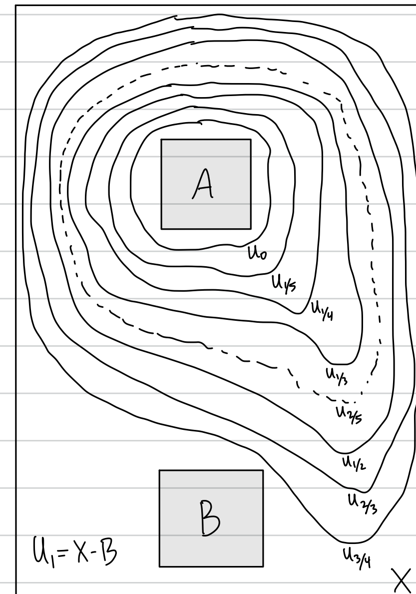
Section 35: The Tietze extension theorem

First, let us recall the Urysohn lemma from Section 34 (last semester):



Thm (Urysohn lemma) Let X be a normal space, A, B disjoint closed subsets, and $[a,b] \subset \mathbb{R}$ $(a<b)$. Then $\exists$ continuous $f: X \to [a,b]$ with $f(x) = a$ $\forall x \in A$ and $f(x) = b$ $\forall x \in B$.

Pf It suffices to consider $[a,b] = [0,1]$. Order the countable set $\mathbb{Q} \cap [0,1]$, starting with $1, 0$.

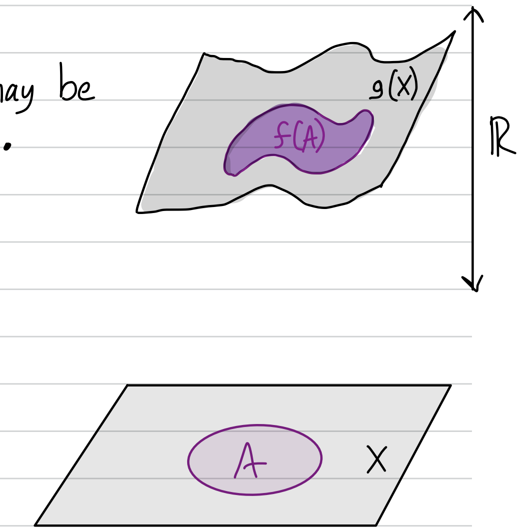
Let $U_1 = X - B$ (open).
Apply normality to get U_0 open with $A \subset U_0 \subset \overline{U_0} \subset U_1$.
Continue inductively, obtaining open sets U_p $\forall p \in \mathbb{Q} \cap [0,1]$ satisfying $p < q \Rightarrow \overline{U_p} \subset U_q$.

Now, define $f: X \to [0,1]$ by $f(x) = \inf \{ p \in \mathbb{Q} \mid x \in U_p \}$.
If $x \in A$, then $x \in U_p$ $\forall p \geq 0$, so $f(x) = 0$ as required.
If $x \in B$, then $x \notin U_p$ $\forall p \leq 1$, so $f(x) = 1$ as required.

Thm (Tietze extension theorem)

