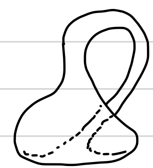
Chapter 8: Baire Spaces and Dimension Theory

For X compact, the space $\mathcal{C}(X, \mathbb{R}^n)$ of continuous functions is complete in the sup (or uniform) metric, and hence is a Baire space.

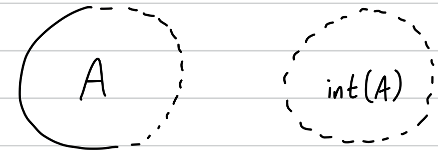
Embedding theorem Every compact metrizable space X of topological dimension m can be embedded in $\mathbb{R}^{2m+1}$.



(Baire spaces can be used to prove the existence of a continuous nowhere-differential real-valued function, though we won't cover this.)

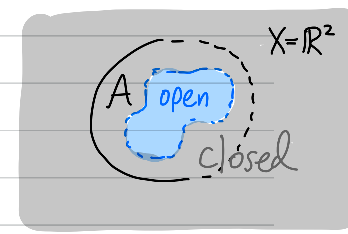
Section 48: Baire Spaces

Recall for X a topological space and $A \subset X$, the interior of A is the union of all open sets contained in A.



A has empty interior in X if the only open set contained in A is the empty set.

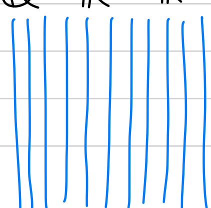
(Equivalently, if the only closed set containing $X-A$ is all of X, i.e. the complement of A is dense in X: $\overline{X-A} = X$.)



Ex $\mathbb{Q} \subset \mathbb{R}$

$[0,1] \times \{0\} \subset \mathbb{R}^2$

$\mathbb{Q} \times \mathbb{R} \subset \mathbb{R}^2$



Non-Ex $\mathbb{Q} \subset \mathbb{Q}$

$[0,1] \subset \mathbb{R}$

Def A space X is a **Baire space** if given any countable collection $\{A_n\}$ of closed sets with empty interiors in X, their union $\bigcup_n A_n$ also has empty interior in X.

Non-Ex

$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$

Ex $\mathbb{Z}_+$, since only the empty set has empty interior.

Ex Any closed subset of $\mathbb{R}^n$ (complete metric space).

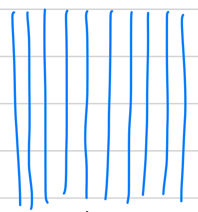
Ex Any open subset of a Baire space (Lemma 48.4)

Ex $\mathbb{R} \setminus \mathbb{Q}$

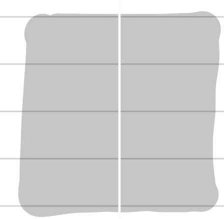
Pic $X = \mathbb{R}^2$

$\{A_q\}_{q \in \mathbb{Q}}$

$A_q = \{q\} \times \mathbb{R}$



$X - A_q$



We frequently use the following reformulation with dense open sets:

Lemma 48.1 X is a Baire space $\iff$ given any countable collection $\{U_n\}$ of dense open sets in X, their intersection $\bigcap_n U_n$ is also dense in X.

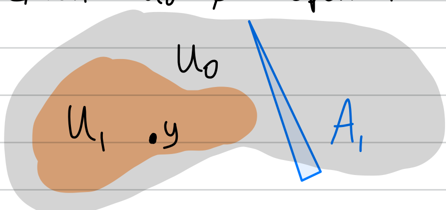
Pf Follows since (1) A closed in $X \iff X - A$ open in X

(2) A has empty interior in $X \iff X - A$ dense in X.

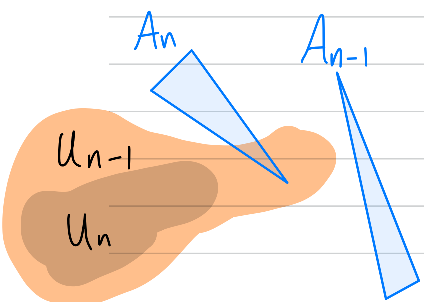
Baire category theorem If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Pf Note X compact Hausdorff or metrizable $\Rightarrow$ X normal $\Rightarrow$ X regular.

Given a countable collection $\{A_n\}$ of closed sets with empty interiors, we must show $\bigcup_n A_n$ also has empty interior in X.
Given $U_0 \neq \emptyset$ open in X, we must find $x \in U_0$ with $x \notin \bigcup_n A_n$, i.e. $x \notin A_n$ $\forall n$.



