Chapter 7: Complete metric spaces and function spaces

When do Cauchy sequences converge?

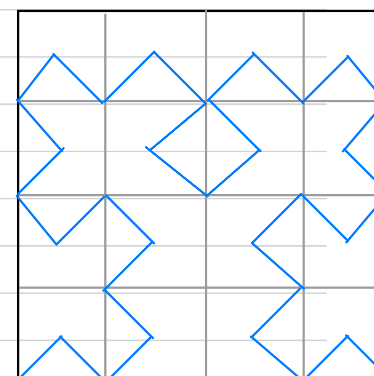
Completeness is a metric property, but related to topological properties like compactness

Question: Is this metrizable space X complete?
Answer: With respect to which metric?

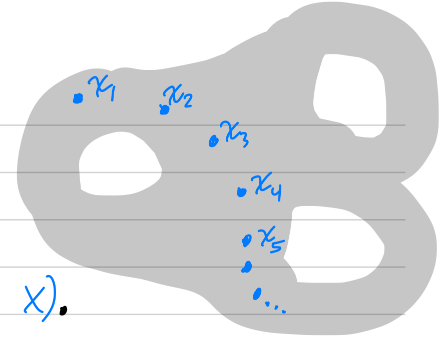
(A metric space is compact $\Longleftrightarrow$ it is complete and totally bounded.)

When does a sequence of continuous or bounded functions converge to a continuous or bounded function?

Space-filling curves



Section 43: Complete metric spaces



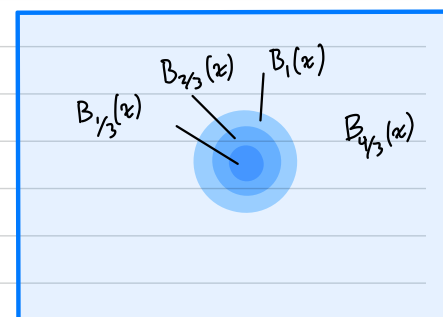
Def (X,d) a metric space. A sequence (x_n) in X is a Cauchy sequence if $\forall \varepsilon>0$, $\exists N$ s.t. $d(x_n, x_m) < \varepsilon$ whenever $n,m \geq N$.
(X,d) is complete if every Cauchy sequence converges (to a point in X).

Rmk Every convergent sequence is Cauchy.

Ex $\mathbb{Q}$ is not complete: 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, is Cauchy but does not converge to a point in $\mathbb{Q}$ (note $\pi \notin \mathbb{Q}$).

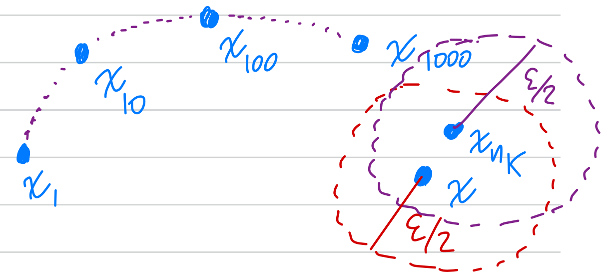
Ex We'll see that $\mathbb{R}$ is complete. But $(0,1)$ is not. Not a topological property!

Rmk A closed subset of a complete metric space is complete.

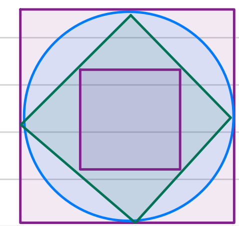


Recall (from HW #4) the standard bounded metric
$\bar{d}: X \times X \to \mathbb{R}$ defined by $\bar{d}(x,y) = \min\{d(x,y), 1\}$.
A sequence is Cauchy in $(X,\bar{d}) \iff$ it is Cauchy in (X,d),
and $(X,\bar{d})$ is complete $\iff$ (X,d) is.

Lemma If every Cauchy sequence has a convergent subsequence, then (X,d) is complete.

