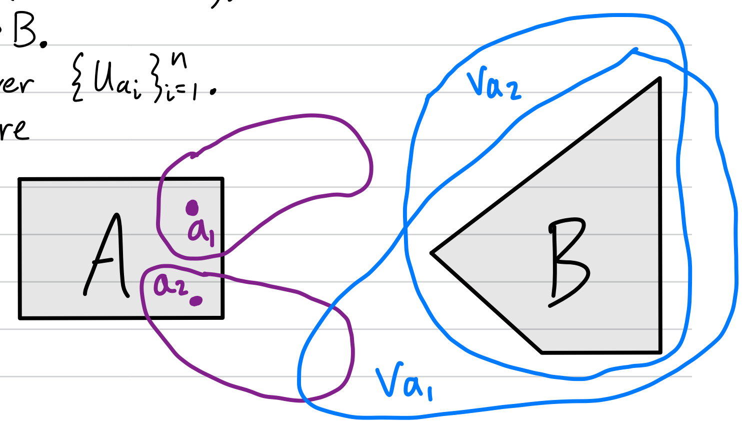
Thm Every compact Hausdorff space X is normal.



Pf Hausdorff $\Rightarrow$ one-point sets closed.

Recall our proof that compact subsets of a Hausdorff space are closed: given Y compact and $x \notin Y$, we found disjoint opens $U \ni x$ and $V \supset Y$.

Let $A, B \subset X$ be closed and disjoint.
B closed and X compact $\Rightarrow$ B compact (and same for A).
Hence $\forall a \in A$ $\exists$ disjoint open sets $U_a \ni a$, $V_a \supset B$.
$\{U_a\}$ covers the compact set $A \Rightarrow$ finite subcover $\{U_{a_i}\}_{i=1}^n$.
Note $U = U_{a_1} \cup \dots \cup U_{a_n}$ and $V = V_{a_1} \cap \dots \cap V_{a_n}$ are disjoint opens containing A and B.



Section 33: The Urysohn Lemma

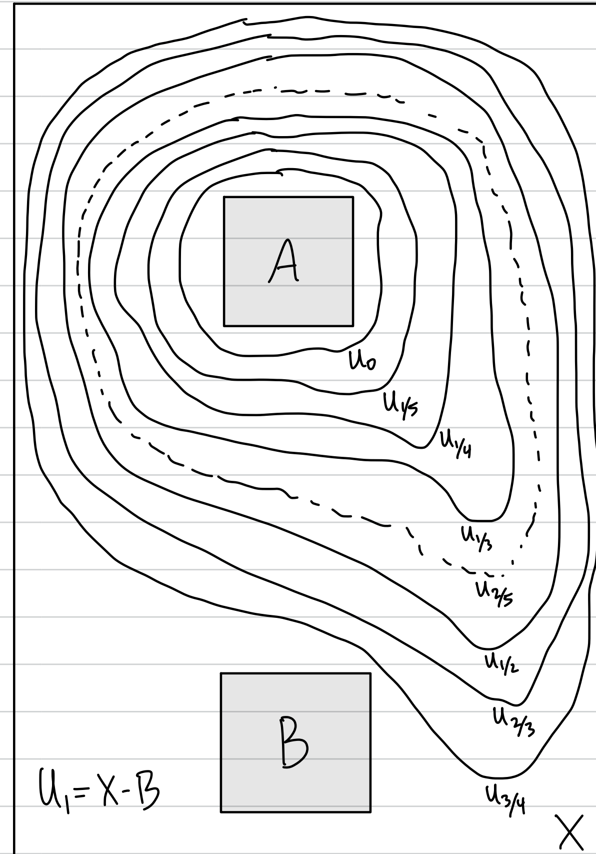
Thm (Urysohn lemma) Let X be a normal space, A, B disjoint closed subsets, and $[a,b] \subset \mathbb{R}$ $(a<b)$. Then $\exists$ continuous $f: X \to [a,b]$ with $f(x)=a$ $\forall x \in A$ and $f(x)=b$ $\forall x \in B$.

Pf It suffices to consider the case $[a,b]=[0,1]$.

Order the countable set $\mathbb{Q} \cap [0,1]$, starting with $1, 0$.
For example: $1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

Let $U_1 = X - B$ (open).
Apply normality to get U_0 open with $A \subset U_0 \subset \overline{U_0} \subset U_1$.

Continue inductively, obtaining open sets U_p $\forall p \in \mathbb{Q} \cap [0,1]$ satisfying $p<q \Rightarrow \overline{U_p} \subset U_q$.

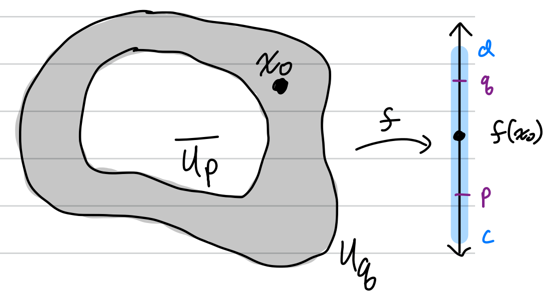


(For example, when constructing $U_{2/5}$, apply normality to get $U_{2/5}$ open with $U_{1/3} \subset U_{2/5} \subset \overline{U_{2/5}} \subset U_{1/2}$.)

For $p \in \mathbb{Q} \cap (-\infty, 0)$, define $U_p = \emptyset$.
For $p \in \mathbb{Q} \cap (1, \infty)$, define $U_p = X$.

Now, define $f: X \to [0,1]$ by $f(x) = \inf\{p \in \mathbb{Q} \mid x \in U_p\}$.
If $x \in A$, then $x \in U_p$ $\forall p \geq 0$, so $f(x) = 0$ as required.
If $x \in B$, then $x \notin U_p$ $\forall p \leq 1$, so $f(x) = 1$ as required.

We now check f is continuous.
Given $x_0 \in X$ and open $(c,d) \ni f(x_0)$, pick $p,q \in \mathbb{Q}$ with
$c < p < f(x_0) < q < d$.
We claim $U_q - \overline{U_p}$ is an open ✓ neighborhood about x_0 with $f(U_q - \overline{U_p}) \subset (c,d)$.



Note $f(x_0) < q \Rightarrow x_0 \in U_q$ and $f(x_0) > p \Rightarrow x_0 \notin U_s$ for some $s > p \Rightarrow f(x_0) \notin \overline{U_p}$. So $x_0 \in U_q - \overline{U_p}$.
Similar arguments show $f(U_q - \overline{U_p}) \subset [p,q] \subset (c,d)$, as desired.