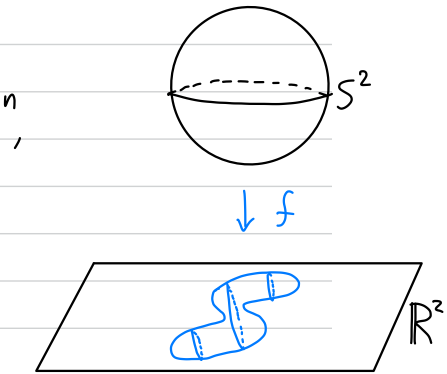
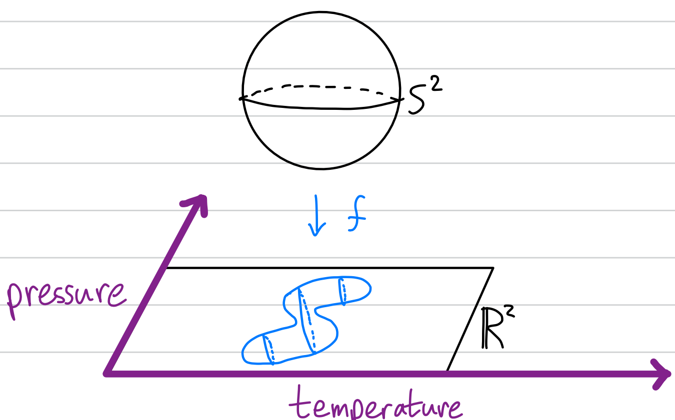
Section 57: The Borsuk-Ulam theorem

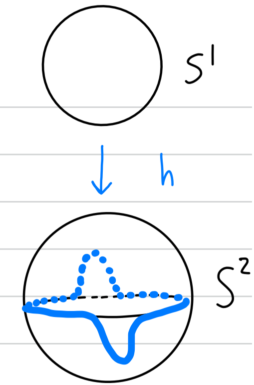


Thm (Borsuk-Ulam) Given a continuous $f: S^n \to \mathbb{R}^n$, there is some $x \in S^n$ with $f(x) = f(-x)$.

The case $n=1$ follows from the intermediate value theorem. We will prove the case $n=2$.

Ex There exist two antipodal points on Earth (S^2) with the same temperature and pressure.



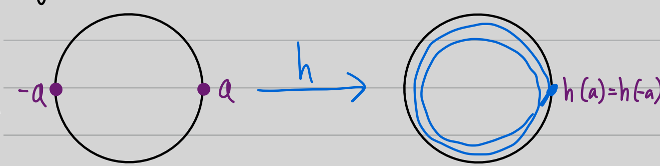


<u>Def</u> If $x \in S^n$, then its <u>antipode</u> is $-x \in S^n$.

<u>Def</u> A map $h: S^n \to S^m$ is <u>antipode-preserving</u> if $h(-x) = -h(x)$ $\forall x \in S^n$
(I.e., h respects the $\mathbb{Z}/2$ actions on S^n and S^m.)

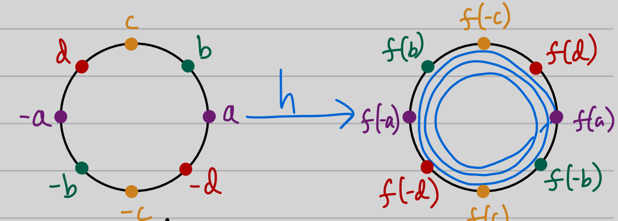
<u>Ex</u> The map $h: S^1 \to S^1$ given by $h(e^{2\pi i t}) = e^{2\pi i qt}$
for $q \in \mathbb{Z}$, is antipode-preserving $\iff$ q is odd.

<u>$q=2$</u>

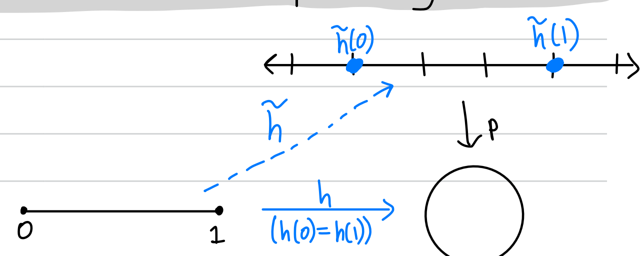


Not antipode-preserving

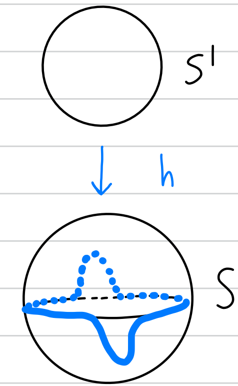
<u>$q=3$</u>



Antipode-preserving



Def If $x \in S^n$, then its <u>antipode</u> is $-x \in S^n$.