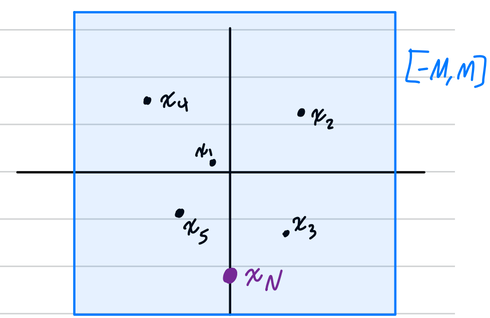


Pf Let (x_n) be Cauchy with $x_{n_k} \to x$.
Let $\varepsilon > 0$.
Choose N s.t. $d(x_n, x_m) < \varepsilon/2 \quad \forall n,m \geq N$.
Choose K with $n_K \geq N$ s.t. $d(x_{n_K}, x) < \varepsilon/2$
Then $\forall n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$



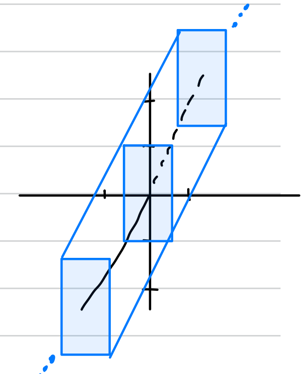
Thm $(\mathbb{R}^k, d_p)$ is complete $\forall\, 1 \leq p \leq \infty$.

Pf Let (x_n) be a Cauchy sequence.
So $\exists\, N$ s.t. $d_p(x_n, x_m) < 1 \quad \forall n,m \geq N$.



Let $M = \max\{d_p(x_1, 0), \ldots, d_p(x_{N-1}, 0), d_p(x_N, 0) + 1\}$
Sequence (x_n) contained in $\overline{B_M^{d_p}(0)} \subset [-M,M]^k$, which is compact, hence sequentially compact (we're in a metric space).
So (x_n) has a convergent subsequence, and $(\mathbb{R}^k, d_p)$ is complete.

What about $\mathbb{R}^\omega$?



$B_{\frac{1}{2}}((0,0,0,\ldots))$

<u>Thm</u> There is a metric D for $\mathbb{R}^\omega$ so that $(\mathbb{R}^\omega, D)$ is complete, namely $D(x,y) = \sup_i \left\{ \frac{\bar{d}(\pi_i(x), \pi_i(y))}{i} \right\}$ where $\bar{d}(a,b) = \min\{|a-b|, 1\}$.

<u>Recall</u> (Section 20, HW#4) D metrizes the product topology.

<u>Recall</u> (Exam 2, HW#4)

Let (x_n) be a sequence in $\mathbb{R}^\omega$.

Then $(x_n) \to x \iff \pi_i(x_n) \to \pi_i(x) \quad \forall i$.

<u>Pf of Thm</u>

Let (x_n) be a Cauchy sequence in $(\mathbb{R}^\omega, D)$.

Since $\bar{d}(\pi_i(x_n), \pi_i(x_m)) \leq i \cdot D(x_n, x_m)$,

we see $(\pi_i(x_n))$ is a Cauchy sequence in $\mathbb{R}$ $\forall i$.

Hence $\forall i$, $\pi_i(x_n) \to a_i$ for some $a_i \in \mathbb{R}$.

Hence $(x_n) \longrightarrow a \in \mathbb{R}^\omega$, defined as $a = (a_1, a_2, a_3, \ldots)$.

Trick question: Is $\mathbb{R}^J$ complete in general?
No, not even metrizable for J uncountable.

Recall from Section 21, for J uncountable, we found $A \subset \mathbb{R}^J$ and $x \in \mathbb{R}^J$ with $y \in \overline{A}$, but no sequence in A converges to y.

$A = \left\{ (x_\alpha) \,\middle|\, \begin{array}{l} x_\alpha = 0 \text{ for finitely many } \alpha \\ x_\alpha = 1 \text{ otherwise} \end{array} \right\}$ $\quad y = (y_\alpha)$ with $y_\alpha = 0 \ \forall \alpha$

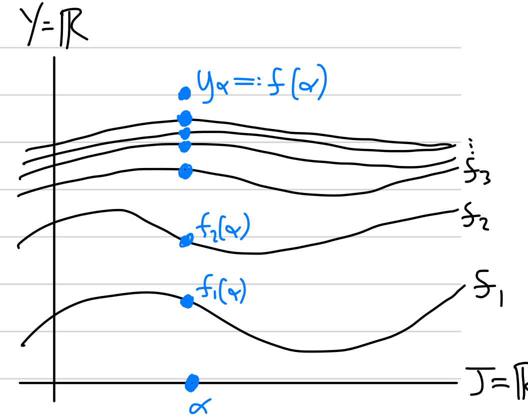
Let (Y,d) be a metric space and let J be a set.
Then Y^J can be equipped with the uniform metric
$\bar{\rho}(x,y) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha)$ (which may not metrize the product topology).

