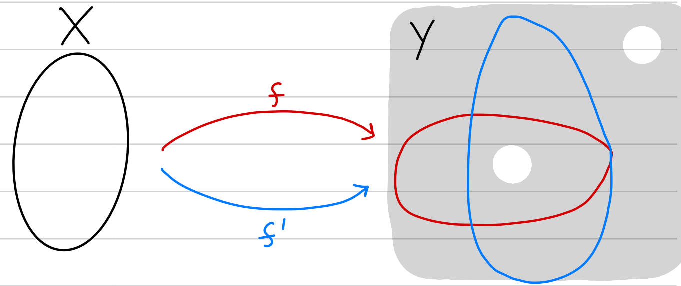
Chapter 9: The Fundamental Group

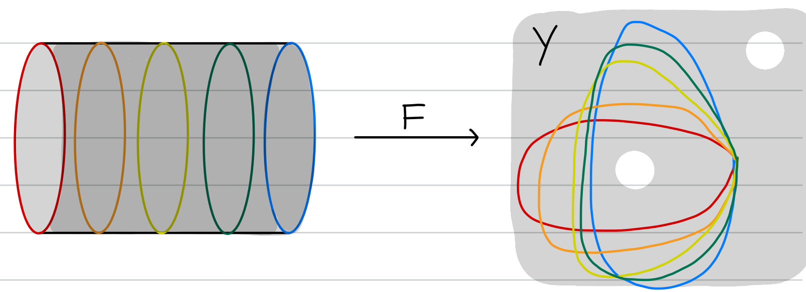
Section 51: Homotopy of paths



Let X, Y be topological spaces.
Let $I = [0,1]$ be the unit interval.

Def Maps $f, f' : X \to Y$ are homotopic ($f \simeq f'$)
if there is a continuous map $F : X \times I \to Y$
with $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ $\forall x \in X$.

Language F is a homotopy.
If f is homotopic to a constant map,
then f is nullhomotopic.

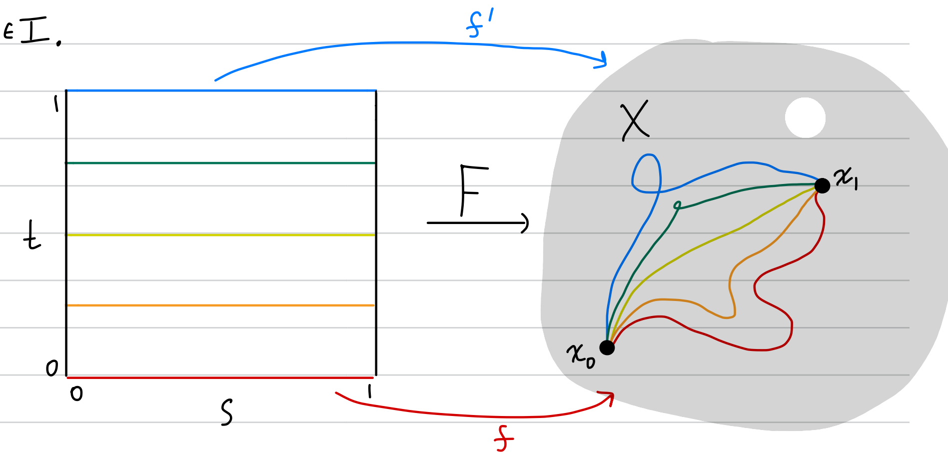


For now, the most relevant case is when $f: I \to X$ is a path, and we add a condition on endpoints.

<u>Def</u> Two paths $f, f': I \to X$ are <u>path homotopic</u> $(f \simeq_p f')$ if $f(0) = x_0 = f'(0)$, $f(1) = x_1 = f'(1)$, and there is a continuous $F: I \times I \to X$ with

$F(s,0) = f(s)$ $\quad F(s,1) = f'(s)$ $\quad \forall s \in I$ and

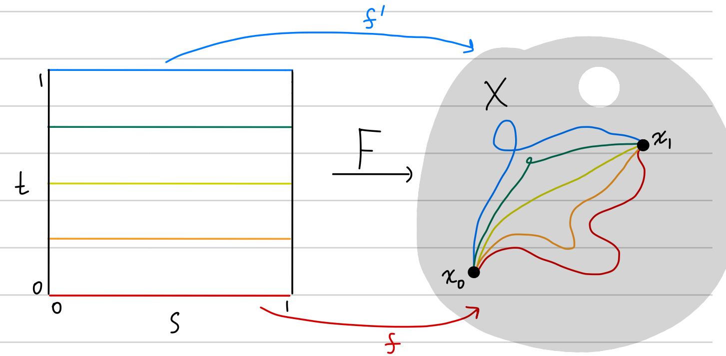
$F(0,t) = x_0$ $\quad F(1,t) = x_1$ $\quad \forall t \in I$.



Lemma The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Notation Let $[f]$ denote the equivalence class of a path f.

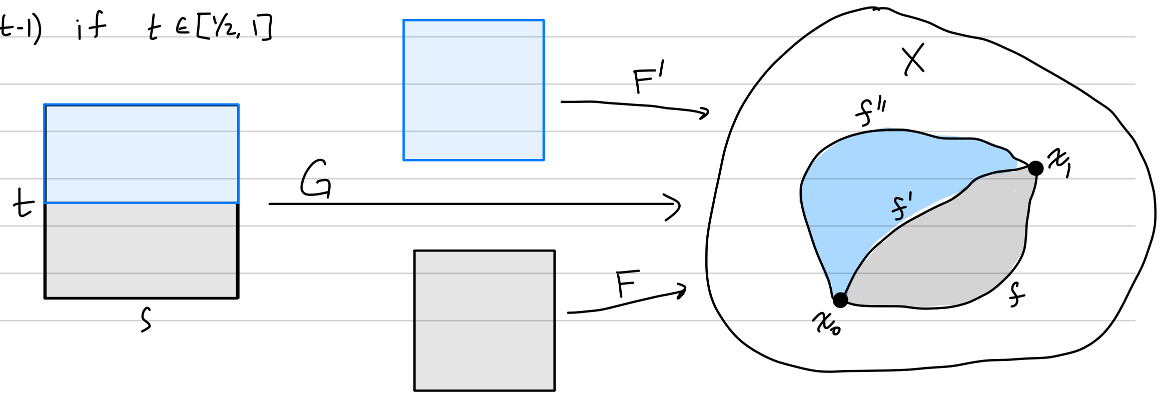
Pf $f \simeq_p f$ via $F(s,t) = f(s)$



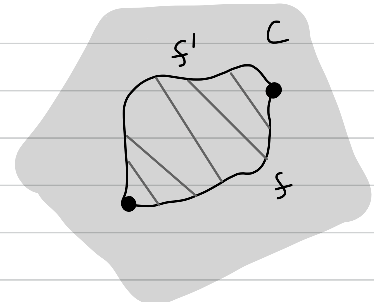
If $f \simeq_p f'$ via the homotopy F, then the homotopy $G(s,t) = F(s,1-t)$ shows $f' \simeq_p f$.