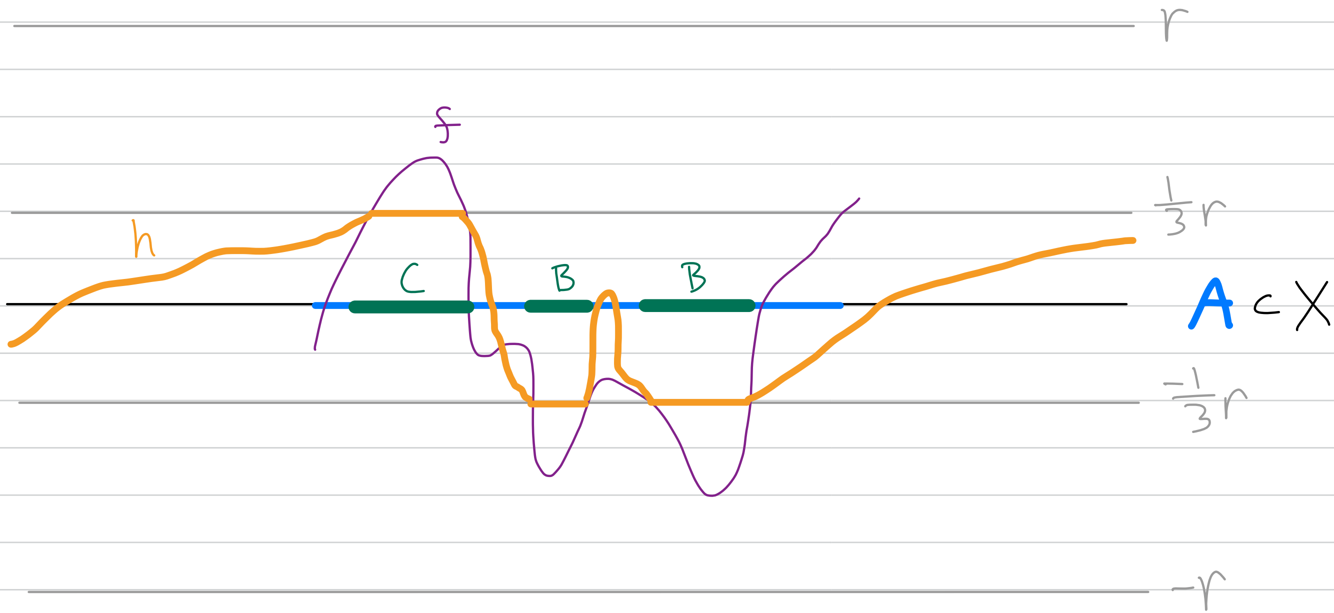
Let X be a normal space; let $A \subset X$ be closed.
Then any continuous map $f: A \to [a,b]$ (resp. $f: A \to \mathbb{R}$) may be extended to a continuous map $g: X \to [a,b]$ (resp. $g: X \to \mathbb{R}$).

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & {[a,b]} \\ X \arrow[ur, dashed, "\exists g"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & \mathbb{R} \\ X \arrow[ur, dashed, "\exists g"'] & \end{tikzcd}$$



Idea Construct a sequence of functions $s_n: X \to \mathbb{R}$ (defined via $s_n(x) = \sum_{i=1}^{n} g_i(x)$) that converges uniformly (so that the limit $g(x) = \sum_{i=1}^{\infty} g_i(x)$ is continuous) and that approximates f better and better on A (so that $g|_A = f|_A$).

Step 1 Let $f : A \to [-r, r]$. Then $\exists h : X \to [\frac{-1}{3} r, \frac{1}{3} r]$ with $|f(a) - h(a)| \le \frac{2}{3} r \quad \forall a \in A$.