Notation: Instead of tuple notation $x \in Y^J$ with $x = (x_\alpha)_{\alpha \in J}$,
let's use functional notation $f \in Y^J$ with $f : J \longrightarrow Y$, $\alpha \longmapsto f(\alpha) = x_\alpha$

So $\bar{\rho}(f,g) = \sup_{\alpha \in J} \bar{d}(f(\alpha), g(\alpha))$.

Recall the standard bounded metric $\bar{d} : Y \times Y \to \mathbb{R}$ is defined by
$\bar{d}(y,y') = \min\{d(y,y'), 1\}$ $\forall y,y' \in Y$.
We use this metric when defining $\bar{\rho}$ so that necessarily $\bar{\rho}(f,g) < \infty$.

Thm (Y,d) complete $\Rightarrow$ $(Y^J, \bar{\rho})$ complete.



Pf Let (f_n) be a Cauchy sequence in $(Y^J, \bar{\rho})$.
$\forall \alpha \in J$, $\bar{d}(f_n(\alpha), f_m(\alpha)) \leq \bar{\rho}(f_n, g_n)$
So $(f_n(\alpha))$ is a Cauchy sequence in Y.
Y complete $\Rightarrow$ $\exists\, y_\alpha \in Y$ with $f_n(\alpha) \to y_\alpha$ $\forall \alpha$.
Define $f: J \to Y$ by $f(\alpha) = y_\alpha$.

We claim $f_n \to f$. Let $\varepsilon > 0$.
Choose N large enough so that $\bar{\rho}(f_n, f_m) < \varepsilon/2$ when $n,m \geq N$.
So $\bar{d}(f_n(\alpha), f_m(\alpha)) < \varepsilon/2$ for $n,m \geq N$ and $\alpha \in J$.
Fix α and $n \geq N$ momentarily, and note $f_m(\alpha) \to f(\alpha)$ implies

$$\bar{d}(f_n(\alpha), f(\alpha)) \leq \varepsilon/2 \text{ for } n \geq N.$$

Since this holds for all α and $n \geq N$, we get
$\bar{\rho}(f_n, f) \leq \varepsilon/2 < \varepsilon$ for $n \geq N$, as desired.

Thm X topological space, (Y,d) metric space.
The sets $\mathcal{C}(X,Y)$ of all continuous functions $X \to Y$
and $\mathcal{B}(X,Y)$ of all bounded functions $X \to Y$
are closed in $(Y^X, \bar{\rho})$, and hence complete if Y is complete.

Remark For bounded functions, we can replace the uniform metric $\bar{\rho}$ with the sup metric $\rho(f,g) = \sup_{\alpha \in J} d(f(\alpha), g(\alpha))$.

Thm A metric space (X,d) has an isometric embedding into a complete metric space.

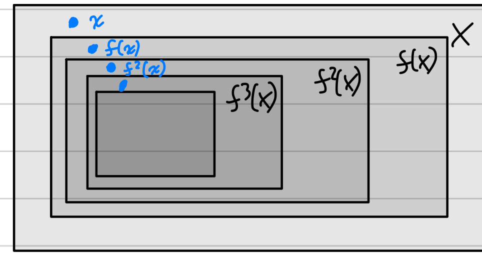
Pf sketch Let $x_0 \in X$. Define $X \longrightarrow \mathcal{B}(X,\mathbb{R})$ by
$$a \longmapsto \phi_a(x) = d(x,a) - d(x_0,a).$$
Check this is an isometric embedding.

Contraction mapping theorem / Banach fixed point theorem §43 #5

Let (X,d) be complete.

Let $f: X \to X$ be a contraction, meaning $\exists\ c<1$ with $d(f(x),f(y)) \leq c\, d(x,y)\ \ \forall x,y \in X$.

Prove $\exists!$ fixed point $x \in X$ satisfying $f(x)=x$.



Remark §28 #7 was instead with X compact.

Pf Let $x \in X$.