If $f \simeq_p f'$ via F, and $f' \simeq_p f''$ via F', then $f \simeq_p f''$ via $G: I \times I \to X$ defined by

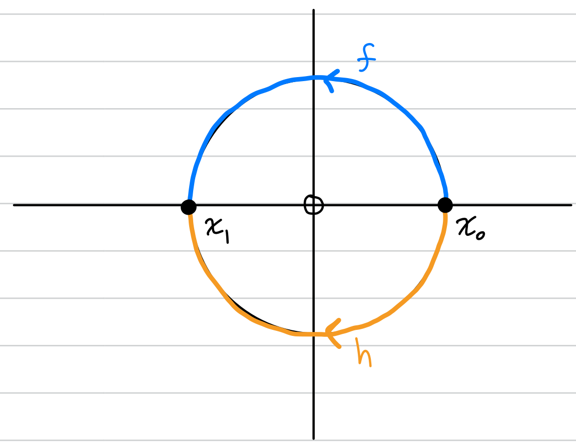
$$G(s,t) = \begin{cases} F(s,2t) & \text{if } t \in [0, 1/2] \\ F'(s, 2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$



Ex If $C \subset \mathbb{R}^n$ is convex, then any two paths f, f' from x_0 to x_1 in C are path homotopic, via $F(s,t) = (1-t)\,f(s) + t\,f'(s)$.



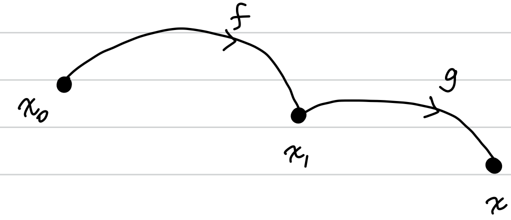
Ex The paths $f, h : I \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $f(s) = (\cos \pi s, \sin \pi s)$ and $h(s) = (\cos \pi s, -\sin \pi s)$ are not path-homotopic. But how to prove this?



(Answer: We'll introduce algebra.)

Def If f is a path in X from x_0 to x_1, and g is a path in X from x_1 to x_2, then the <u>product</u> $f * g$ is given by

$$(f*g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, 1/2] \\ g(2s-1) & \text{if } s \in [1/2, 1]. \end{cases}$$

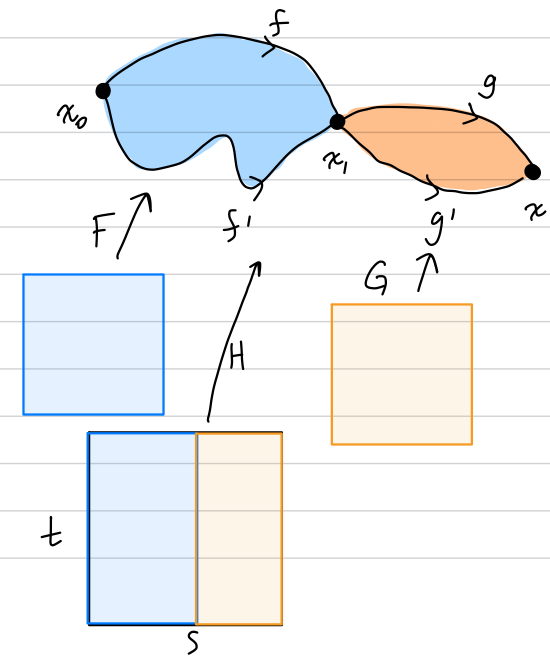


We get a well-defined product on path-homotopy classes, defined by $[f] * [g] = [f * g]$.

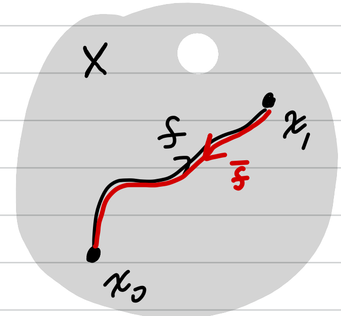
Indeed, if $f \simeq_p f'$ via F, and $g \simeq_p g'$ via G, then

$$H(s,t) = \begin{cases} F(2s, t) & \text{if } s \in [0, 1/2] \\ G(2s-1, t) & \text{if } s \in [1/2, 1] \end{cases}$$

shows $f * g \simeq_p f' * g'$.



For $x \in X$, let e_x be the constant path at x.
Let f be a path from x_0 to x_1.
Define path $\bar{f}$ by $\bar{f}(s) = f(1-s)$.



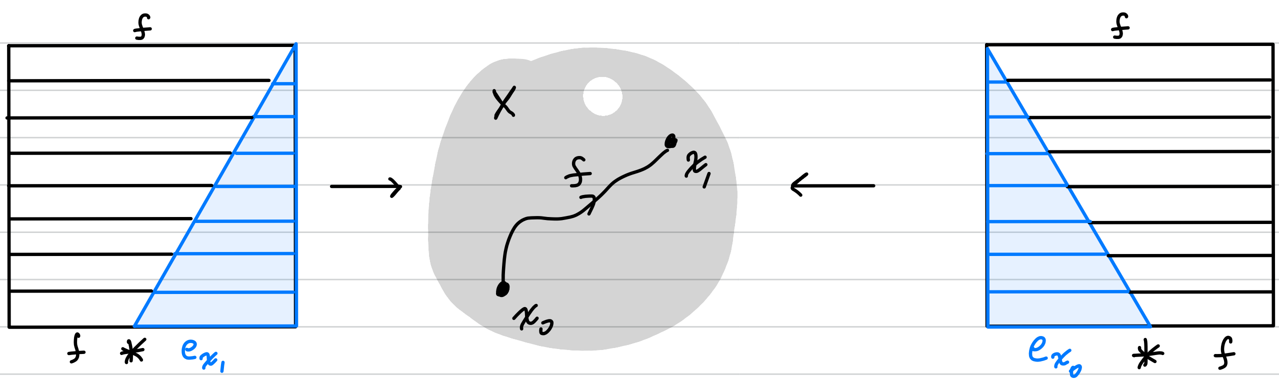
<u>Theorem</u> The product $*$ satisfies

(1) Associativity: If they are defined, then $[f]*([g]*[h]) = ([f]*[g])*[h]$.

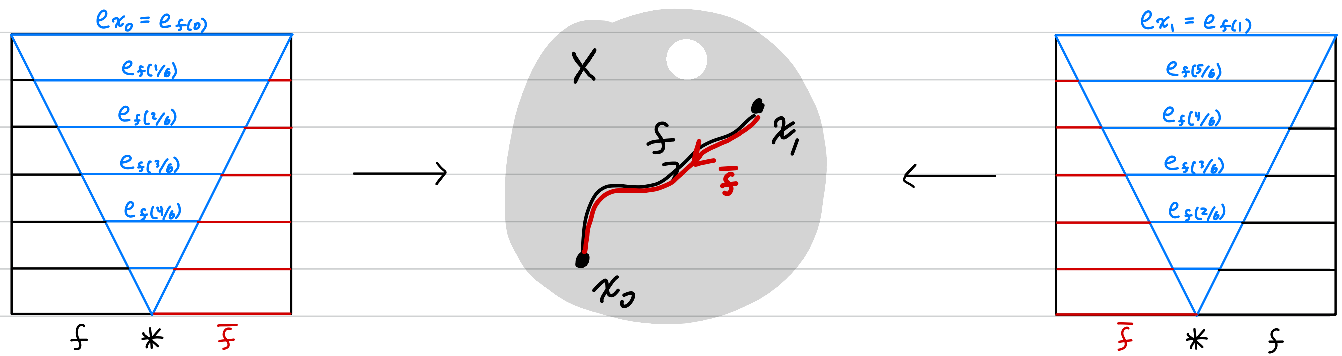
(2) Right and left identities: $[f]*[e_{x_1}] = [f]$ and $[e_{x_0}]*[f] = [f]$.

(3) Inverse: $[f]*[\bar{f}] = [e_{x_0}]$ and $[\bar{f}]*[f] = [e_{x_1}]$.

