Chapter 4: Homotopy Theory

- The higher homotopy groups $\pi_n(X)$ generalize the fundamental group $\pi_1(X)$.
- There are formal similarities with homology:
  - $\pi_n(X)$ abelian for $n \geq 2$.
  - Long exact sequence $\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X,A) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots$
  - The Hurewicz Theorem says if X is simply connected, then the first nonzero homotopy group $\pi_n(X)$ is isomorphic to the first nonzero homology group $H_n(X)$.
- Homotopy groups are much harder to compute than homology: no excision or Mayer-Vietoris.

- Homotopy groups are still of great theoretical significance:
  - <u>Whitehead's theorem</u> says if a map $f: X \to Y$ between CW complexes induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ $\forall n$, then f is a homotopy equivalence.

    Careful: There must be a map f inducing the isomorphisms. $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$ have isomorphic homotopy groups but are not homotopy equivalent.
  - Fiber bundles $\begin{array}{c} F \to E \\ \quad\downarrow \\ \quad B \end{array}$ give long exact sequences

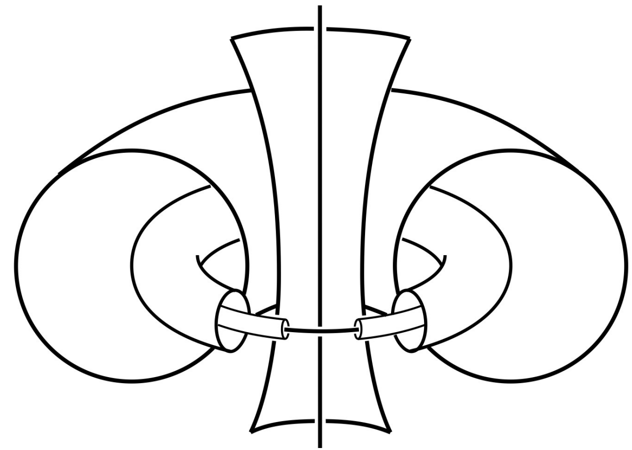
$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

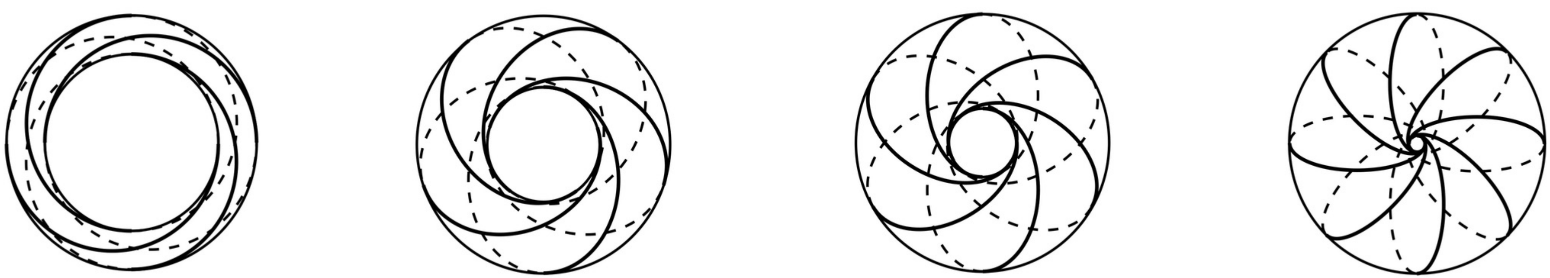
- The treatment in these notes is relatively informal and in reality, care needs to be taken with basepoints.

<u>Ex</u> The Hopf fibration is a fiber bundle $S^1 \to S^3 \to S^2$,
giving $\cdots \to \underset{\substack{\| \\ 0}}{\pi_3(S^1)} \to \underset{\substack{\cong \\ \mathbb{Z}}}{\pi_3(S^3)} \xrightarrow{\cong} \pi_3(S^2) \to \underset{\substack{\| \\ 0}}{\pi_2(S^1)} \to \cdots$

by Hurewicz

We see $\pi_3(S^2) \cong \mathbb{Z}$.