Note $d(f^n(x), f^{n+1}(x)) \leq c \cdot d(f^{n-1}(x), f^n(x))$

$$\vdots$$

$$\leq c^n\, d(x, f(x))$$

Hence $(f^n(x))$ is a Cauchy sequence, since for $n<m$,

$$d(f^n(x), f^m(x)) \leq c^n d(x,f(x)) + \ldots + c^{m-1} d(x,f(x))$$

$$= c^n d(x,f(x))(1+c+\ldots+c^{m-n-1})$$

$$< c^n d(x,f(x))(1+c+c^2+\ldots)$$

$$= \underbrace{c^n}_{\substack{\to 0 \\ \text{as } n\to\infty}}\ \underbrace{d(x,f(x))\frac{1}{1-c}}_{\text{constant}}$$

Let z be the limit of this Cauchy sequence $(f^n(x))$.

Hence $f(z) = f(\lim_{n\to\infty} f^n(x)) \underset{\uparrow}{=} \lim_{n\to\infty} f^{n+1}(x) = z$, as desired.

since f continuous

And z must be the only fixed point, since if $z' \in X$ also satisfied $f(z') = z'$, then we'd have

$$d(z,z') = d(f(z), f(z')) \leq c\, d(z,z')$$
$$\implies d(z,z') = 0$$
$$\implies z = z'.$$

Section 44: The Peano space-filling curve

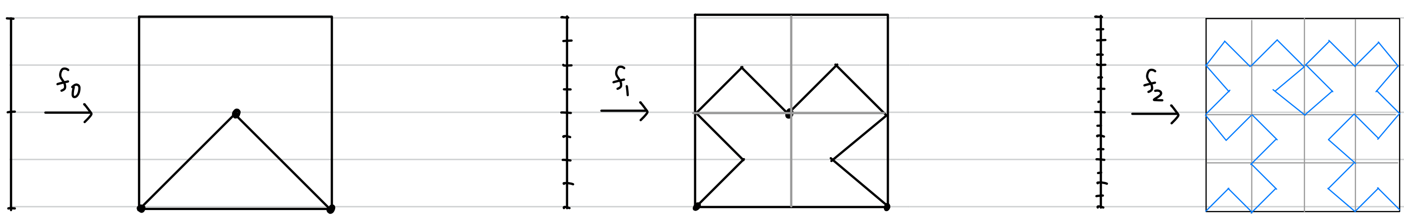
Let $I = [0,1]$.

Thm There exists a continuous surjective function $f : I \to I^2$

Pf (I^2, d_∞) is complete as a closed subset of the complete space $(\mathbb{R}^2, d_\infty)$, where $d_\infty((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$.

Since I^2 is bounded (or alternatively, since I is compact), the space of continuous functions $\mathcal{C}(I, I^2)$ is complete under the sup metric $\rho(f,g) = \sup_{t \in I} d_\infty(f(t), g(t))$.

Construct a sequence of piecewise linear curves



Function f_n consists of 4^n "triangular paths" each in a square of side length $1/2^n$.

Obtain f_{n+1} by applying the construction $f_0 \rightsquigarrow f_1$ to each such square.

To see that (f_n) is a Cauchy sequence, note $f_n(t)$ and $f_{n+1}(t)$ are in the same square of side length $1/2^n$, for all t.
Hence $\rho(f_n, f_{n+1}) = \sup_{t \in I} d_\infty(f_n(t), f_{n+1}(t)) \leq 1/2^n$.

So $\forall n<m$, $\rho(f_n, f_m) \leq \rho(f_n, f_{n+1}) + \rho(f_{n+1}, f_{n+2}) + \dots + \rho(f_{m-1}, f_m)$

$$\leq 1/2^n + 1/2^{n+1} + \dots + 1/2^{m-1}$$

$$\leq 1/2^{n-1}$$

Therefore $f_n \longrightarrow f \in \mathcal{C}(I, I^2)$.

To see that $f : I \to I^2$ is surjective, let $x \in I^2$. $\forall \varepsilon > 0$ $\exists N$ s.t. $1/2^N < \varepsilon$.
Divide I^2 into squares of side length $1/2^N$. Note $f_N(I)$ and therefore $f(I)$ intersect each such square, so $B(x, \varepsilon)$ intersects $f(I)$. Hence $x \in \overline{f(I)}$.
I compact, f continuous $\Rightarrow$ $f(I)$ compact $\Rightarrow$ $f(I)$ closed. So $x \in f(I)$.