<u>Pf</u> (2)

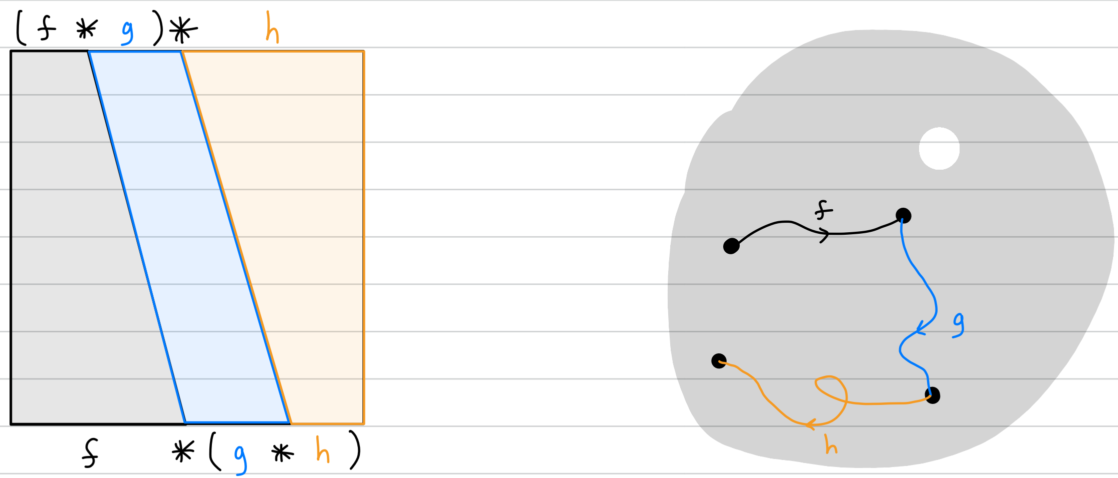


(3)



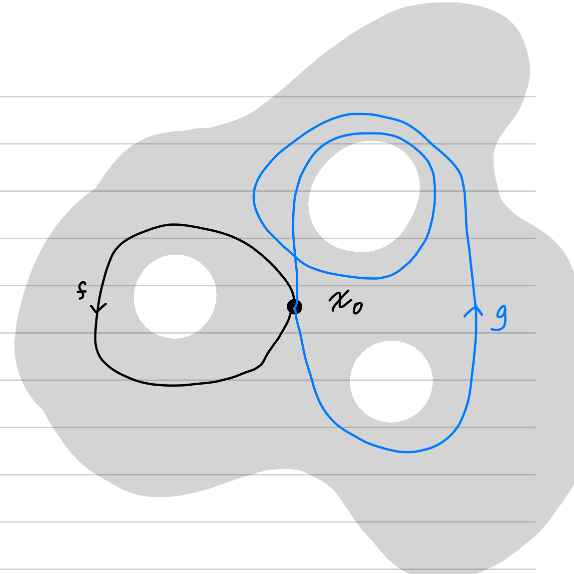
$e_{x_0} = e_{f(0)}$
$e_{f(1/6)}$
$e_{f(2/6)}$
$e_{f(3/6)}$
$e_{f(4/6)}$
$f * \bar{f}$
X
f
x_1
$\bar{f}$
x_0
$e_{x_1} = e_{f(1)}$
$e_{f(5/6)}$
$e_{f(4/6)}$
$e_{f(3/6)}$
$e_{f(2/6)}$
$\bar{f} * f$

(1)



$(f * g) * h$
$f * (g * h)$
f
g
h

Section 52: The fundamental group



Let X be a topological space and $x_0 \in X$.
A <u>loop based at x_0</u> is a path that begins and ends at x_0.

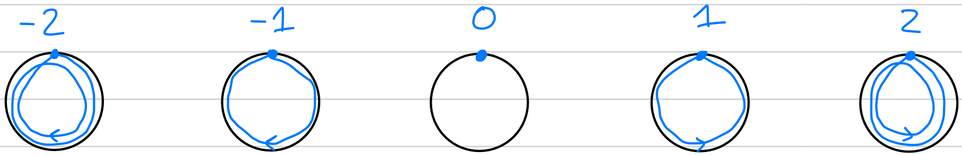
<u>Def</u> <u>The fundamental group $\pi_1(X, x_0)$</u> has as its elements the homotopy classes of loops in X based at x_0, with group operation $*$.

Recall $[f] * [g] := [f * g]$. The identity is $[e_{x_0}]$.

The inverse of $[f]$ is $[\bar{f}]$.

The fundamental group is also called the first homotopy group.
Roughly speaking, the <u>n-th homotopy group $\pi_n(X, x_0)$</u> measures the n-dimensional holes in X via the homotopy classes of (based) maps $S^n \longrightarrow X$.

<u>Ex</u> Let $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$.
We will use covering spaces to show $\pi_1(S^1) = \mathbb{Z}$.

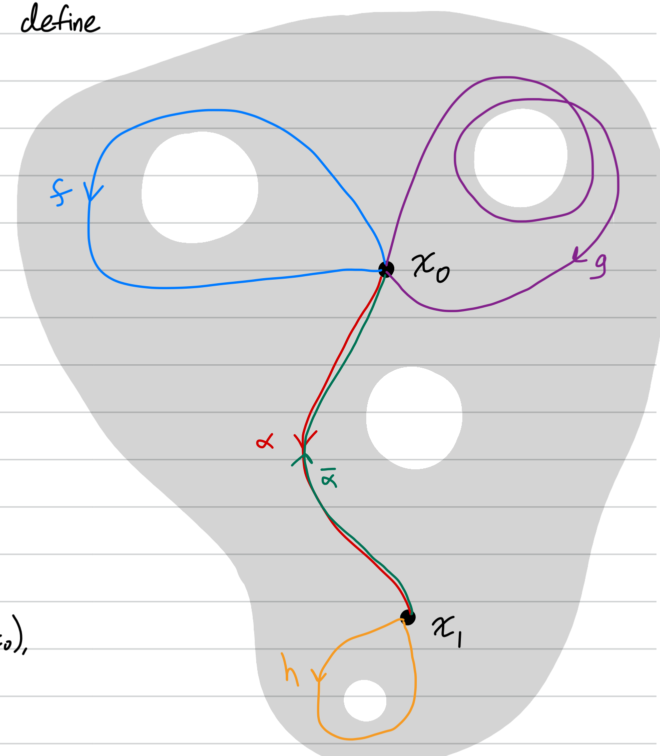


For X path connected, different basepoints yield isomorphic fundamental groups.