Section 4.1 Homotopy groups

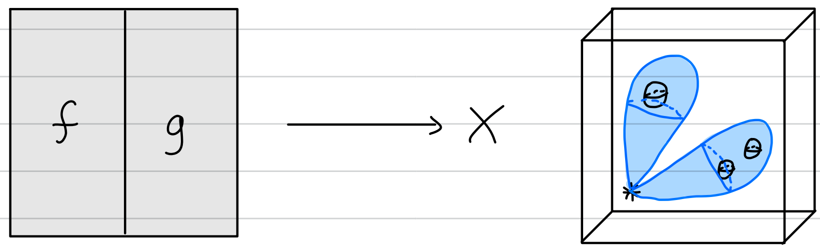
More generally, the higher homotopy groups $\pi_i(S^n)$ for $i>n$ are complicated, whereas $H_i(S^n)=0$ for $i>n$.

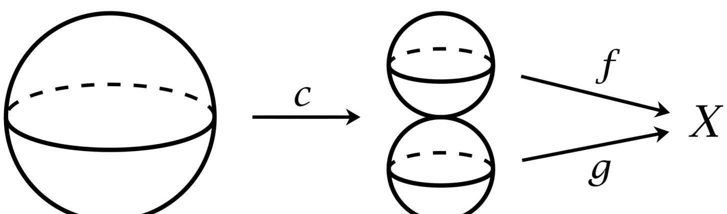
$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Def Let X be a space and $x_0 \in X$. For $n \geq 1$, the <u>n-th homotopy group $\pi_n(X)$</u> is the homotopy classes of maps $(I^n, \partial I^n) \longrightarrow (X, x_0)$, with group operation given by

$$(f+g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \tfrac{1}{2}] \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\tfrac{1}{2}, 1]. \end{cases}$$





Equivalently, $\pi_n(X)$ is the homotopy classes of maps $(S^n, s_0) \longrightarrow (X, x_0)$, with addition the composition

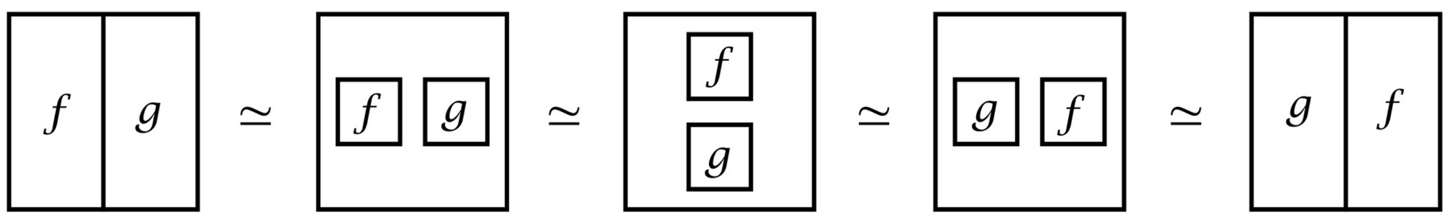
$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$$

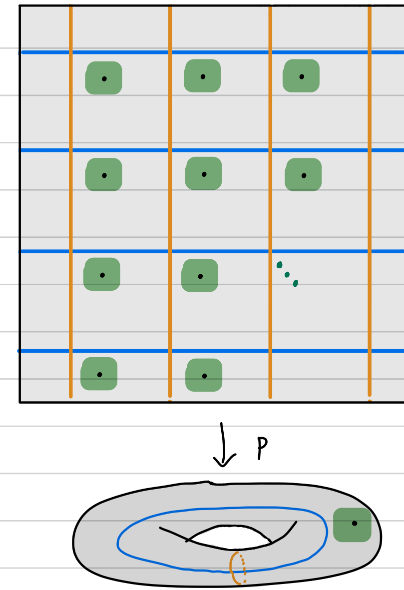
where c collapses $S^{n-1} \subseteq S^n$ to a point.