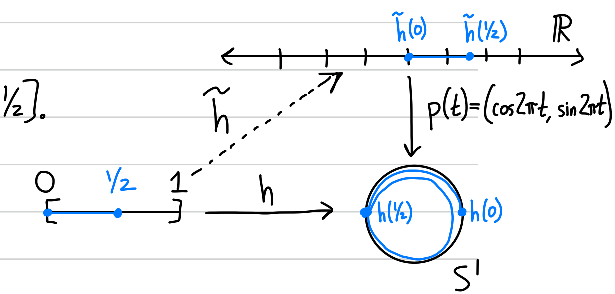
Def A map $h: S^n \to S^m$ is <u>antipode-preserving</u> if $h(-x) = -h(x)$ $\forall x \in S^n$
(I.e., h respects the $\mathbb{Z}/2$ actions on S^n and S^m.)

Thm If $h: S^1 \to S^1$ is continuous and antipode-preserving, then $h_*: \mathbb{Z} \to \mathbb{Z}$ is given by $h_*(i) = qi$ $\forall i \in \mathbb{Z}$ with q odd, and hence h is not nullhomotopic.

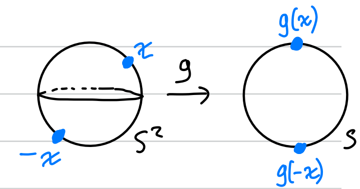
Pf (We follow Hatcher's book, page 33.)
Think of $h: S^1 \to S^1$ equivalently as $h: [0,1] \to S^1$ with $h(0) = h(1)$.
Antipode-preserving means $h(s + 1/2) = -h(s)$ $\forall s \in [0, 1/2]$ $(\star)$.
Consider the lift $\tilde{h}: [0,1] \to \mathbb{R}$ with $p \circ \tilde{h} = h$.
$(\star)$ gives $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ for some odd $q \in \mathbb{Z}$, $\forall s \in [0, 1/2]$.
Since $\tilde{h}$ is continuous, q cannot depend on s.
In particular,

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q.$$

Hence $h_*: \mathbb{Z} \to \mathbb{Z}$ is given by $h_*(i) = qi$ $\forall i \in \mathbb{Z}$, with q odd.

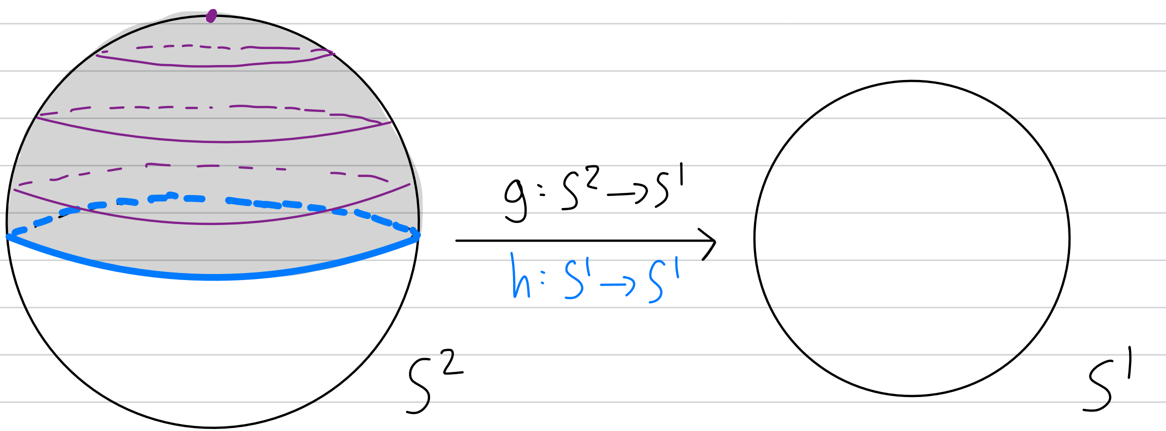


Thm There is no continuous antipode-preserving map $g: S^2 \to S^1$.

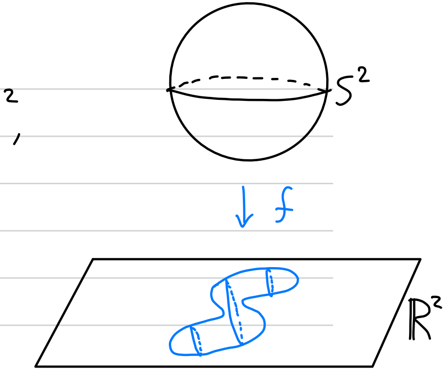


(**Rmk** There is no continuous antipode-preserving map $S^k \to S^n$ for any $k > n$, though we won't prove this.)