<u>Def</u> For α a path in X from x_0 to x_1, we define

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \quad \text{by}$$
$$[f] \longmapsto [\bar{\alpha}] * [f] * [\alpha]$$

<u>Thm</u> $\hat{\alpha}$ is an isomorphism



<u>Pf</u> To see that $\hat{\alpha}$ is a homomorphism, note

$$\begin{aligned}\hat{\alpha}([f]*[g]) &= [\bar{\alpha}]*[f]*[g]*[\alpha] \\ &= [\bar{\alpha}]*[f]*[\alpha]*[\bar{\alpha}]*[g]*[\alpha] \\ &= \hat{\alpha}([f])*\hat{\alpha}([g]).\end{aligned}$$

To see that $\hat{\alpha}$ is an isomorphism, note that its inverse is $\hat{\bar{\alpha}}$, since

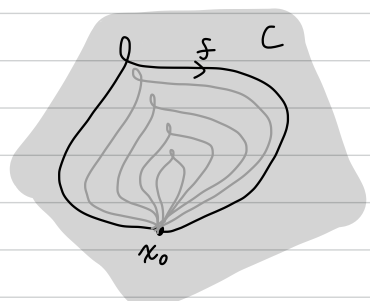
$(\hat{\bar{\alpha}} \circ \hat{\alpha})([f]) = [\alpha]*([\bar{\alpha}]*[f]*[\alpha])*[\bar{\alpha}] = [f] \quad \forall [f] \in \pi_1(X, x_0)$,

and similarly $(\hat{\alpha} \circ \hat{\bar{\alpha}})([h]) = [h] \quad \forall [h] \in \pi_1(X, x_1)$. $\square$

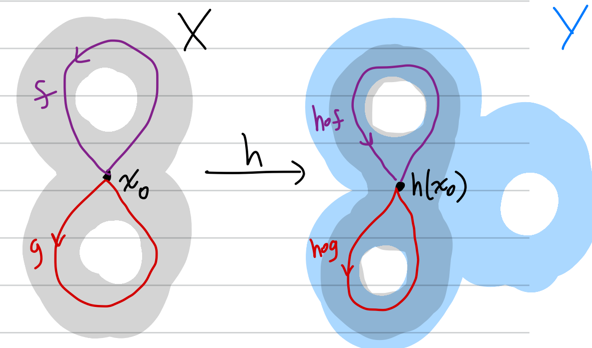
Def A space X is <u>simply connected</u> if it is path connected and $\pi_1(X, x_0) = 0$ for some (and hence all) $x_0 \in X$.

(Here 0 denotes the trivial group $\{e\}$.)

Ex If $C \subset \mathbb{R}^n$ is convex, then C is simply connected.



Def If $h: X \to Y$ is a continuous map between topological spaces, then the <u>homomorphism induced by h</u> is $h_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, h(x_0))$ via $h_*([f]) = [h \circ f]$.



Well-defined: If $f \simeq_p f'$ via F,
then $h \circ f \simeq_p h \circ f'$ via $h \circ F$.

Homomorphism: Follows since $h \circ (f * g) = (h \circ f) * (h \circ g)$.

$$\pi_1(X) \xrightarrow{h_*} \pi_1(Y)$$
$$\pi_1(X) \cong \langle [f], [g] \rangle$$

Thm The fundamental group is a functor from the category of pointed spaces to the category of groups, meaning that $(k \circ h)_* = k_* \circ h_*$ and $(id_X)_* = id_{\pi_1(X)}$.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "k \circ h"'] & Y \arrow[r, "k"] & Z \end{tikzcd} \qquad X \xrightarrow{id_X} X$$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[rr, bend right, "(k \circ h)_* = k_* \circ h_*"'] & \pi_1(Y, h(x_0)) \arrow[r, "k_*"] & \pi_1(Z, k(h(x_0))) \end{tikzcd} \qquad \pi_1(X, x_0) \xrightarrow[(id_X)_* = id_{\pi_1(X, x_0)}]{} \pi_1(X, x_0)$$

<u>Algebra terminology</u> Let G, G' be groups.

A <u>homomorphism</u> $f: G \to G'$ satisfies $f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in G$.

Its <u>kernel</u> is $f^{-1}(e')$, where e' is the identity in G'.

A homomorphism is an <u>isomorphism</u> if it is bijective.

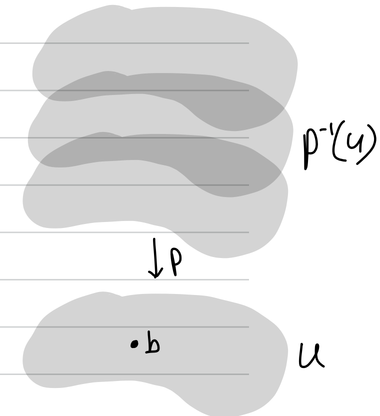
A subgroup H of G is <u>normal</u> if $xhx^{-1} \in H \quad \forall x \in G$ and $\forall h \in H$, or equivalently, if $xH = Hx \quad \forall x \in G$.

If so, the <u>quotient group</u> G/H has elements the cosets $xH \quad \forall x \in G$, with group operation $(xH) \cdot (yH) = (x \cdot y) H$.

Note
$$\begin{aligned} f: G &\to G/H \\ x &\mapsto xH \end{aligned}$$
is a surjective homomorphism with kernel H.

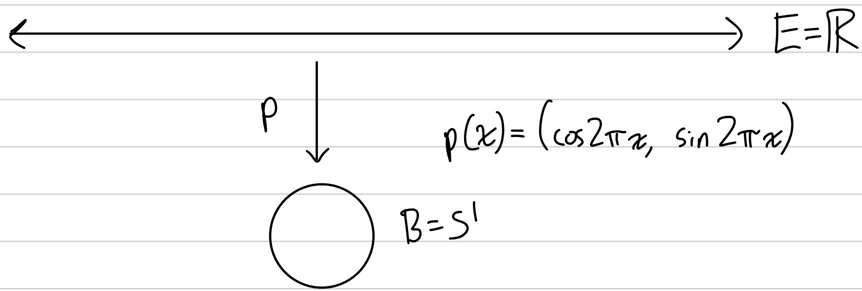
Conversely, if homomorphism $f: G \to G'$ is surjective, then its kernel N is normal in G, and the induced map
$$\begin{aligned} G/N &\longrightarrow G' \\ xN &\longmapsto f(x) \end{aligned}$$
is an isomorphism.

Section 53: Covering spaces

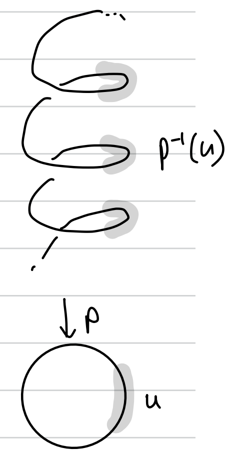


Let $p: E \to B$ be a continuous surjection between topological spaces. Suppose each $b \in B$ has a neighborhood U s.t. $p^{-1}(U)$ is a union of disjoint sets $V_\alpha \subset E$ with $p|_{V_\alpha} : V_\alpha \to U$ a homeomorphism $\forall \alpha$. Then p is a covering map and E is a covering space of B.

We say U is evenly covered.