<u>Def</u> Let X be a space and $x_0 \in X$. For $n \geq 1$, the <u>n-th homotopy group $\pi_n(X)$</u> is the homotopy classes of maps $(I^n, \partial I^n) \longrightarrow (X, x_0)$, with group operation given by

$$(f+g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{for } s_1 \in [0, 1/2] \\ g(2s_1-1, s_2, \ldots, s_n) & \text{for } s_1 \in [1/2, 1]. \end{cases}$$

We use additive notation since $\pi_n(X)$ is abelian for $n \geq 2$:





Proposition 4.1 A covering space projection $p : \tilde{X} \to X$ induces isomorphisms $p_* : \pi_n(\tilde{X}) \longrightarrow \pi_n(X)$ for all n.

Ex Since $(S^1)^n$ has a universal cover $\mathbb{R}^n$ that is not only simply-connected but also contractible, we have $\pi_i((S^1)^n) = 0 \;\; \forall \;\; n \geq 2$.

(Spaces X with $\pi_i(X) = 0 \;\; \forall n \geq 2$ are called asphérical.)

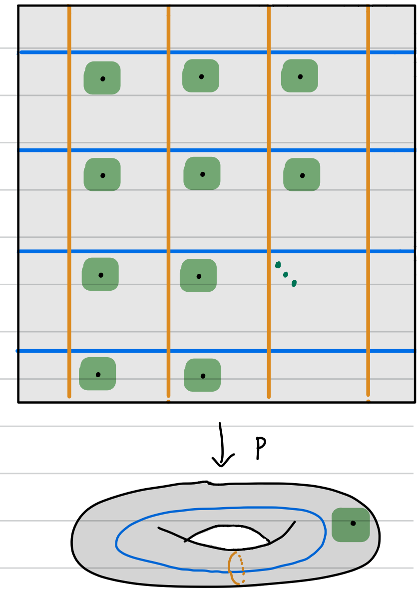
By contrast, $H_i((S^1)^n) \cong \mathbb{Z}^{\binom{n}{i}}$ is nontrivial $\forall i \leq n$.
Tori are generators of higher homology, not of higher homotopy groups.

Pf 1 Surjectivity of p_* is by the lifting criterion (Prop 1.33): S^n simply-connected for $n \geq 2$ gives $f_*(\pi_1(S^n)) \subseteq p_*(\pi_1(\tilde{X}))$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, dashed] \arrow[r, "f"'] & X \end{tikzcd}$$

Injectivity of p_* is by the homotopy lifting property (Prop 1.30).

<u>Proposition 4.1</u> A covering space projection $p: \tilde{X} \to X$ induces isomorphisms $p_*: \pi_n(\tilde{X}) \longrightarrow \pi_n(X)$ for all n.



<u>Ex</u> Since $(S^1)^n$ has a universal cover $\mathbb{R}^n$ that is not only simply-connected but also contractible, we have $\pi_i((S^1)^n) = 0 \ \forall\ n \geq 2$.

(Spaces X with $\pi_i(X) = 0 \ \forall n \geq 2$ are called <u>aspherical</u>.)

By contrast, $H_i((S^1)^n) \cong \mathbb{Z}^{\binom{n}{i}}$ is nontrivial $\forall i \leq n$.
Tori are generators of higher homology, not of higher homotopy groups.

<u>Pf 2</u> A covering space is a fibration

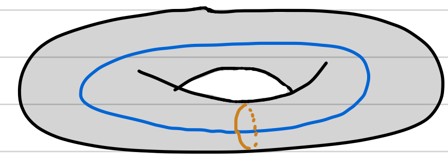
$$\begin{array}{ccc} F & \longrightarrow & \tilde{X} \\ & & \downarrow \\ & & X \end{array}$$

where F has the discrete topology, giving a long exact sequence

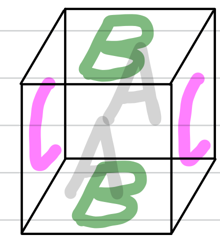
$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(\tilde{X}) \xrightarrow[p_*]{\cong} \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

with $\pi_n(F)$ trivial for $n \geq 1$ and hence p_* an isomorphism for $n \geq 2$.