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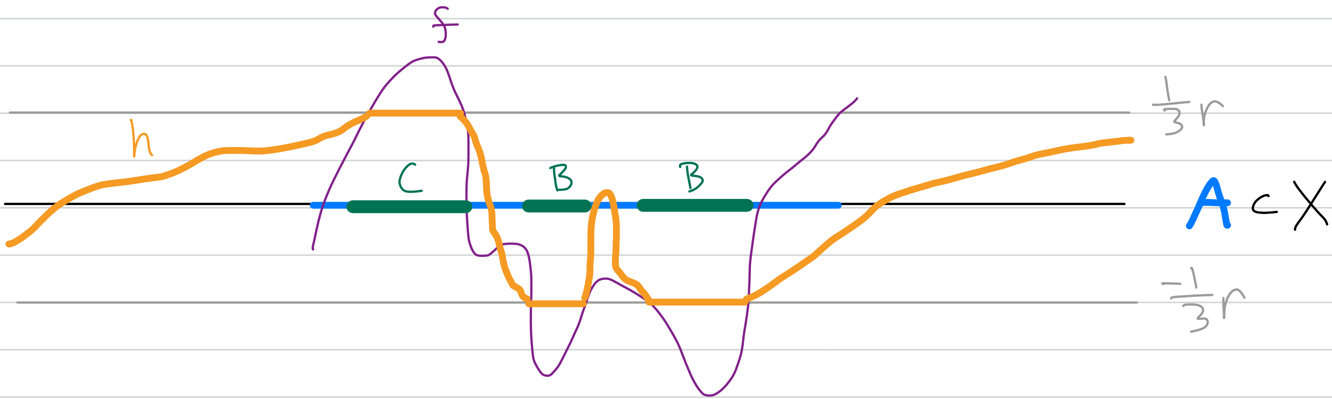
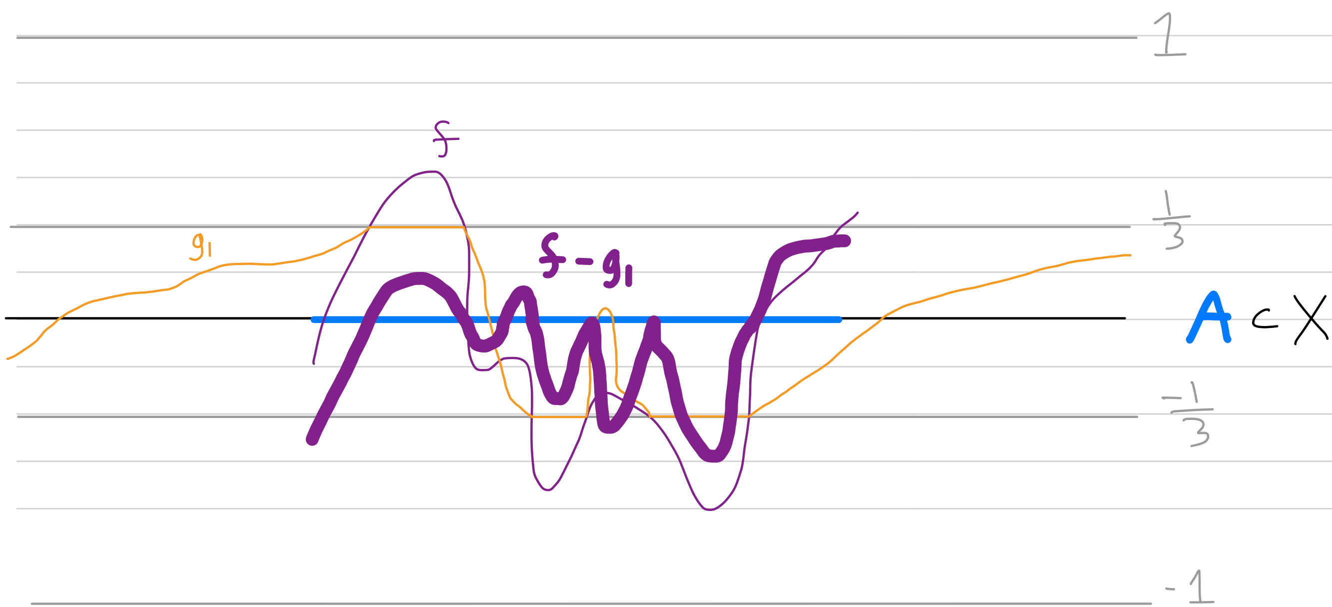


To prove Step 1, let $B = f^{-1}([-r, \frac{-1}{3}r])$ and $C = f^{-1}([\frac{1}{3}r, r])$.
Note B and C are closed in X (since f is continuous and A is closed in X).
By the Urysohn lemma, $\exists$ cont. $h: X \to [\frac{-1}{3}r, \frac{1}{3}r]$ with $h(x) = \frac{-1}{3}r \quad \forall\, x \in B$
and $h(x) = \frac{1}{3}r \quad \forall\, x \in C$.
To see $|f(a) - h(a)| \leq \frac{2}{3}r \quad \forall a \in A$, simply consider the three cases $x \in B$, $x \in C$, $x \in X \setminus (B \cup C)$ separately.

<u>Step 2</u> Prove the Tietze extension theorem for $f: A \to [a,b]$.

Note it suffices to consider $f: A \to [-1,1]$, which satisfies Step 1 with $r=1$. This gives $g_1: X \to [\frac{-1}{3}, \frac{1}{3}]$ with $|f(a) - g_1(a)| \leq \frac{2}{3}$ $\forall a \in A$.