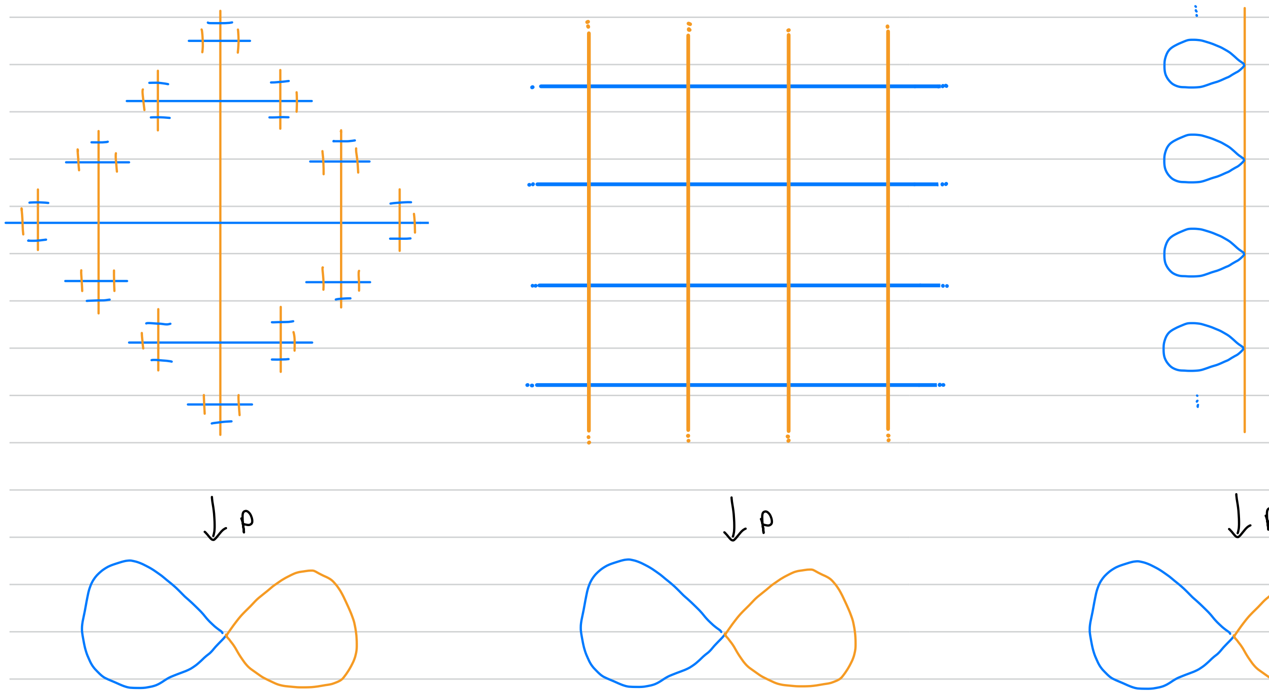


$p(x) = (\cos 2\pi x, \sin 2\pi x)$

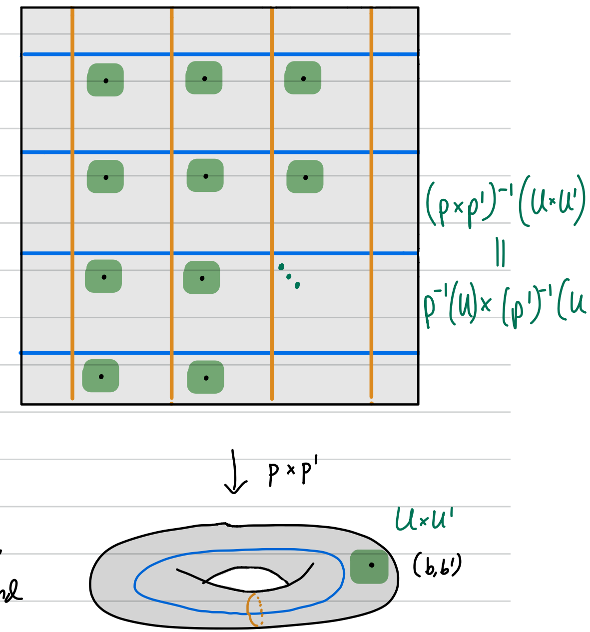


Thm The map $p: \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

<u>Ex</u> The following are covering spaces of the figure-eight:



Ex The map $p \times p : \mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$ defined by
$(p \times p)(x,y) = (p(x), p(y)) = (\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y)$
is a covering map.



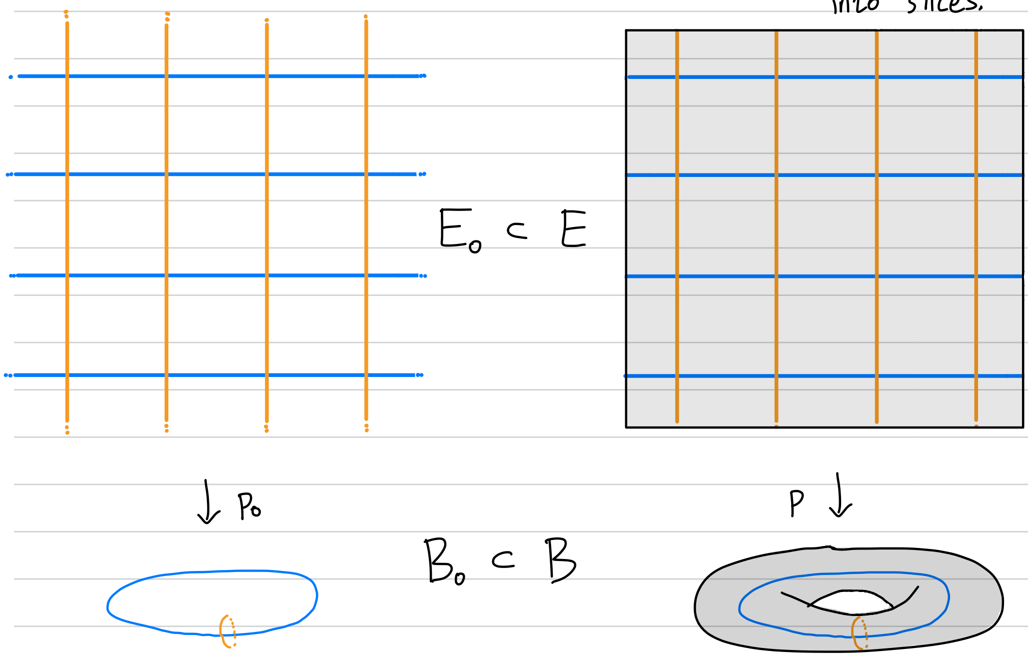
More generally...

Thm If $p : E \to B$ and $p' : E' \longrightarrow B'$ are covering maps, then $p \times p' : E \times E' \longrightarrow B \times B'$ is a covering map.

Pf Given $(b,b') \in B \times B'$, let $U \ni b$ and $U' \ni b'$ be neighborhoods evenly covered by p and p', resp. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be the partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ into slices. Then the partition of $(p \times p')^{-1}(U \times U') = p^{-1}(U) \times (p')^{-1}(U')$ into the slices $\{V_\alpha \times V'_\beta\}$ shows that $U \times U'$ is a neighborhood of (b,b') evenly covered by $p \times p'$, as required.

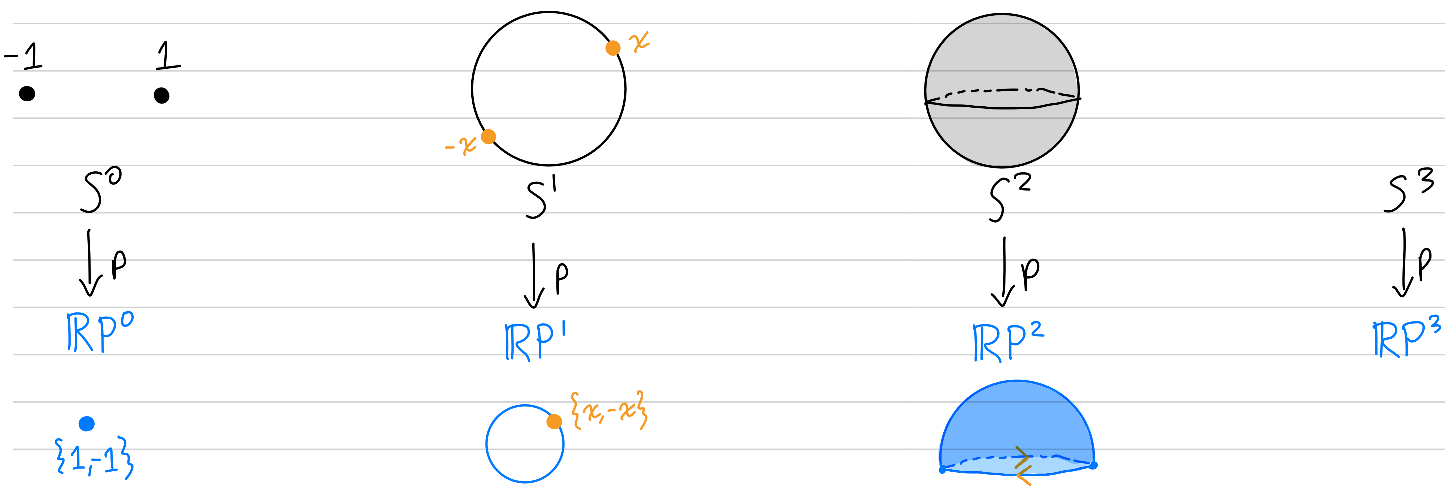
Thm Let $p: E \to B$ be a covering space. If $B_0 \subset B$ and $E_0 := p^{-1}(B_0)$, then $p_0 := p|_{B_0} : E_0 \longrightarrow B_0$ is a covering map.