Prop 4.2 For a product $\prod_\alpha X_\alpha$ of path-connected spaces X_α there are isomorphisms $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ $\forall n$.



Pf A map $f: Y \to \prod_\alpha X_\alpha$ is the same as a collection of maps $\{f_\alpha : Y \to X_\alpha\}$. Taking $Y = S^n$ and $Y = S^n \times I$ gives the result.

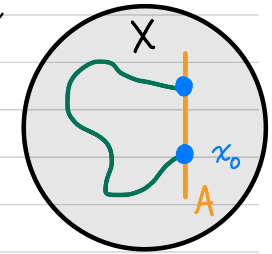


Ex $\pi_1((S^1)^n) \cong (\pi_1(S^1))^n \cong \mathbb{Z}^n$ and

$\pi_i((S^1)^n) \cong (\pi_i(S^1))^n \cong 0$ $\forall n \geq 2$.

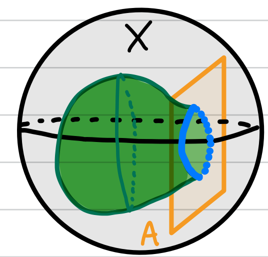
Thm 4.3 For (X,A) a pair of spaces with basepoint $x_0 \in A \subseteq X$, we have a long exact sequence

$$\cdots \to \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots$$

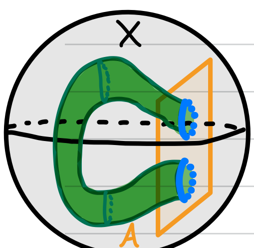


Element of set $\pi_1(X, A, x_0)$.

The relative homotopy groups $\pi_n(X, A, x_0)$ can be defined as homotopy classes of maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$



Element of group $\pi_2(X, A, x_0)$.



An element of abelian group $H_2(X,A)$ that is not an element of $\pi_2(X,A)$.

Thm 4.3 For (X,A) a pair of spaces with basepoint $x_0 \in A \subseteq X$, we have a long exact sequence

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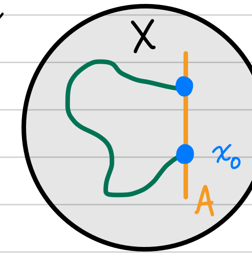

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The relative homotopy groups $\pi_n(X,A,x_0)$ can be defined as homotopy classes of maps
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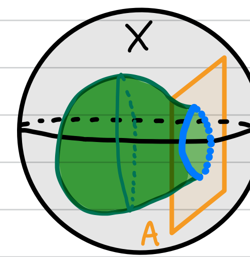
with addition defined via the map $D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subseteq D^n$ to a point.



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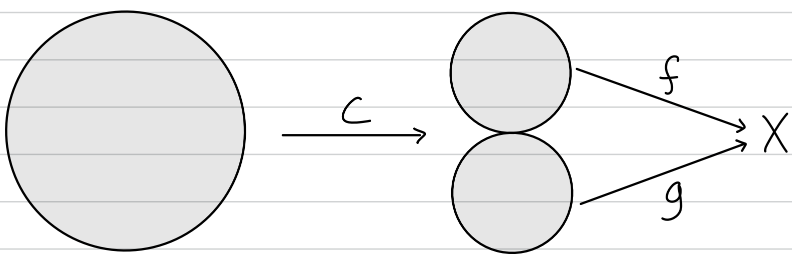

$$(D^n, S^n) \xrightarrow{c} (D^n \vee D^n, S^{n-1} \vee S^{n-1}) \xrightarrow{f \vee g} (X, A)$$

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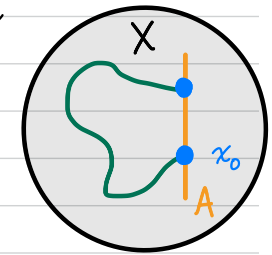