Pf Suppose $g: S^2 \to S^1$ were continuous and antipode-preserving. Restricting to the equator of S^2 would give $h: S^1 \to S^1$ that is antipode-preserving. The extension of h to the northern hemisphere of S^2 (via g) shows h is nullhomotopic, contradicting the prior theorem.



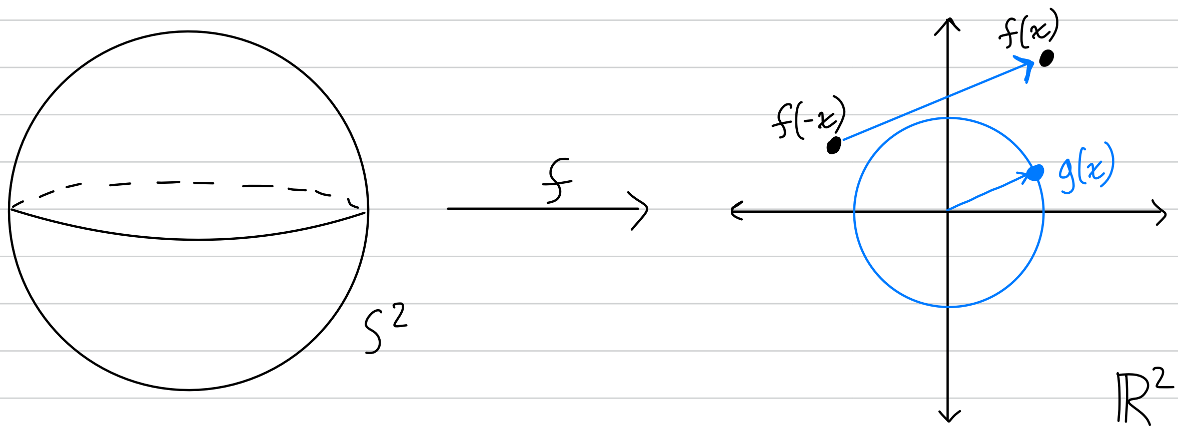
Thm (Borsuk-Ulam) Given a continuous $f: S^2 \to \mathbb{R}^2$, there is some $x \in S^2$ with $f(x) = f(-x)$.



Pf Suppose $f(x) \neq f(-x)$ $\forall x \in S^2$. Then

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a continuous map $g: S^2 \to S^1$ with $g(-x) = -g(x)$ $\forall x \in S^2$, contradicting the prior theorem.

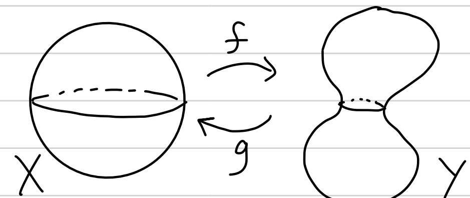


Section 58: Deformation retracts and homotopy type

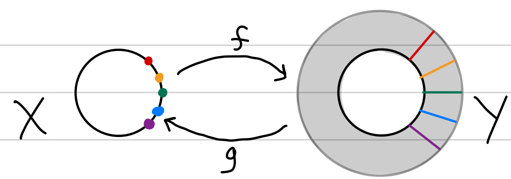
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Let X, Y be topological spaces and $f: X \to Y$, $g: Y \to X$ be continuous maps.

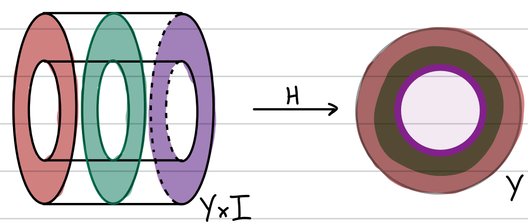
Recall f, g are homeomorphisms if
$g \circ f = id_X$ and $f \circ g = id_Y$.
We say X and Y are homeomorphic, denoted $X \cong Y$.



Def The maps f, g are <u>homotopy equivalences</u> if $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.
We say X and Y are <u>homotopy equivalent</u>, denoted $X \simeq Y$.



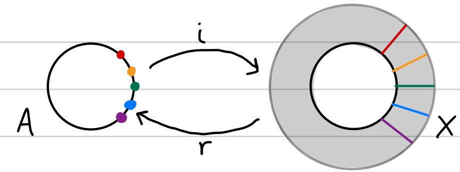
In this example, we have $g \circ f = id_X$ and $f \circ g \simeq id_Y$ via a homotopy $H: Y \times I \to Y$ with $H(\,, 0) = id_Y$ and $H(\,, 1) = f \circ g$.