Pf Given $b_0 \in B_0$, let U be an open set in B containing b_0 that is evenly covered by p. Let $\{V_\alpha\}$ partition $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $\{V_\alpha \cap E_0\}$ show that $U \cap B_0$ is evenly covered by p_0.



Ex The n-sphere S^n is $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.
Real projective space $\mathbb{R}P^n$ is $\mathbb{R}^n = S^n/\sim$, where $x \sim -x$ $\forall x \in S^n$.

The map $p : S^n \longrightarrow \mathbb{R}P^n$ via
$$x \longmapsto \{x, -x\}$$
is a covering map.



Later, we will see that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$.

Section 54: The fundamental group of the circle

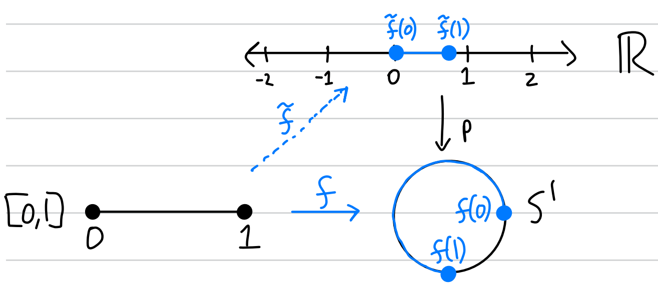
To prove $\pi_1(S^1) \cong \mathbb{Z}$, we will first need some properties about lifting paths to covering spaces.

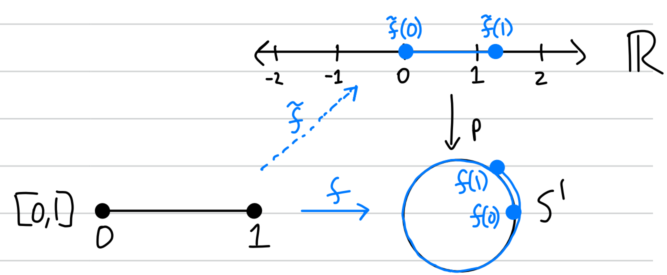
$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & B \end{tikzcd}$$

__Def__ Let $p: E \to B$ be a map. Given a map $f: X \to B$, a __lifting of f__ is a map $\tilde{f}: X \to E$ such that $p \circ \tilde{f} = f$.

A main example will be:

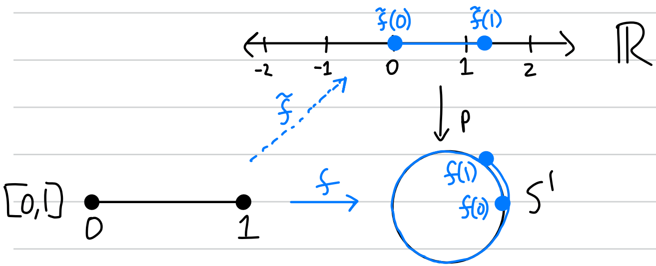
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"'] \\ {[0,1]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd} \qquad p(x) = (\cos 2\pi x, \sin 2\pi x)$$





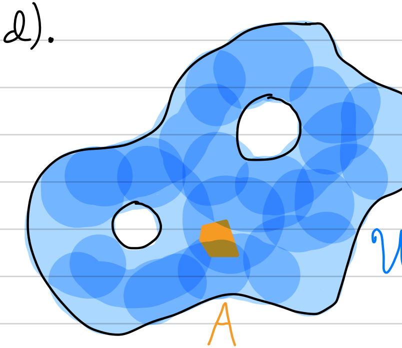
Path lifting lemma Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Any path $f: [0,1] \to B$ with $f(0) = b_0$ has a unique lift $\tilde{f}: [0,1] \to E$ with $\tilde{f}(0) = e_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\exists! \tilde{f}"] \arrow[r, "f"'] & B \arrow[ul, phantom, "\circlearrowright"] \end{tikzcd}
\qquad
\begin{tikzcd} & e_0 \arrow[d, mapsto] \\ 0 \arrow[ur, dashed, mapsto] \arrow[r, mapsto] & b_0 \end{tikzcd}$$

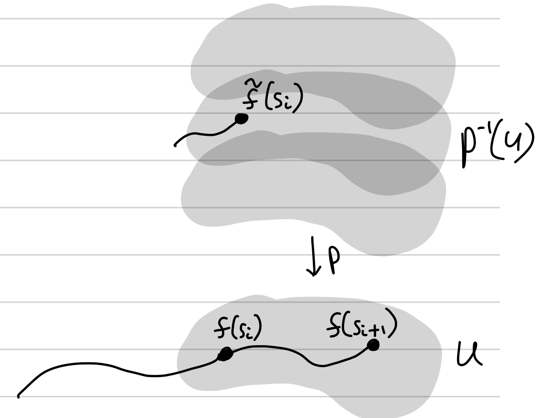


The proof will use the **Lebesgue number lemma**:
Let $\mathcal{U}$ be an open cover of a compact metric space (X, d).
Then $\exists \delta > 0$ s.t. for each $A \subset X$ with $\operatorname{diam}(A) < \delta$,
$\exists U \in \mathcal{U}$ with $A \subset U$.

$$\operatorname{diam}(A) = \sup\{d(a,a') : a, a' \in A\}$$



<u>Proof of path lifting lemma</u> Let $\mathcal{U}$ be an open cover of B by sets evenly covered by p. Apply the Lebesgue number lemma to $\{f^{-1}(U) : U \in \mathcal{U}\}$ to get a subdivision $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$ of $[0,1]$ s.t. each $f([s_i, s_{i+1}])$ is contained in some $U \in \mathcal{U}$.



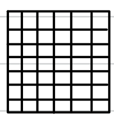
Let $\tilde{f}(0) = e_0$. Assume $\tilde{f}$ is defined on $[0, s_i]$. To define $\tilde{f}$ on $[s_i, s_{i+1}]$, suppose $\tilde{f}(s_i) \in V_\alpha$ with $p|_{V_\alpha} : V_\alpha \to U$ a homeomorphism, and let
$\tilde{f}(t) = (p|_{V_\alpha})^{-1}(f(t)) \quad \forall t \in [s_i, s_{i+1}]$.

Then $\tilde{f} : [0,1] \to E$ is continuous
(by the pasting lemma)
and $p \circ \tilde{f} = f$ by definition of $\tilde{f}$.