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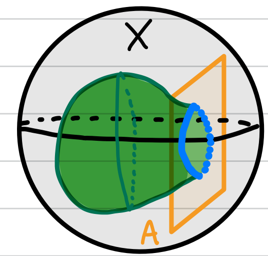
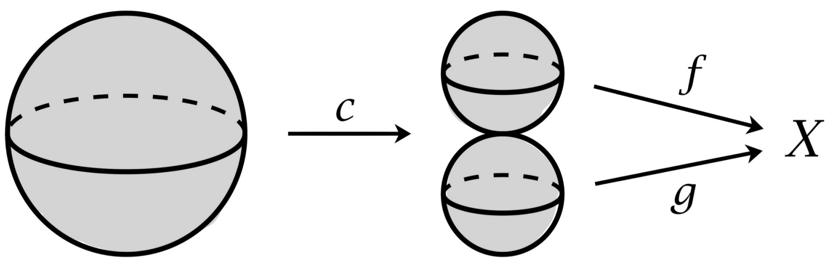


Element of group $\pi_2(X, A, x_0)$.



$$(D^n, S^n) \xrightarrow{c} (D^n \vee D^n, S^{n-1} \vee S^{n-1}) \xrightarrow{f \vee g} (X, A)$$

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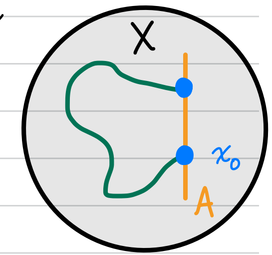
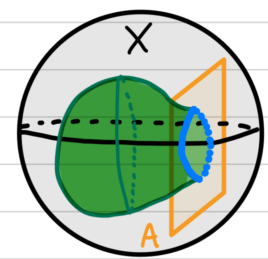
$$\cdots \to \pi_n(A,x_0) \longrightarrow \pi_n(X,x_0) \longrightarrow \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots$$



Element of set $\pi_1(X,A,x_0)$.

The relative homotopy groups $\pi_n(X,A,x_0)$ can be defined as homotopy classes of maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$.

The connecting homomorphism or boundary map ∂ is defined by restricting maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ to S^{n-1}.

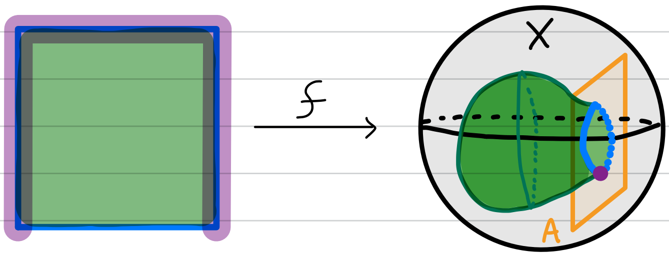


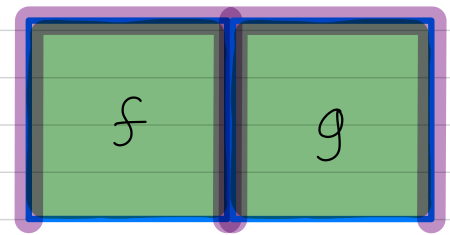
Element of group $\pi_2(X,A,x_0)$.

Remark $\pi_n(X,A,x_0)$ is a
- set for $n=1$,
- group for $n=2$,
- abelian group for $n \geq 3$.

This can best be seen in the definition using cubes.

Alternatively, the relative homotopy groups $\pi_n(X, A, x_0)$ can be defined as homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$

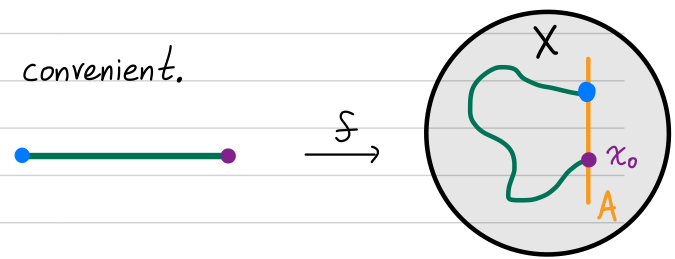




where $J^{n-1} = \partial I^n \setminus \{\text{face with last coordinate zero}\}$

<u>Remark</u> This again makes the definition of ∂ convenient.