A_1 empty interior $\Rightarrow$ $U_0 \not\subset A_1$.
Let $y \in U_0 - A_1$, which is open.
X regular $\Rightarrow$ $\exists$ open U_1 with $y \in U_1 \subset \overline{U_1} \subset U_0 - A_1$



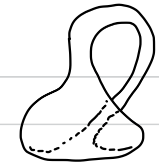
Inductively, have nonempty open U_{n-1} with $\overline{U_{n-1}} \subset U_{n-2} - A_{n-1}$.
A_n empty interior $\Rightarrow$ $U_{n-1} \not\subset A_n$.
X regular $\Rightarrow$ $\exists$ open U_n with $\emptyset \neq U_n \subset \overline{U_n} \subset U_{n-1} - A_n$.

We'll show $\bigcap_n \overline{U_n} \neq \emptyset$. This will complete the proof since then $x \in \overline{U_1} \subset U_0$ and $x \notin A_n$ $\forall n$ since $\overline{U_n}$ is disjoint from A_n.

Case X compact Hausdorff. The closed sets $\overline{U_1} \supset \overline{U_2} \supset \overline{U_3} \supset \ldots$ have the finite intersection property, and hence a nonempty intersection since X compact.

Case X a complete metric space. Furthermore choose U_n with $\operatorname{diam}(\overline{U_n}) < 1/n$. Apply Lemma 48.3 (or §43 Exercise #4) which says that if $C_1 \supset C_2 \supset C_3 \supset \ldots$ is a nested sequence of closed sets with $\operatorname{diam}(C_n) \to 0$, then $\bigcap_n C_n \neq \emptyset$. $\square$

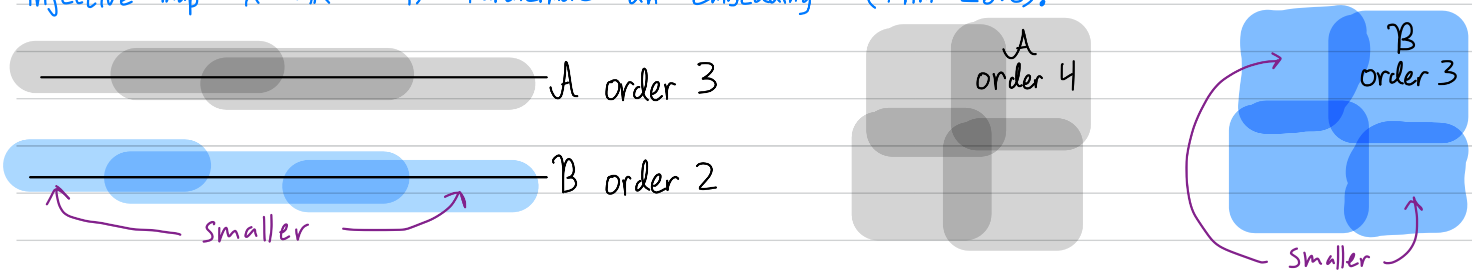
<u>Remark</u> The Baire category theorem implies that $[0,1]$ is uncountable. Indeed, as a complete metric space, it is a Baire space.
Each singleton $\{x\}$ is closed with empty interior, and $[0,1]$ has nonempty interior.
So if the union $\bigcup_{x \in [0,1]} \{x\} = [0,1]$ were countable, this would contradict the Baire category theorem.



Section 50: Introduction to dimension theory

Thm Every compact metric space X of topological dimension m can be embedded in $\mathbb{R}^{2m+1}$.

Recall "Embedded" means a continuous injective map $X \hookrightarrow \mathbb{R}^{2m+1}$ that is a homeomorphism onto its image (pg 105). Since X is compact and $\mathbb{R}^{2m+1}$ is Hausdorff, any continuous injective map $X \hookrightarrow \mathbb{R}^{2m+1}$ is furthermore an embedding (Thm 26.6).



Def The <u>topological dimension</u> of a space X is the smallest m such that for every open cover $\mathcal{A}$ of X, $\exists$ an open cover $\mathcal{B}$ that refines it with order at most $m+1$.

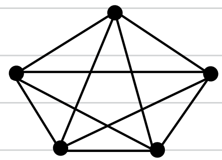
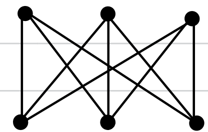
$\forall B \in \mathcal{B} \ \exists A \in \mathcal{A}$ with $B \subset A$

No more than $m+1$ sets intersect at a point.

The topological dimension of

- $\mathbb{R}^m$ is m.
- an m-dimensional manifold is m.
- a finite graph is 1.