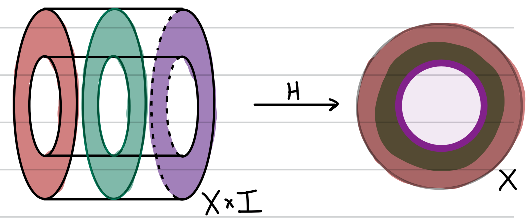
Rmk Showing $\simeq$ is an equivalence relation is fun.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

The prior example is furthermore a deformation retract.

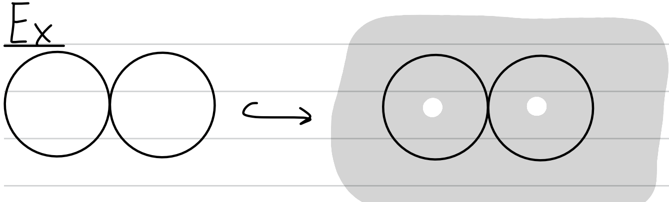


__Def__ Let $A \subset X$ with $i: A \to X$ the inclusion. Then A is a __deformation retract__ of X if $\exists$ a retraction $r: X \to A$ (meaning $r \circ i = id_A$) and a homotopy $H: X \times I \to X$ with $H(-,0) = id_X$, $H(-,1) = i \circ r$, and $H(a,t) = a$ $\forall a \in A$ and $\forall t \in I$.

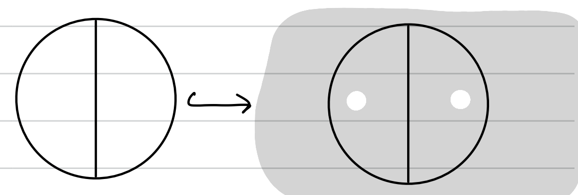


Note that deformation retracts are special cases of homotopy equivalences and special cases of retracts.

__Ex__



and



Fuchs showed $X \simeq Y \Leftrightarrow \exists$ a third space Z that deformation retracts onto both X and Y.

Now we show that homotopy equivalent spaces have isomorphic fundamental groups. This is easiest for deformation retracts, where the homotopy H fixes basepoints.

<u>Thm</u> If A is a deformation retract of X, then $i: A \hookrightarrow X$ induces an isomorphism $i_*: \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ $\forall x_0 \in A$.

<u>Pf</u> Since $r \circ i = id_A$, we have

$r_* \circ i_* = (r \circ i)_* = (id_A)_* = id_{\pi_1(A)}$,

so i_* is injective.

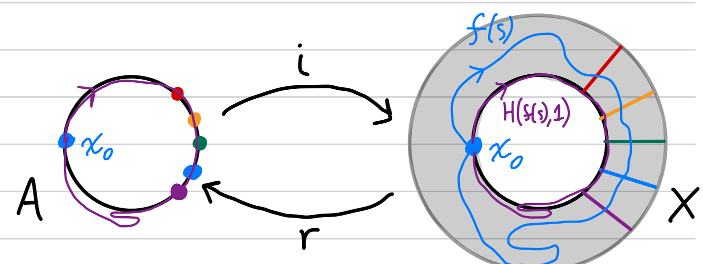
$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend right, "id_A"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(A) \arrow[r, "i_*"] \arrow[rr, bend right, "id_{\pi_1(A)}"'] & \pi_1(X) \arrow[r, "r_*"] & \pi_1(A) \end{tikzcd}$$

And given a loop $f: I \to X$ based at $x_0 \in A$, the composition

$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} X$$

gives a path homotopy from $H(f(s), 0) = f(s)$ to a loop $H(f(s), 1)$ in A, showing that i_* is surjective.



Lemma Let $h, k : X \to Y$ be continuous with $h(x_0) = y_0$, $k(x_0) = y_1$. If $h \simeq k$, then $\exists$ a path α in Y from y_0 to y_1 with $k_* = \hat{\alpha} \circ h_*$ as maps $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \arrow[dl, phantom, "\circlearrowright"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

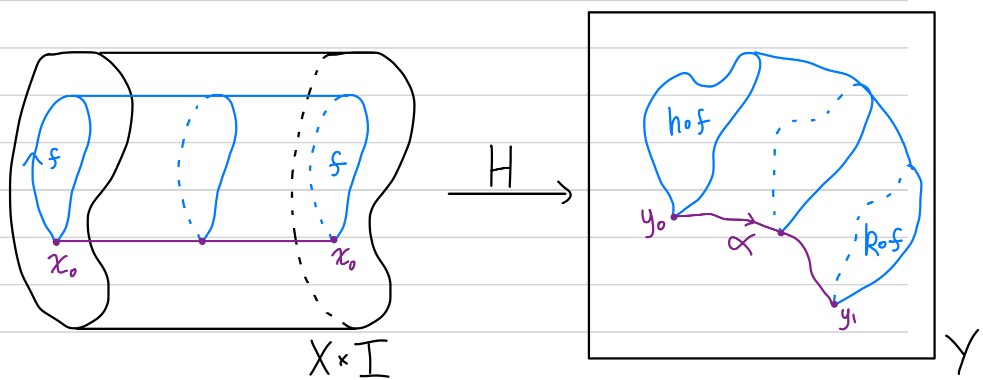
(Indeed, if $H : X \times I \to Y$ is a homotopy with $H(-, 0) = h$ and $H(-, 1) = k$, then $\alpha = H(x_0, -)$.)