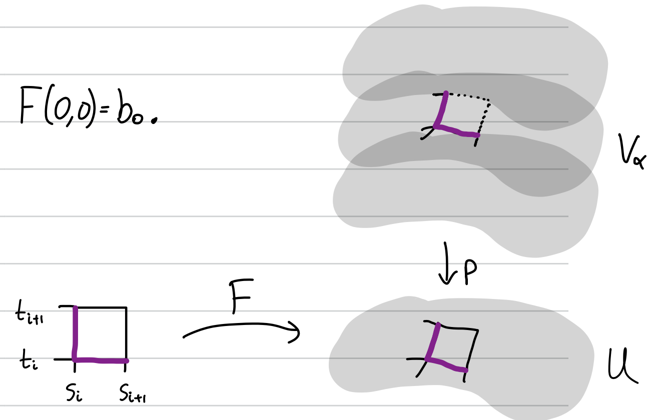
Is $\tilde{f}$ unique?
Assume unique on $[0, s_i]$.
Suppose $\tilde{\tilde{f}}$ were a different lift on $[s_i, s_{i+1}]$.
Note $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is connected, and hence contained in one V_α.
But $\forall x \in U$ $\exists!$ point in $p^{-1}(x) \cap V_\alpha$, so $\tilde{\tilde{f}} = \tilde{f}$ on $[s_i, s_{i+1}]$ too.

<u>Path homotopy lifting lemma</u> Let $p: E \to B$ be a covering map with $p(e_0) = b_0$. Let $F: I \times I \to B$ be continuous with $F(0,0) = b_0$. Then $\exists!$ lift $\tilde{F}: I \times I \to E$ with $\tilde{F}(0,0) = e_0$.

If F is a path homotopy, so is $\tilde{F}$.



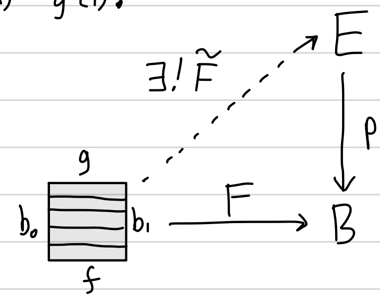
$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \times I \arrow[ur, dashed, "\exists! \tilde{F}"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\circlearrowright"] & B \end{tikzcd}$$

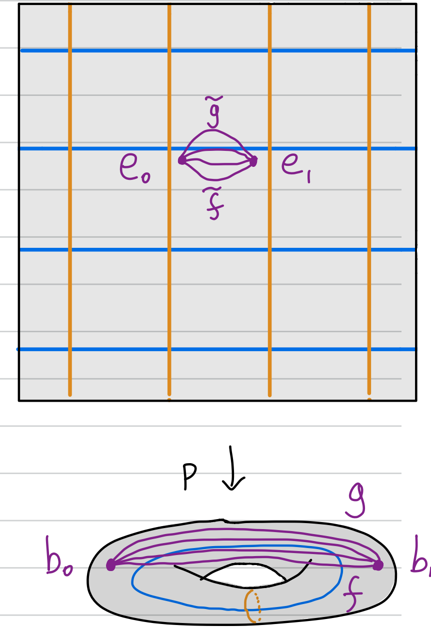


<u>Pf</u> Use path lifting lemma to extend $\tilde{F}$ to $\{0\} \times I$ and $I \times \{0\}$.
Using Lebesgue number lemma, subdivide $I \times I$ into small squares so that each $F(\text{small square})$ is contained in some open $U \subset B$ evenly covered by p.
Extend $\tilde{F}$ lexicographically to each new small square.
If $\tilde{F}(\text{L}) \subset V_\alpha$, then define $\tilde{F}(s,t) = (p|_{V_\alpha})^{-1}(F(s,t))$ $\forall (s,t) \in [s_i, s_{i+1}] \times [t_i, t_{i+1}]$.

If F is a path homotopy, then $F(\{0\} \times I) = b_0$ and $\tilde{F}(\underbrace{\{0\} \times I}_{\text{connected}}) \subset \underbrace{p^{-1}(b_0)}_{\text{discrete topology}}$, so $\tilde{F}(\{0\} \times I) = e_0$. Similarly for the right edge $\{1\} \times I$.

Thm Let $p: E \to B$ be a covering map with $p(e_0) = b_0$.
Let $f, g: I \to B$ be paths from b_0 to b_1.
Let $\tilde{f}, \tilde{g}: I \to E$ be lifts starting at e_0.
If f, g are path homotopic, then so are $\tilde{f}, \tilde{g}$,
and hence $\tilde{f}(1) = \tilde{g}(1)$.

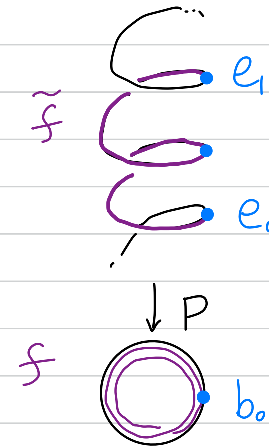




Pf Let $F: I \times I \to B$ be a path homotopy from f to g.
By the path homotopy lifting lemma, we have a lift
$\tilde{F}: I \times I \to E$ that is also a path homotopy.
Note $\tilde{F}|_{\text{bottom edge}}$ is a lift of f, which by uniqueness of path lifting is $\tilde{f}$.
Note $\tilde{F}|_{\text{top edge}}$ " " " " g, " " " " " " " $\tilde{g}$.
Hence $\tilde{F}$ is a path homotopy from $\tilde{f}$ to $\tilde{g}$.

Def Let $p: E \to B$ be a covering map with $p(e_0) = b_0$.
Then $\phi: \pi_1(B, b_0) \to p^{-1}(b_0)$ defined by
$$\phi([f]) = \tilde{f}(1) \quad \text{where } \tilde{f} \text{ is the lift of } f \text{ with } \tilde{f}(0) = e_0$$
is a well-defined set map, called the <u>lifting correspondence</u>.

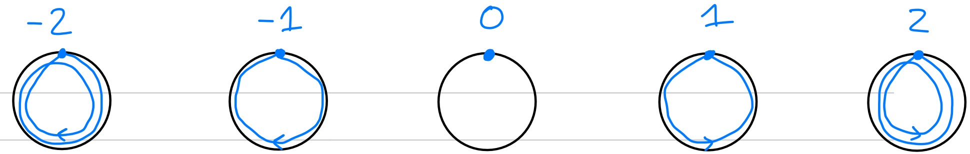
(Note ϕ depends on the choice of e_0.)



Thm If E is path connected, then ϕ is surjective.
If E is simply connected, then ϕ is bijective.

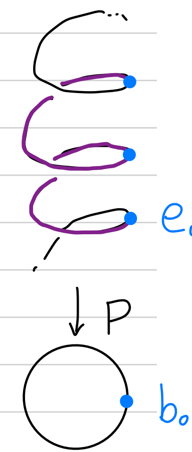
Pf Let $e_1 \in p^{-1}(b_0)$. Since E is path connected,
let $\tilde{f}$ be a path in E from e_0 to e_1.
Then $f = p \circ \tilde{f}$ is a loop in B with $\phi([f]) = \tilde{f}(1) = e_1$.

Let $[f], [g] \in \pi_1(B, b_0)$ with $\phi([f]) = \phi([g])$.
So the lifts $\tilde{f}, \tilde{g}$ starting at e_0 satisfy $\tilde{f}(1) = \tilde{g}(1)$.
E simply connected $\Rightarrow \exists$ path homotopy $\tilde{F}$ between $\tilde{f}$ and $\tilde{g}$.
The path homotopy $p \circ \tilde{F}$ shows $[f] = [g]$.