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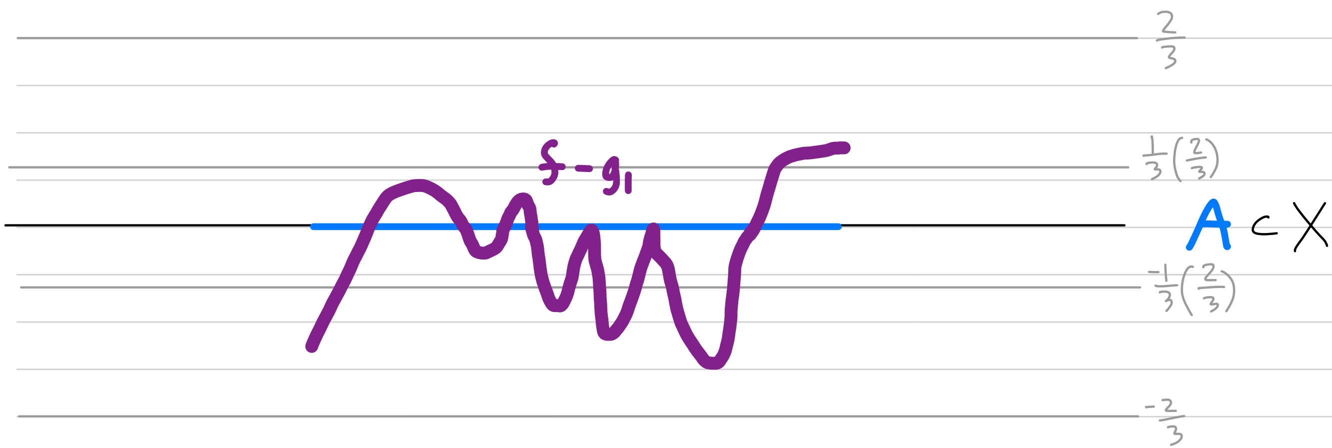

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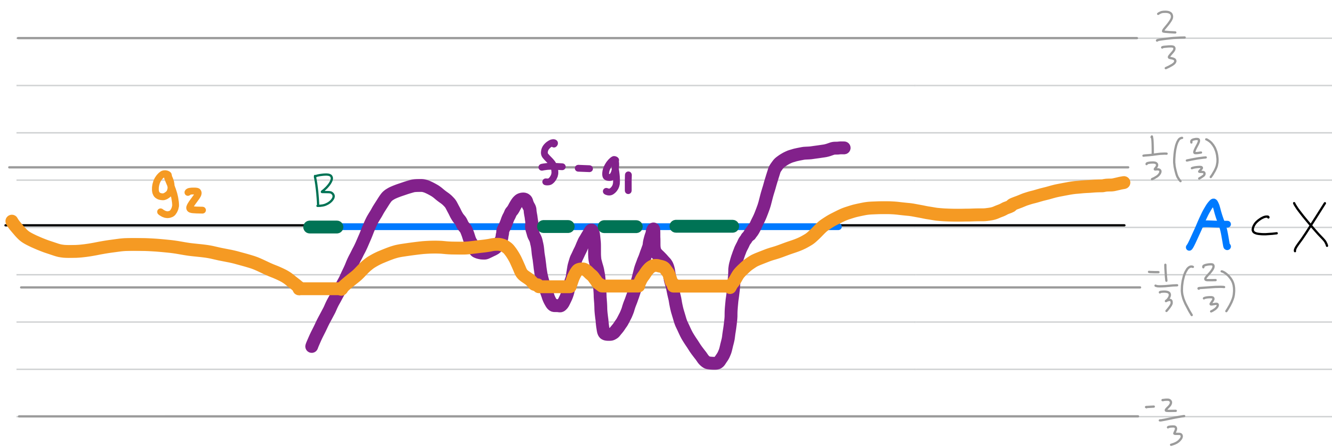
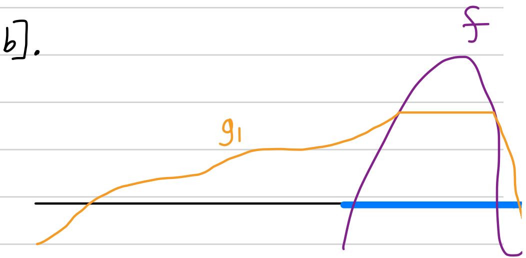
Note it suffices to consider $f: A \to [-1,1]$, which satisfies Step 1 with $r = 1$. This gives $g_1 : X \to [\frac{-1}{3}, \frac{1}{3}]$ with $|f(a) - g_1(a)| \leq \frac{2}{3} \quad \forall a \in A$.

