Course overview

Section 2.3: The formal viewpoint
Section 2.C: Simplicial approximation & Lefschetz fixed point theorem

Chapter 3: Cohomology (Idea of cohomology pg. 186)
Section 3.1: Cohomology groups

- Universal coefficient theorem
- Cohomology of spaces, pg. 197

swap

Section 3.2: Cup product
Section 3.3: Poincaré duality

For M an n-dimensional orientable closed manifold, we have $H^k(M) \cong H_{n-k}(M)$ for all k.

Two key differences between homology and cohomology are

- Homology is a covariant functor while cohomology is contravariant. A continuous function $f: X \to Y$ induces $f_*: H_i(X) \longrightarrow H_i(Y)$ on homology, but $f^*: H^i(Y) \longrightarrow H^i(X)$ on cohomology.

- Cohomology has a natural product, the cup product $H^i(X) \times H^j(X) \xrightarrow{\smile} H^{i+j}(X)$, defined as a composition

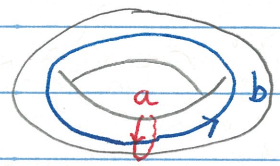
$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \longrightarrow H^{i+j}(X).$$

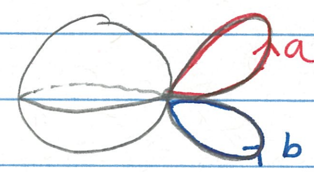
In both cohomology and homology, one can construct natural maps $H^i(Z) \times H^j(W) \longrightarrow H^{i+j}(Z \times W)$ and $H_i(Z) \times H_j(W) \longrightarrow H_{i+j}(Z \times W)$.

By contravariance, the diagonal map $X \to X \times X$ via $x \mapsto (x,x)$ induces this map on cohomology. No nice analogue for homology, in general.

Example of the cup product

Homology can't tell the difference between the torus $S^1 \times S^1$ and $S^2 \vee S^1 \vee S^1$.





$$H_i(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases} \qquad H_i(S^2 \vee S^1 \vee S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases}$$

Follows from $\tilde{H}_i(X \vee Y) \cong \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$

The cup product structure on cohomology can tell the difference!

$$H^i(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\times\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases} \qquad H^i(S^2 \vee S^1 \vee S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\times\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases}$$

Let $[a]$ and $[b]$ be generators for H^1, with $[T]$ a generator for H^2. We have

$$[a] \cup [b] = [T]$$

Let $[a]$ and $[b]$ be generators for H^1. We have

$$[a] \cup [b] = 0$$

$$H^1(X) \times H^1(X) \xrightarrow{\cup} H^2(X)$$

Section 2.3: The formal viewpoint

Consider the category of CW complexes and continuous maps.
A <u>reduced homology theory</u> is a functor assigning

- to each nonempty CW complex X a sequence of abelian groups $\tilde{h}_n(X)$, and
- to each map $f: X \to Y$ a sequence of homomorphisms $f_*: \tilde{h}_n(X) \to \tilde{h}_n(Y)$

(functor means $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$)

satisfying the following three axioms:

(1) If $f \simeq g$, then $f_* = g_*$.

(2) There are natural boundary homomorphisms $\partial: \tilde{h}_n(X/A) \to \tilde{h}_n(A)$ $\forall$ CW pairs (X, A), fitting into an exact sequence

$$\cdots \longrightarrow \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \longrightarrow \cdots$$

(Natural means $f: (X, A) \to (Y, B)$ inducing $\bar{f}: X/A \to Y/B$ gives

$$\begin{tikzcd} \tilde{h}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{h}_{n-1}(A) \arrow[d, "f_*"] \\ \tilde{h}_n(Y/B) \arrow[r, "\partial"'] & \tilde{h}_{n-1}(B) \end{tikzcd}$$

)

(3) For $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha: X_\alpha \hookrightarrow X$, the map $\bigoplus_\alpha i_\alpha: \bigoplus_\alpha \tilde{h}_n(X_\alpha) \to \tilde{h}_n(X)$ is an isomorphism $\forall n$.

