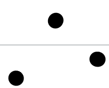
Introduction to manifolds

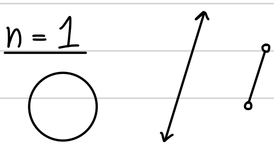
Def An n-manifold is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Means each $x \in M$ has a neighborhood $U \ni x$ homeomorphic to $\mathbb{R}^n$, or equivalently,
" " " " " " " " some open set in $\mathbb{R}^n$.

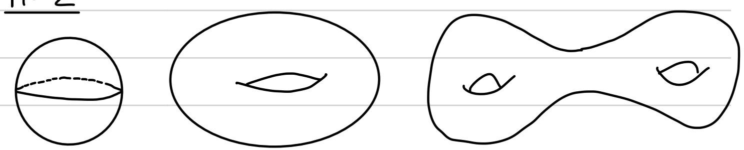
n=0

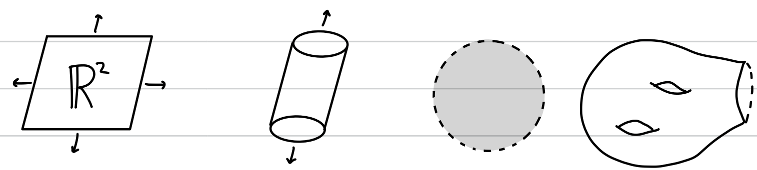


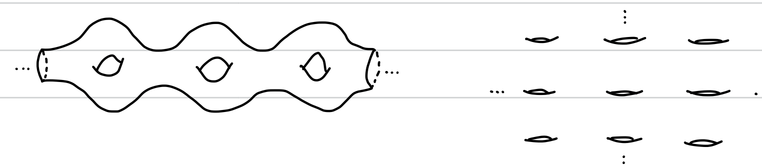
n=1



n=2



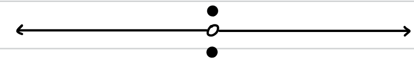




Dim n
- $\mathbb{R}^n$
- S^n
- $(S^1)^n$
- $\mathbb{R}P^n$
- many others!

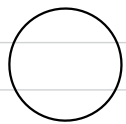
Dim 2n
- $\mathbb{C}P^n$

Second countable (topology has a countable basis) rules out the long line.

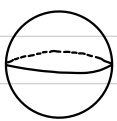
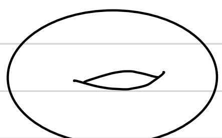
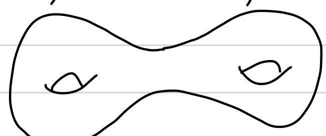
Hausdorff rules out the line with two origins 

The only closed connected n-dimensional manifolds (up to homeomorphism) are

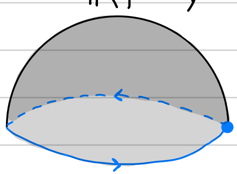
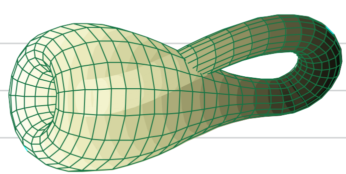
<u>$n=0$</u> The point 

<u>$n=1$</u> The circle 

<u>$n=2$</u> S^2, torus $\mathbb{T}^2 = S^1 \times S^1$, $\mathbb{T}^2 \# \mathbb{T}^2$, all genus g tori $M_g := \underbrace{\mathbb{T}^2 \# \dots \# \mathbb{T}^2}_{g \text{ times}}$,

$\mathbb{RP}^2$, Klein bottle $\mathbb{RP}^2 \# \mathbb{RP}^2$, all genus g nonorientable surfaces $N_g := \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{g \text{ times}}$.

$\underline{n=3}$ Hard! $\boxed{\text{Every closed connected 2-manifold can be given a metric with constant curvature (positive, zero, or negative).}}$

Proven by Perelman in 2006, declined Fields medal

Thurston's geometrization conjecture (now theorem) says each closed 3-manifold can be canonically decomposed into pieces with one of eight types of geometric structure. It implies the ...

Poincaré conjecture (now theorem) Every closed connected 3-manifold with trivial fundamental group is homeomorphic to S^3.

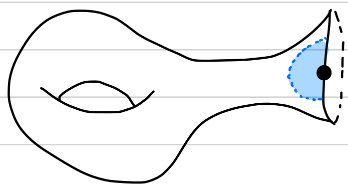
$\underline{n=4}$ Hard!

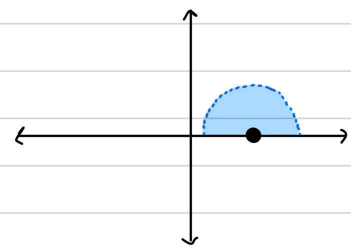
$\underline{n \geq 5}$ Hard, but some things get easier.

In 1961, Smale proved a generalized Poincaré conjecture (a homotopy n-sphere is homeomorphic to S^n) for $n \geq 5$.
In 1982, Freedman proved it for $n=4$.
The trivial π_1 assumption does not suffice for $n \geq 4$.

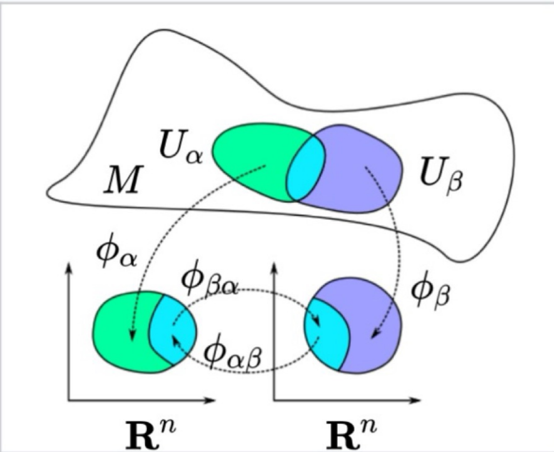
Invariance of dimension For $n \neq m$, a space cannot be both an n-manifold and an m-manifold.

Def An **n-manifold with boundary** (need not be a manifold!) is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$ or to $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.





From Wikipedia:



The transition map of two charts. $\phi_{\alpha\beta}$ denotes $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_{\beta\alpha}$ denotes $\phi_\beta \circ \phi_\alpha^{-1}$.

Given a topological space M...				
a C^k atlas	is a collection of charts	$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$	such that $\{U_\alpha\}_{\alpha\in A}$ covers M, and such that for all α and β in A, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is	a C^k map
a smooth or C^∞ atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$		a smooth map
an analytic or C^ω atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$		a real-analytic map
a holomorphic atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{C}^n\}_{\alpha\in A}$		a holomorphic map

A <u>differentiable (C^k or C^∞)</u> manifold is a second-countable Hausdorff space M equipped with a maximal differentiable atlas.

<u>Whitney embedding theorem</u> A C^∞ n-manifold can be smoothly embedded in $\mathbb{R}^{2n}$.
(For n a power of 2, $\mathbb{RP}^n$ cannot be embedded in $\mathbb{R}^{2n-1}$.)