Pf Let $[f] \in \pi_1(X, x_0)$. We must show

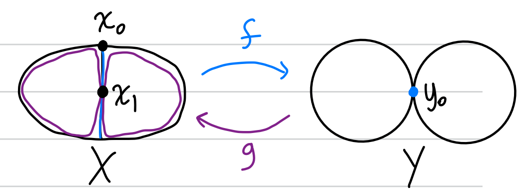
$$k_*([f]) = \hat{\alpha}(h_*([f]))$$

i.e. $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$,

which can be seen in the figure on the right.



Thm If $f : X \to Y$ is a homotopy equivalence with $f(x_0) = y_0$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.



Pf Let $g : Y \to X$ be a homotopy inverse (satisfying $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$).
We have

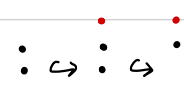
$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)$$

where $y_0 = f(x_0)$, $x_1 = g(y_0)$, $y_1 = f(x_1)$.

Since $g \circ f \simeq id_X$, the prior lemma implies
$$g_* \circ (f_{x_0})_* = (g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}$$
for some path α in X.
Since $\hat{\alpha}$ is an isomorphism, $(f_{x_0})_*$ is injective.

Since $f \circ g \simeq id_Y$, the prior lemma implies
$$(f_{x_1})_* \circ g_* = (f \circ g)_* = \hat{\beta} \circ (id_Y)_* = \hat{\beta}$$
for some path β in Y.
Since $\hat{\beta}$ is an isomorphism, g_* is injective.

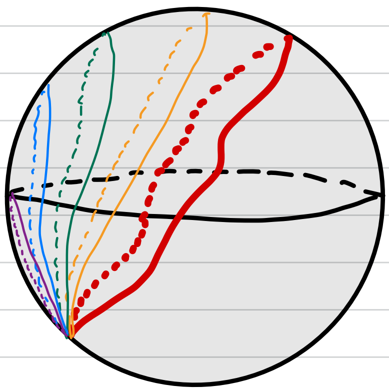
Since $(f_{x_0})_*$ and g_* are injective with $g_* \circ (f_{x_0})_*$ an isomorphism, this implies $(f_{x_0})_*$ is also surjective.



Section 59: The fundamental group of S^n

Thm S^n is simply connected for $n \geq 2$.

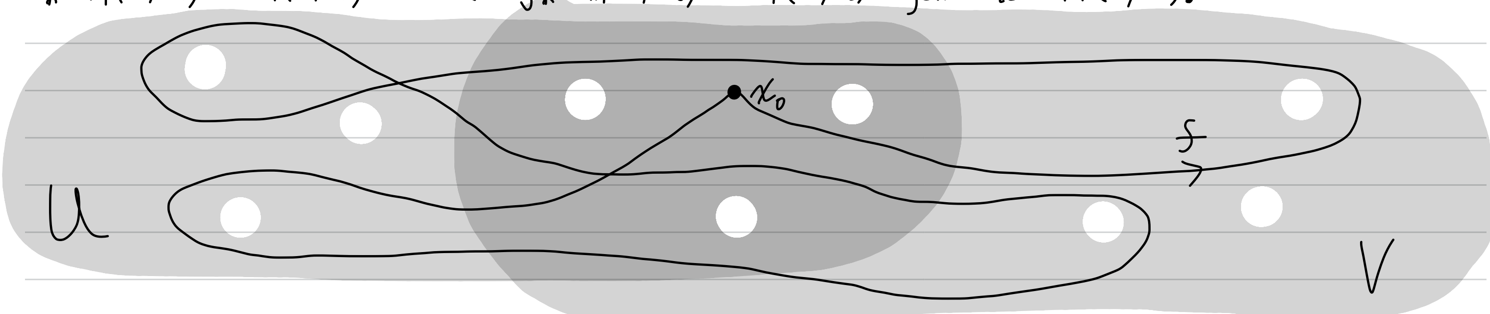
In particular, $\pi_1(S^n) = 0$ for $n \geq 2$.



This will allow us, in the next section, to show that the sphere, torus, and double torus are not homotopy equivalent.