<u>Remark</u> $\pi_n(X, A, x_0)$ is a



- set for $n=1$,
- group for $n=2$,
- abelian group for $n \geq 3$.

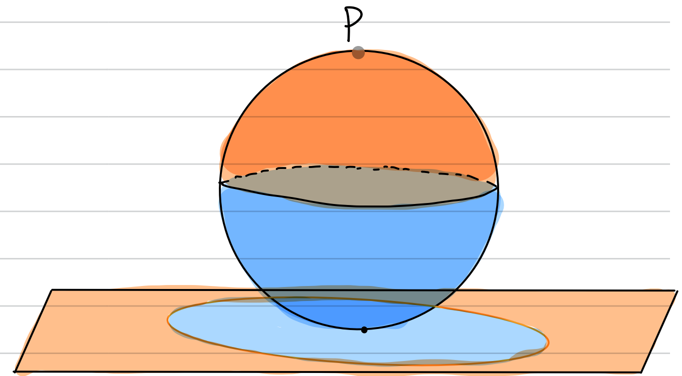
This can best be seen in the definition using cubes.

Whitehead's theorem, Thm 4.5 If a map $f: X \to Y$ between CW complexes induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ $\forall n$, then f is a homotopy equivalence.

Since CW complexes are built using attaching maps whose domains are spheres, perhaps it is not surprising homotopy groups say much about them.

Careful: There must be a map f inducing the isomorphisms.
$S^2 \times \mathbb{R}P^3$ and $S^3 \times \mathbb{R}P^2$ have isomorphic homotopy groups but are not homotopy equivalent.

How do we see $\pi_i(S^n) = 0$ for $i < n$?
An appealing proof would homotope $f: S^i \to S^n$ to miss some $p \in S^n$, since $S^n \setminus \{p\} \simeq *$.



(One can find surjective such maps $f: S^i \to S^n$ using space-filling curves.)

This follows from cellular approximation.

Cellular Approximation Thm 4.8 Every map $f: X \to Y$ between CW complexes is homotopic to a cellular map, meaning $f(X^n) \subseteq Y^n$ $\forall n$.

Aside
For comparison, recall...

Thm 2C.1 (Simplicial Approximation)
If K, L are simplicial complexes with K finite, then any map $f: K \to L$ is homotopic to a simplicial map $g: sd^m(K) \to L$ for some m.

iterated barycentric subdivision

Section 4.2 Elementary methods of calculation

Def A space X is n-connected if $\pi_i(X)=0$ $\forall i \le n$.

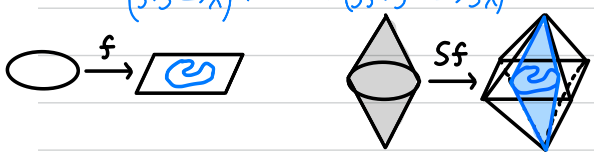
- 0-connected means path-connected.
- 1-connected means simply-connected.

Freudenthal suspension theorem, Corollary 4.24

If X is an $(n-1)$-connected CW complex, then the suspension map

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \quad \text{is an isomorphism } \forall i < 2n-1.$$

$$(f: S^i \to X) \longmapsto (Sf: S^{i+1} \to SX)$$



Pf sketch

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \xrightarrow{\cong} \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$$

A version of excision holds for X sufficiently connected

Corollary 4.25 $\pi_n(S^n) \cong \mathbb{Z}$ $\forall n \ge 1$.