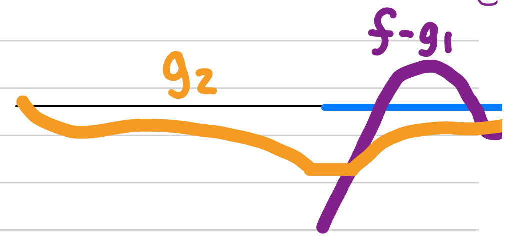
Apply Step 1 to $f - g_1 : A \to [\frac{-2}{3}, \frac{2}{3}]$, with $r = \frac{2}{3}$, to get $g_2 : X \to [\frac{-1}{3}(\frac{2}{3}), \frac{1}{3}(\frac{2}{3})]$ with $|f(a) - g_1(a) - g_2(a)| \leq (\frac{2}{3})^2 \quad \forall a \in A$.



Step 2 Prove the Tietze extension theorem for $f: A \to [a,b]$.



Note it suffices to consider $f: A \to [-1,1]$, which satisfies Step 1 with $r=1$. This gives $g_1: X \to [\frac{-1}{3}, \frac{1}{3}]$ with $|f(a) - g_1(a)| \le \frac{2}{3}$ $\forall a \in A$.



Apply Step 1 to $f - g_1 : A \to [\frac{-2}{3}, \frac{2}{3}]$, with $r = \frac{2}{3}$, to get $g_2 : X \to [\frac{-1}{3}(\frac{2}{3}), \frac{1}{3}(\frac{2}{3})]$ with $|f(a) - g_1(a) - g_2(a)| \le (\frac{2}{3})^2$ $\forall a \in A$.

$\vdots$

At the general step, we have $g_i : X \to [-\frac{1}{3}(\frac{2}{3})^{i-1}, \frac{1}{3}(\frac{2}{3})^{i-1}]$ for $1 \le i \le n$ with $|f(a) - \sum_{i=1}^{n} g_i(a)| \le (\frac{2}{3})^n$ $\forall a \in A$.

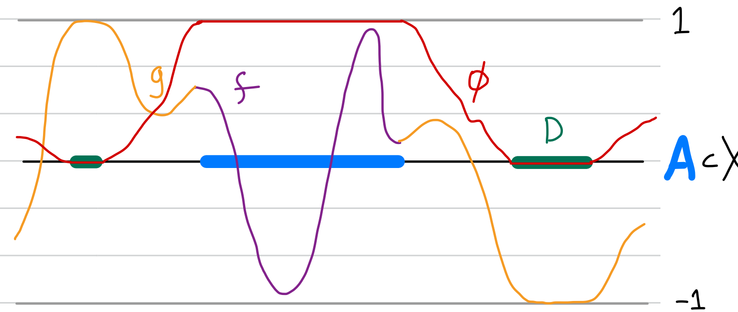
Apply Step 1 to $f - \sum_{i=1}^{n} g_i : A \to [-(\frac{2}{3})^n, (\frac{2}{3})^n]$, with $r = (\frac{2}{3})^n$, to get $g_{n+1} : X \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n]$ with $|f(a) - \sum_{i=1}^{n+1} g_i(a)| \le (\frac{2}{3})^{n+1}$ $\forall a \in A$.

Define $g: X \to [-1,1]$ by $g(x) = \sum_{i=1}^{\infty} g_i(x)$ (note $\frac{1}{3} \sum_{i=1}^{\infty} (\frac{2}{3})^i = 1$).
By the Weierstrass M-test, the partial sums converge uniformly to a continuous function g, which furthermore satisfies $g|_A = f$.

Step 3 Prove the Tietze extension theorem for $f: A \to \mathbb{R}$.

Since $\mathbb{R}$ is homeomorphic to $(-1,1)$, assume $f: A \to (-1,1)$.
We must find an extension $h: X \to (-1,1)$.

By Step 2, $\exists$ an extension $g: X \to [-1,1]$.
Let $D = g^{-1}(\{-1,1\})$, which is closed in X.
Note A and D are disjoint since $g(A) = f(A) \subset (-1,1)$.
By the Urysohn lemma, $\exists$ cont. $\phi: X \to [0,1]$
with $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$.
Define $h: X \to (-1,1)$ by $h(x) = \phi(x)\, g(x)$,
which lands in $(-1,1)$ since $|g(x)| = 1 \Rightarrow \phi(x) = 0$,
and which extends f since
$h(a) = \phi(a)\, g(a) = 1 \cdot g(a) = f(a) \quad \forall a \in A$.



Section 38 (from Chapter 5): The Čech-Stone compactification

Covered by Prof. Dana Bartošová

Section 37 (from Chapter 5): The Tychonoff theorem

Covered by Prof. Jeremy Booher