A second-countable Hausdorff space s.t. each point has a neighborhood homeomorphic to $\mathbb{R}^m$

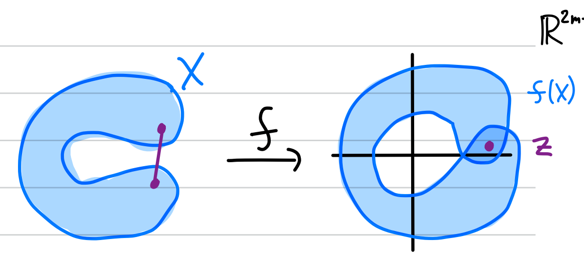
The graphs K_5  and $K_{3,3}$  can be embedded in $\mathbb{R}^3 = \mathbb{R}^{2m+1}$ but not $\mathbb{R}^2$, showing the theorem cannot be improved in general.

Whitney's theorem says any (smooth) m-manifold can furthermore be embedded in $\mathbb{R}^{2m}$.

Thm Every compact metric space X of topological dimension m can be embedded in $\mathbb{R}^{2m+1}$.

Pf Denote the square metric on $\mathbb{R}^{2m+1}$ by $|v-w| = \max\{|v_i - w_i| : 1 \le i \le 2m+1\}$.
(Other common notations are $\|v-w\|_\infty$ or $d_\infty(v,w)$.)

Since $\mathbb{R}^{2m+1}$ is complete, $\mathcal{C}(X, \mathbb{R}^{2m+1})$ is complete with the sup metric $\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|$, and hence a Baire space.

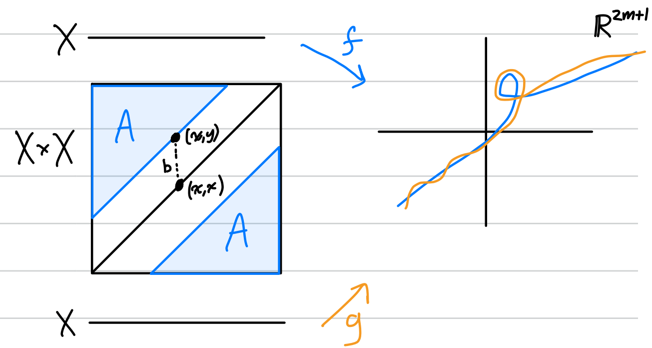


Given $f \in \mathcal{C}(X, \mathbb{R}^{2m+1})$, define $\Delta(f) = \sup_{z \in f(X)} \operatorname{diam} f^{-1}(z)$.

If $\Delta(f) = 0$, then each set $f^{-1}(z)$ is a singleton, so f is injective.

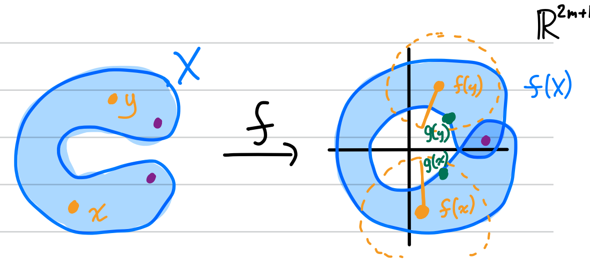
For $\varepsilon > 0$, let U_ε be the set of all $f \in \mathcal{C}(X, \mathbb{R}^{2m+1})$ with $\Delta(f) < \varepsilon$.
We will show U_ε is (1) open and (2) dense in the Baire space $\mathcal{C}(X, \mathbb{R}^{2m+1})$.
Hence the countable intersection $\bigcap_{n \in \mathbb{Z}_+} U_{1/n}$ is dense, thus nonempty.
And $f \in \bigcap_n U_{1/n}$ satisfies $\Delta(f) = 0$, so is injective.

(1) Why is U_ε open in $\mathcal{C}(\overset{(X,d)}{X}, \mathbb{R}^{2m+1})$?