Rmks

- In the case of finite wedge sums, (3) can be deduced from (1) and (2).

- Can deduce the Mayer-Vietoris LES, for example.

- Could give axioms instead for unreduced homology (Eilenberg & Steenrod 1952).

- The original 'dimension axiom' $\tilde{h}_n(S^0) = h_n(pt) = 0$ for $n \neq 0$ is no longer included!
  Ex In the bordism homology theory, $\tilde{h}_n(S^0) = h_n(pt) \neq 0$ for infinitely many n.

- Any sequence of coefficient groups $\tilde{h}_n(S^0) = h_n(pt)$ is possible, for example via $\tilde{h}_n(X) = \bigoplus_i \tilde{H}_{n-i}(X; G_i)$. Negative n is fine.

- A general homology theory is not uniquely identified by its coefficient groups, though singular homology is (Thm 4.59).

Categories and functors

Def A *category* $\mathcal{C}$ consists of

(1) A collection $Ob(\mathcal{C})$ of objects.

(2) Sets $Mor(X,Y)$ of morphisms $\forall X,Y \in Ob(\mathcal{C})$, including a distinguished 'identity' $\mathbb{1}_X \in Mor(X,X)$ $\forall X$.

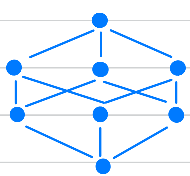
(3) A composition $\circ : Mor(X,Y) \times Mor(Y,Z) \longrightarrow Mor(X,Z)$ $\forall X,Y,Z$,

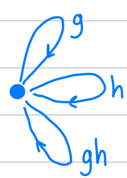
$$(f, g) \longmapsto gf$$

satisfying $f\mathbb{1} = f$, $\mathbb{1}f = f$, and $(fg)h = f(gh)$.

In the category of sets, groups, spaces, the morphisms are functions, perhaps with extra structure (group homomorphisms, continuous maps).

Consider also posets, a group G, a homotopy category, chain complexes.





$$X \xrightarrow{[f]} Y$$

Composition well-defined since $fg \simeq f'g'$ if $f \simeq f'$ and $g \simeq g'$.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[d] \\
C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] & C_{n-1}
\end{tikzcd}$$

Def A *(covariant) functor* $F: \mathcal{C} \to \mathcal{D}$
assigns to each $X \in Ob(\mathcal{C})$ some $F(X) \in Ob(\mathcal{D})$ and
to each $f \in Mor(X,Y)$ some $F(f) \in Mor(F(X), F(Y))$
such that $F(\mathbb{1}) = \mathbb{1}$ and $F(fg) = F(f)F(g)$.

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

$$\begin{tikzcd} F(X) \arrow[r, "F(g)"] \arrow[rr, bend right, "F(fg)=F(f)F(g)"'] & F(Y) \arrow[r, "F(f)"] & F(Z) \end{tikzcd}$$

Ex Homology. $\mathbb{1}_* = \mathbb{1}$ and $(fg)_* = f_* g_*$.

A *contravariant functor* instead assigns
to each $f \in Mor(X,Y)$ some $F(f) \in Mor(F(Y), F(X))$
with $F(\mathbb{1}) = \mathbb{1}$ and $F(fg) = F(g)F(f)$.

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(g)"'] & F(Z) \arrow[l, "F(f)"'] \arrow[ll, bend left, "F(fg)=F(g)F(f)"] \end{tikzcd}$$

Ex The dual vector space functor $F(V) = V^*$
assigning the vector space V to V^*,
the space of linear maps $V \to \mathbb{R}$.

$$V \xrightarrow{f} W$$

$V^* \xleftarrow{f^*} W^*$ since
$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed] & W \arrow[d] \\ & \mathbb{R} \end{tikzcd}$$

Ex Cohomology