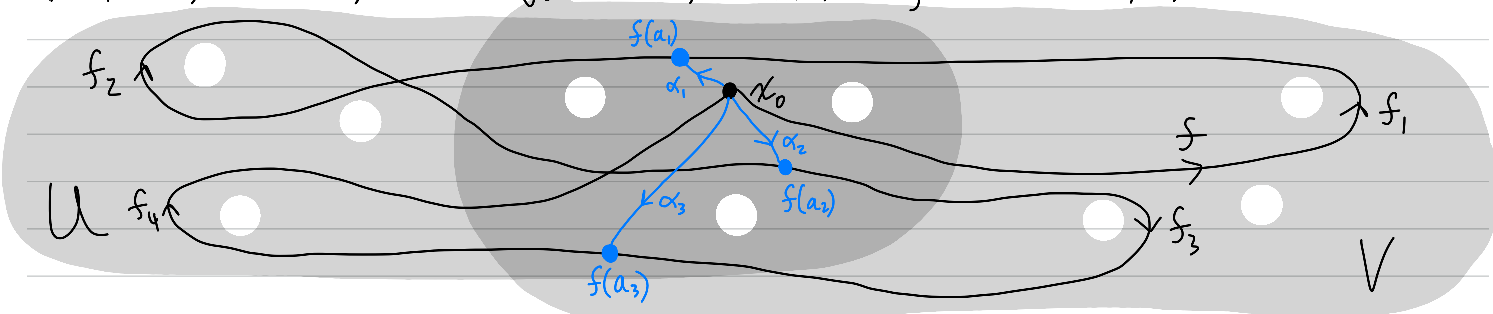
We will use the following special case of the Seifert-van Kampen theorem:

<u>Thm</u> Let $X = U \cup V$ with U, V open, $U \cap V$ path connected, $x_0 \in U \cap V$, and $i : U \hookrightarrow X$, $j : V \hookrightarrow X$ the inclusion maps. Then the images of $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$.



Here, "generates $\pi_1(X, x_0)$" means any loop f in X based at x_0 is path homotopic to one of the form $g_1 * g_2 * \ldots * g_n$, with each g_i a loop based at x_0 lying entirely in U or in V.

<u>Thm</u> Let $X = U \cup V$ with U, V open, $U \cap V$ path connected, $x_0 \in U \cap V$, and $i : U \hookrightarrow X$, $j : V \hookrightarrow X$ the inclusion maps. Then the images of $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$.

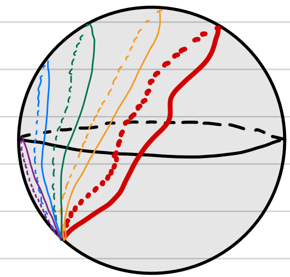


<u>Pf</u> Pick $0 = a_0 < a_1 < \dots < a_n = 1$ with $f(a_i) \in U \cap V$ and with $f([a_{i-1}, a_i])$ contained in U or in V $\forall i$. (Use Lebesgue Number Lemma in $[0,1]$.)
Let f_i be the path attained by restricting f to $[a_{i-1}, a_i]$.

Since $U \cap V$ is path connected, pick paths α_i in $U \cap V$ from x_0 to a_i, letting α_0 and α_n be constant paths at x_0. Define $g_i = \alpha_{i-1} * f_i * \overline{\alpha_i}$.
Note each g_i is a loop based at x_0 lying entirely in U or in V, with
$[g_1] * \dots * [g_n] = [f_1] * \dots * [f_n] = [f]$.

Corollary If $X = U \cup V$ where U and V are open and simply connected and $U \cap V$ is nonempty and path connected, then X is simply connected.

Thm S^n is simply connected for $n \geq 2$.



Pf Let $0 < \varepsilon \leq 1$.

Let $U = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} > -\varepsilon\}$

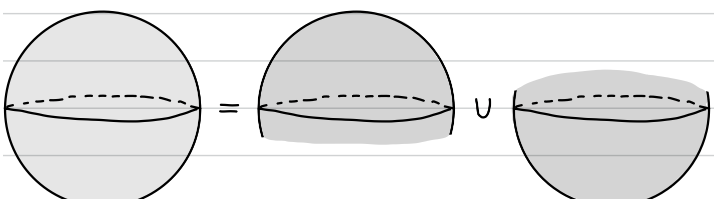
and $V = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < \varepsilon\}$.

Note $S^n = U \cup V$.

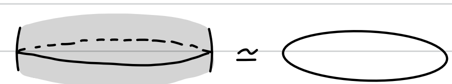
Note U and V are open and simply connected (they're contractible).

Note $U \cap V \simeq S^{n-1}$ is path connected for $n \geq 2$.

Hence by the corollary, S^n is simply connected.