Pf Consider $\mathbb{Z} \cong \underbrace{\pi_2(S^2) \longrightarrow \pi_3(S^3) \longrightarrow \pi_4(S^4) \longrightarrow \cdots}_{\text{Isomorphisms by Cor. 4.24}}$

Isomorphism by Hopf fiber bundle

$$\cdots \longrightarrow \underset{0}{\underset{\shortparallel}{\pi_2(S^3)}} \longrightarrow \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \longrightarrow \underset{0}{\underset{\shortparallel}{\pi_1(S^3)}} \longrightarrow \cdots$$

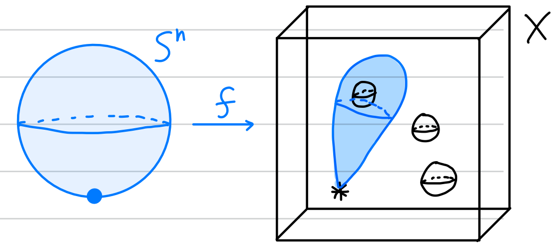
Hurewicz theorem, Thm 4.32

If a space X is $(n-1)$-connected for $n \geq 2$, then $\tilde{H}_i(X)=0$ for $i<n$ and $\pi_n(X) \cong H_n(X)$.

The first nonzero homotopy and homology groups of a simply-connected space are isomorphic.

Recall $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$.

The Hurewicz map $h: \pi_n(X) \longrightarrow H_n(X)$ is defined by: $[f: S^n \to X] \longmapsto f_*(\alpha)$ for α a fixed generator of $H_n(S^n) \cong \mathbb{Z}$.



Since homology groups are typically more computable, the following version of Whitehead's is often easier to apply.

Cor. 4.33 A map $f: X \to Y$ between simply-connected CW complex is a homotopy equivalence if $f_*: H_n(X) \longrightarrow H_n(Y)$ is an iso $\forall n$.

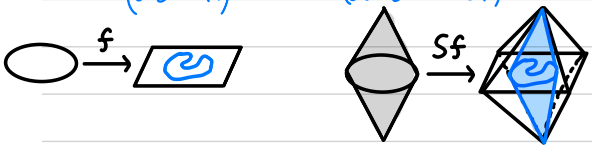
Stable homotopy groups

Freudenthal suspension theorem, Corollary 4.24

If X is an $(n-1)$-connected CW complex, then the suspension map

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \quad \text{is an isomorphism } \forall i < 2n-1.$$

$$(f: S^i \to X) \longmapsto (Sf: S^{i+1} \to SX)$$



In particular, for $n \geq 1$ we have $i < n \Rightarrow i < 2n-1$, so
X $(n-1)$-connected $\Rightarrow SX$ n-connected.

For Y a CW complex, S^nY is $(n-1)$-connected,
so in the sequence of iterated suspensions

$$\pi_i(Y) \longrightarrow \pi_{i+1}(SY) \longrightarrow \pi_{i+2}(S^2Y) \longrightarrow \dots \longrightarrow \pi_{i+n}(S^nY) \xrightarrow{\cong} \pi_{i+n+1}(S^{n+1}Y) \xrightarrow{\cong} \dots$$

all maps are eventually isomorphisms (eventually $i+n < 2n-1$).
This group is the <u>stable homotopy group</u>, denoted $\pi_i^s(Y)$.

$\pi_i(S^n)$

$n \downarrow$ \ $i \to$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

For Y a CW complex, S^nY is $(n-1)$-connected,
so in the sequence of iterated suspensions

$$\pi_i(Y) \longrightarrow \pi_{i+1}(SY) \longrightarrow \pi_{i+2}(S^2Y) \longrightarrow \dots \longrightarrow \pi_{i+n}(S^nY) \xrightarrow{\cong} \pi_{i+n+1}(S^{n+1}Y) \xrightarrow{\cong} \dots$$

all maps are eventually isomorphisms (eventually $i+n < 2n-1$).
This group is the <u>stable homotopy group</u>, denoted $\pi_i^S(Y)$.

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

The stable homotopy groups of spheres, $\pi_i^s := \pi_i^s(S^0) = \pi_{i+n}(S^n)$ for $n > i+1$ are fundamental objects, and known up to $i \approx 60$ or so:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
π_i^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_6$	$\mathbb{Z}_{504}$	0	$\mathbb{Z}_3$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_{480} \times \mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_{264} \times \mathbb{Z}_2$