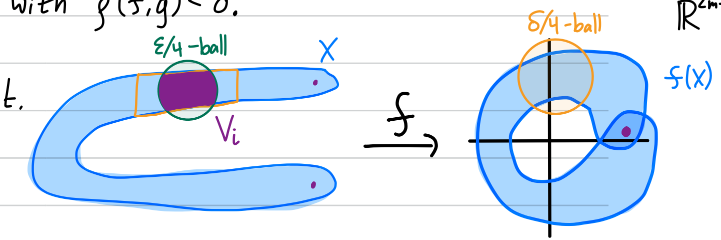
Given $f \in U_\varepsilon$, we'll find $\delta > 0$ s.t. $\rho(f,g) < \delta \Rightarrow g \in U_\varepsilon$.
Fix b with $\Delta(f) < b < \varepsilon$.
Note $f(x) = f(y) = z \Longrightarrow x, y \in f^{-1}(z) \Longrightarrow d(x,y) < b$.
So $|f(x) - f(y)|$ is positive on $A := \{(x,y) \in X \times X \mid d(x,y) \geq b\}$.
A closed in the compact space $X \times X \Longrightarrow A$ compact.
So this positive function attains its minimum;
let $\delta = \frac{1}{2} \min\limits_{(x,y) \in A} \{|f(x) - f(y)|\}$.

Now, suppose $\rho(f,g) < \delta$.
If $(x,y) \in A$, then $|f(x) - f(y)| \geq 2\delta$ (by def^{n}), giving $|g(x) - g(y)| > 0$.
Since $|g(x) - g(y)|$ is positive on A, if $g(x) = g(y)$, then necessarily $d(x,y) < b$.
Hence $\Delta g \leq b < \varepsilon$, so $g \in U_\varepsilon$.



(2) Why is U_ε dense in $\mathcal{C}(X, \mathbb{R}^{2m+1})$?

Let $f \in \mathcal{C}(X, \mathbb{R}^{2m+1})$. Given $\delta > 0$, we must find $g \in U_\varepsilon$ with $\rho(f,g) < \delta$.



Cover X by finitely many open sets $\{V_1, \dots, V_n\}$ s.t.

(1) $\operatorname{diam} V_i < \varepsilon/2$ in X

(2) $\operatorname{diam} f(V_i) < \delta/2$ in $\mathbb{R}^{2m+1}$

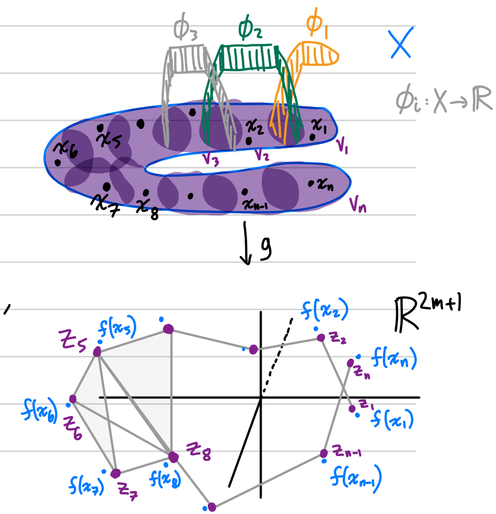
(3) $\{V_1, \dots, V_n\}$ has order $\leq m+1$

This is possible since f is continuous, X has topological dimension m, and X is compact.

Let $\{\phi_i\}$ be a partition of unity dominated by $\{V_1, \dots, V_n\}$.

These are continuous functions $\phi_i : X \to [0,1]$ with

- $\operatorname{supp}(\phi_i) \subset V_i \quad \forall i$, and
- $\sum_{i=1}^n \phi_i(x) = 1 \quad \forall x \in X$.



For each i, pick $x_i \in V_i$.

Though $\{f(x_1), \dots, f(x_n)\}$ may not be in general position in $\mathbb{R}^{2m+1}$,

we can pick $z_i \in \mathbb{R}^{2m+1}$ with $|z_i - f(x_i)| < \delta/2 \quad \forall i$

such that $\{z_1, \dots, z_n\}$ is in general position.

Define $g : X \to \mathbb{R}^{2m+1}$ by $g(x) = \sum_{i=1}^n \phi_i z_i$.

To see $\rho(f,g)<\delta$, note

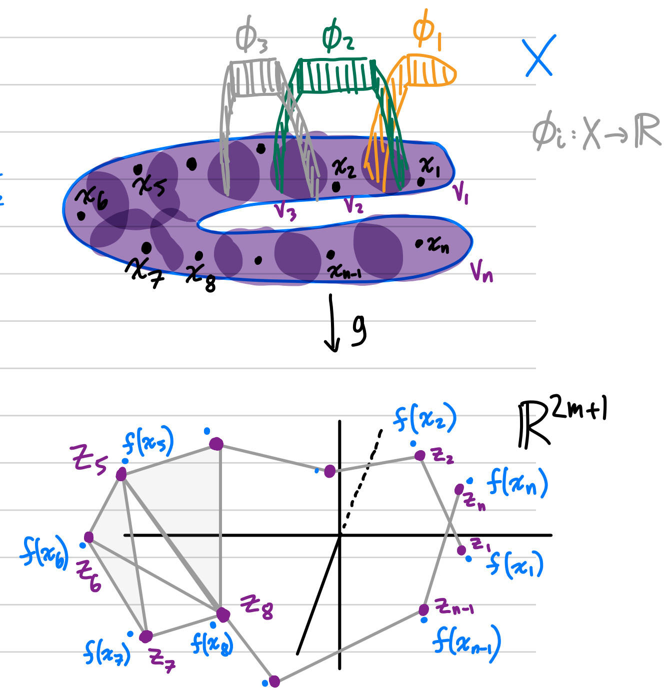
$$g(x)-f(x) = \sum_{i=1}^{n}\phi_i(x)z_i - \sum_{i=1}^{n}\phi_i(x)f(x)$$