$S^n = U \cup V$



$U \cap V \simeq S^{n-1}$

Section 60: Fundamental groups of some surfaces

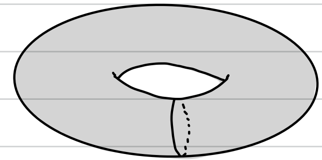
For $T = S^1 \times S^1$ the torus, $\mathbb{RP}^2$ the projective plane, and $T \# T$ the double torus, we will see

$\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

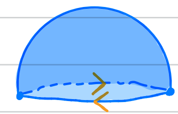
$\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$

$\pi_1(T \# T)$ is not abelian.

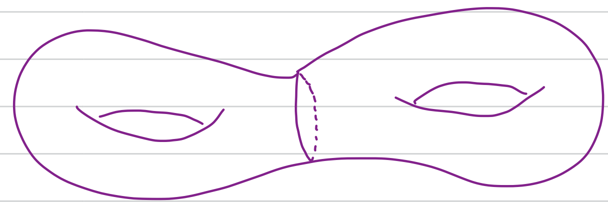
Since $\pi_1(S^2) \cong 0$, this shows that none of $S^2, T, \mathbb{RP}^2, T\#T$ are homotopy equivalent.



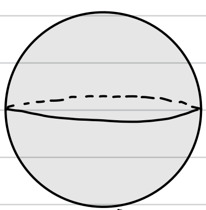
Torus $T = S^1 \times S^1$



Projective plane $\mathbb{RP}^2$

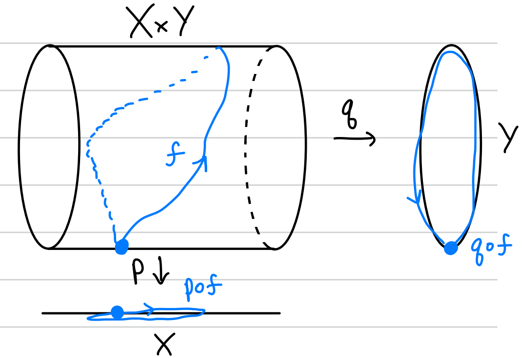


Double torus $T \# T$



S^2

Thm $\pi_1(X\times Y, (x_0,y_0))$ is isomorphic to $\pi_1(X,x_0)\times\pi_1(Y,y_0)$.

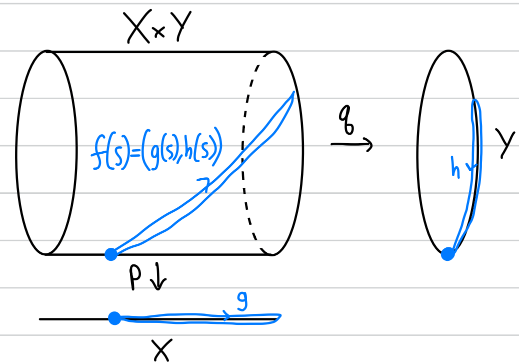


Pf If $p: X\times Y\to X$ and $q: X\times Y\to Y$ are projections, then $\Phi: \pi_1(X\times Y,(x_0,y_0))\longrightarrow \pi_1(X,x_0)\times\pi_1(Y,y_0)$ via

$$\Phi([f]) = (p_*([f]),\ q_*([f]))$$

is a group homomorphism.

Φ is surjective since if $[g]\in\pi_1(X)$, $[h]\in\pi_1(Y)$, then for $f: I\to X\times Y$ via $f(s)=(g(s),h(s))$, we have $\Phi([f]) = (p_*([f]), q_*([f])) = ([p\circ f],[q\circ f]) = ([g],[h])$.



Φ is injective since if

$$([e_{x_0}],[e_{y_0}]) = \Phi([f]) = (p_*([f]), q_*([f])) = ([p\circ f],[q\circ f]),$$

then $e_{x_0}\simeq p\circ f$ via G and $e_{y_0}\simeq q\circ f$ via H, so $e_{(x_0,y_0)}\simeq f$ via

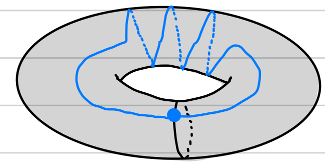
$$I\times I\to X\times Y$$
$$(s,t)\mapsto (G(s,t),H(s,t)).$$

Rmk See also the explanation in Hatcher Prop 1.12 — universal property of $X\times Y$

Rmk Similarly, $\pi_n(X\times Y)\cong\pi_n(X)\times\pi_n(Y)$ $\forall n\geq 1$.

Corollary $\pi_1(T) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

$\qquad\quad \underset{(3,\,1)}{\Psi}$



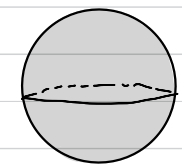
Torus $T = S^1 \times S^1$

Recall

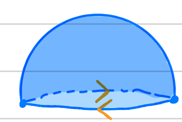
Real projective space $\mathbb{RP}^2$ is $\mathbb{RP}^2 = S^2/\sim$, where $x \sim -x \;\; \forall x \in S^2$.