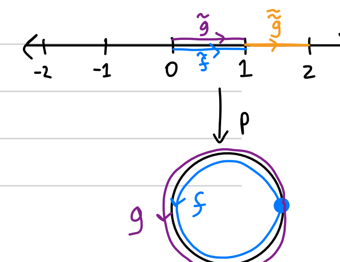


Thm $\pi_1(S^1) \cong \mathbb{Z}$

Pf Let $p: \mathbb{R} \to S^1$ be the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$.
Let $e_0 = 0$ and $b_0 = p(e_0) = (1,0)$, so $p^{-1}(b_0) = \mathbb{Z}$.
Since E is simply connected, the lifting correspondence
$\phi: \pi_1(S^1, b_0) \to \mathbb{Z}$ is a bijection.
It remains to show ϕ is a group homomorphism.

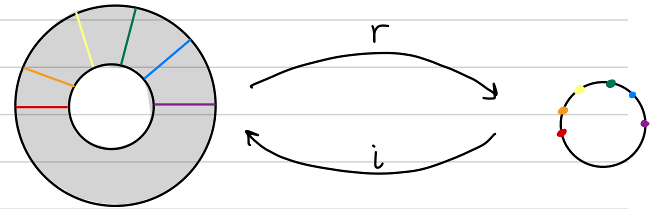


Let $[f], [g] \in \pi_1(B, b_0)$ have lifts $\tilde{f}, \tilde{g}$ starting at 0.
Let $\phi([f]) = \tilde{f}(1) = n$ and $\phi([g]) = \tilde{g}(1) = m$.
Define $\tilde{\tilde{g}}: I \to \mathbb{R}$ by $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$.
Note $\tilde{f} * \tilde{\tilde{g}}$ is a well-defined path (since $\tilde{f}(1) = n = \tilde{\tilde{g}}(0)$)
that lifts $f * g$ (since $p \circ \tilde{f} = f$ and $p \circ \tilde{\tilde{g}} = g$),
beginning at 0 and
ending at $(\tilde{f} * \tilde{\tilde{g}})(1) = \tilde{\tilde{g}}(1) = n + m$.



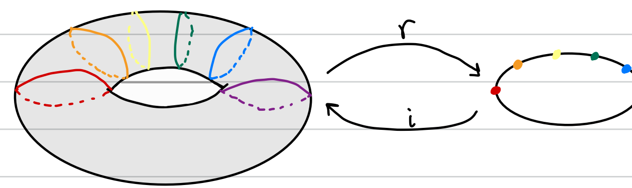
Hence $\phi([f] * [g]) = \phi([f * g]) = (\tilde{f} * \tilde{\tilde{g}})(1) = n + m = \phi([f]) + \phi([g])$,
as desired.

Section 55: Retractions and fixed points



Def For $A \subset X$, a <u>retraction</u> $r: X \to A$ is a continuous map with $r(a) = a \quad \forall a \in A$.

Let $i: A \to X$ be the inclusion defined by $i(a) = a \quad \forall a \in A$.



Lemma If $r: X \to A$ is a retraction, then $r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective and $i_*: \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.

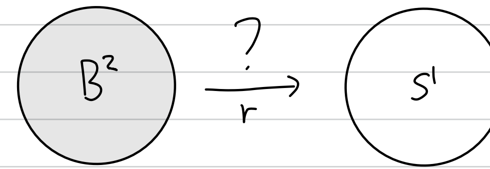
Proof

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "r \circ i = id_A"'] & X \arrow[r, "r"] & A \end{tikzcd} \qquad \begin{tikzcd} \pi_1(A) \arrow[r, "i_*"] \arrow[rr, bend right, "r_* \circ i_* = (r \circ i)_*"'] & \pi_1(X) \arrow[r, "r_*"] & \pi_1(A) \end{tikzcd}$$

Since r is a retraction, $r \circ i = id_A$.
Since π_1 is a functor (Thm 52.4), we get
$r_* \circ i_* = (r \circ i)_* = (id_A)_* = id_{\pi_1(A)}$.
Hence r_* is surjective and i_* is injective.

Recall $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \le 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

<u>Thm</u> There is no retraction $r : B^2 \to S^1$.



<u>Pf</u> $\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "r \circ i = id_{S^1}"'] & B^2 \arrow[r, "r"] & S^1 \end{tikzcd}$ Apply π_1 $\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "id_{\mathbb{Z}}"'] & 0 \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$

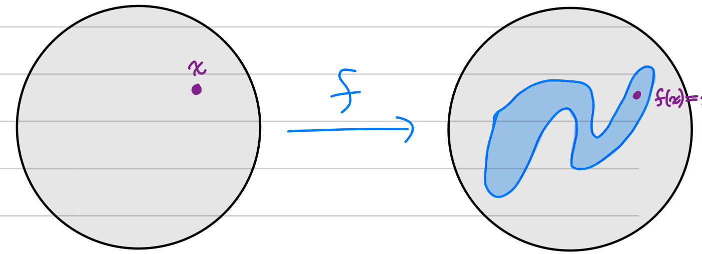
This would contradict the fact there is no surjection $r_* : 0 \to \mathbb{Z}$
(or that there is no injection $i_* : \mathbb{Z} \to 0$,
or that $id_{\mathbb{Z}}$ does not factor through the trivial group 0.)

Aside: It is more generally true that there is no retraction $B^{n+1} \to S^n$.

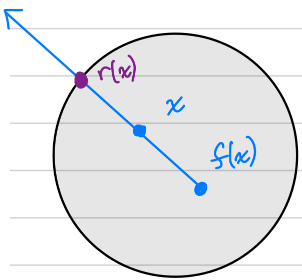
Indeed, to $\begin{tikzcd} S^n \arrow[r, "i"] \arrow[rr, bend right, "r \circ i = id_{S^n}"'] & B^{n+1} \arrow[r, "r"] & S^n \end{tikzcd}$ apply π_n to get $\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "id_{\mathbb{Z}}"'] & 0 \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$

and the same contradictions. Here, π_n is the "n-th homotopy group".

Thm (Brouwer fixed point theorem for the disk)
If $f: B^2 \to B^2$ is continuous, then there is a fixed point $x \in B^2$ satisfying $f(x) = x$.

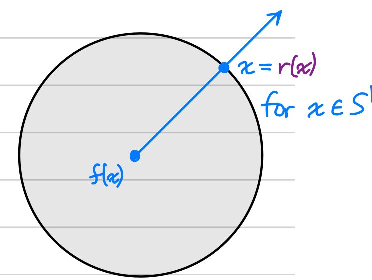


Pf Suppose $f(x) \neq x$ $\forall x \in B^2$.
Then we can define $r: B^2 \to S^1$ by letting $r(x)$ be the unique point of S^1 on the ray from $f(x)$ through x.



Using the fact that f is continuous, with work one can show that r is continuous.

To see that r is a retraction, note that if $x \in S^1$, then $r(x) = x$



Since there is no retraction $r: B^2 \to S^1$, there must be some $x \in B^2$ with $f(x) = x$.

Aside: The same proof shows any continuous $f: B^{n+1} \to B^{n+1}$ has a fixed point.

<u>Corollary</u> Let A be a 3×3 matrix with positive real entries. Then A has a positive real eigenvalue (characteristic value).

<u>Ex</u>

$$A = \begin{bmatrix} 3 & 2.5 & 7 \\ 0.7 & 4 & 5 \\ 1.9 & 3.2 & 6 \end{bmatrix}$$

Section 56: The fundamental theorem of algebra

Covered by Prof. Philip Boyland