$$= \sum_{i=1}^{n}\phi_i(x)\underbrace{\left(z_i - f(x_i)\right)}_{<\,\delta/2} + \sum_{i=1}^{n}\phi_i(x)\underline{\left(f(x_i)-f(x)\right)}$$

by choice of z_i

if $\phi_i(x)\neq 0$, then $x\in V_i$, hence term is at most $\operatorname{diam} f(V_i) < \delta/2$

$$< \delta.$$

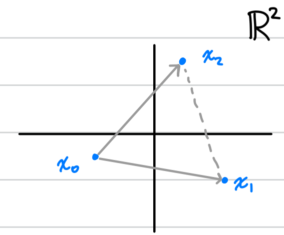


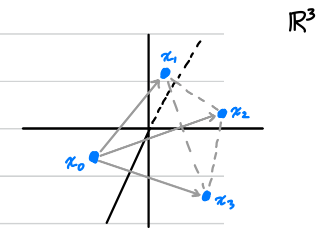
To show $g\in U_\varepsilon$, we will prove if $x,y\in X$ with $g(x)=g(y)$, then $\exists i$ with $x,y\in V_i$, so $d(x,y)\le \operatorname{diam} V_i < \varepsilon/2$.
Hence $\Delta(g)\le \varepsilon/2<\varepsilon$, as desired.

Indeed, if $g(x)=g(y)$, then $\sum_{i=1}^{n}\left(\phi_i(x)-\phi_i(y)\right)z_i = \vec{0}$.
At most $m+1$ of the $\phi_i(x)$ (resp. $\phi_i(y)$) terms are nonzero, since $\{V_i\}$ has order $\le m+1$.
So at most $2m+2$ terms $(\phi_i(x)-\phi_i(y))$ in the sum are nonzero.
And the coefficients sum to zero.
Since the z_i points are in general position in $\mathbb{R}^{2m+1}$, $(\phi_i(x)-\phi_i(y))=0$ $\forall i$, so $\phi_i(x)=\phi_i(y)$ $\forall i$.
And $\phi_i(x)>0$ for some i, meaning $x\in V_i$ and $y\in V_i$ also. $\square$

General position in $\mathbb{R}^N$

In part (2) of the embedding theorem proof, we used properties of general position in $\mathbb{R}^N$ (where $N=2m+1$). Which properties?

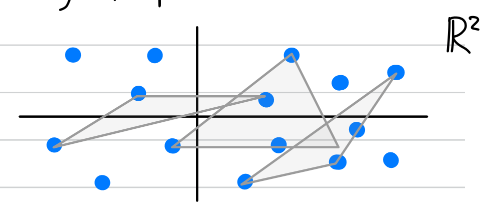




<u>Def</u> A set $\{x_0, \dots, x_k\} \subset \mathbb{R}^N$ is <u>affinely independent</u> if $\underbrace{\sum_{i=0}^k a_i x_i = \vec{0} \text{ and } \sum_{i=0}^k a_i = 0}$ imply $a_i = 0 \ \forall i$.

equivalent to $\sum_{i=1}^k a_i (x_i - x_0) = \vec{0}$

So $\{x_0, \dots, x_k\}$ affinely independent $\iff$ $\{x_1 - x_0, \dots, x_k - x_0\}$ linearly independent.



<u>Def</u> A finite set $S \subset \mathbb{R}^N$ is in <u>general position</u> if every subset of S of size at most $N+1$ is affinely independent.

<u>Lemma</u> Given $\{x_0, \dots, x_k\} \subset \mathbb{R}^N$ and $\delta > 0$, $\exists \{y_0, \dots, y_k\} \subset \mathbb{R}^N$ in general position with $|x_i - y_i| < \delta \ \forall i$