The map $p : S^2 \longrightarrow \mathbb{RP}^2$ via

$$x \longmapsto \{x, -x\}$$ is a covering map.



$$S^2 \xrightarrow{\;p\;} \mathbb{RP}^2$$

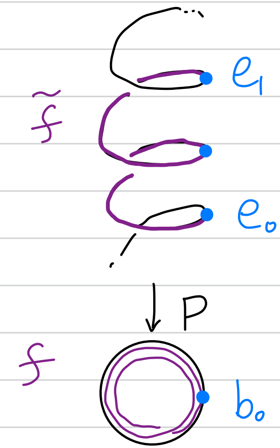


Recall

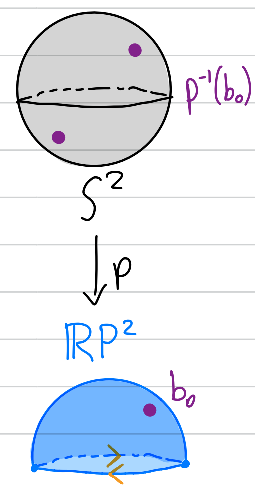
Let $p: E \to B$ be a covering map with $p(e_0) = b_0$.
Then $\phi: \pi_1(B, b_0) \to p^{-1}(b_0)$ defined by

$$\phi([f]) = \tilde{f}(1) \quad \text{where } \tilde{f} \text{ is the lift of } f \text{ with } \tilde{f}(0) = e_0$$

is a well-defined set map, called the <u>lifting correspondence</u>.



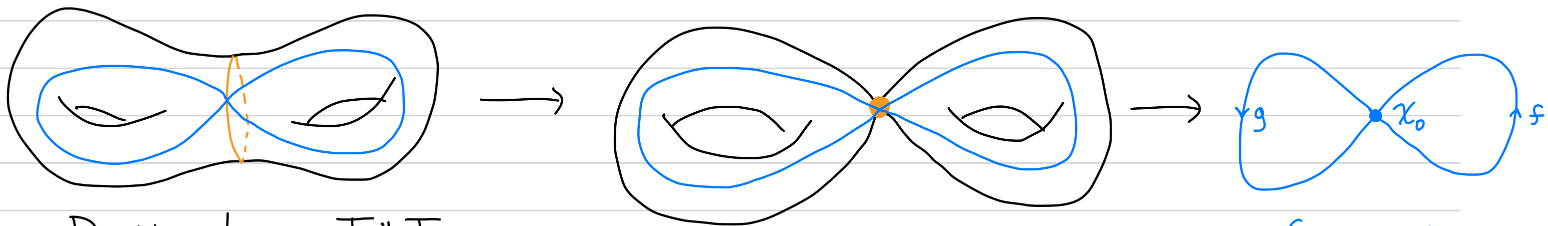
<u>Thm</u> If E is path connected, then ϕ is surjective.
If E is simply connected, then ϕ is bijective.



<u>Corollary</u> $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$.

<u>Pf</u> Since $p: S^2 \to \mathbb{R}P^2$, $x \longmapsto \{x, -x\}$ is a covering map with S^2 simply connected, we have that $\pi_1(\mathbb{R}P^2, b_0)$ is in bijective correspondence with $p^{-1}(b_0)$, a set of size 2. Hence $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$.

To see that $\pi_1(T\#T)$ is not abelian, consider the retraction $T\#T \to$ figure eight.



Double torus $T\#T$

figure eight

Retractions are surjective on π_1.
And π_1(figure eight) is not abelian since $[f]*[g] \neq [g]*[f]$.
Hence $\pi_1(T\#T)$ is not abelian.

To formally see that $[f]*[g] \neq [g]*[f]$ in π_1(figure eight), consider the covering space in Munkres' Figure 60.3.
Note $f*g$ lifts to a path from e_0 to e_1,
whereas $g*f$ lifts to a path from e_0 to e_{-1};
hence $[f]*[g] \neq [g]*[f]$ by Theorem 54.3.

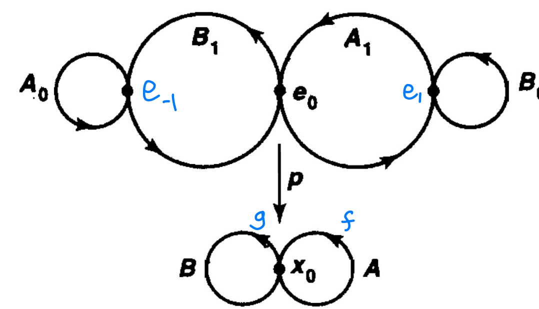


Figure 60.3