Chapter 1: Fundamental group

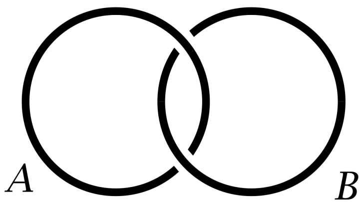
Associates to each space X a group $\pi_1(X)$ measuring the 1-dimensional holes.

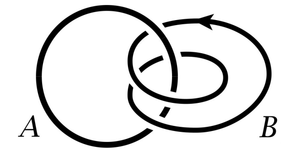
Section 1.1 Basic constructions

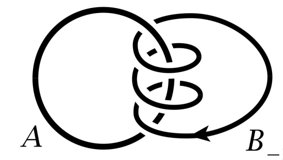
Section 1.2 Van Kampen's theorem

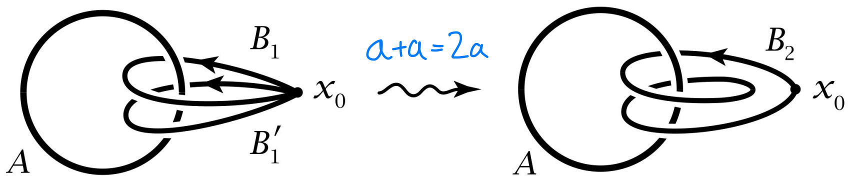
Section 1.3 Covering spaces

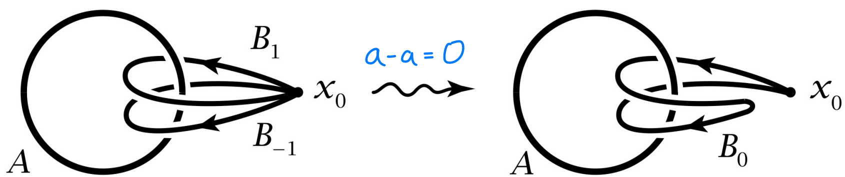
The idea What is the algebraic structure of loops B in the complement of a single loop A ?





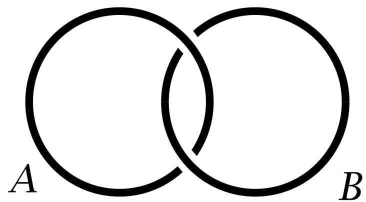


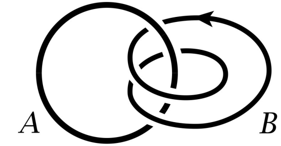


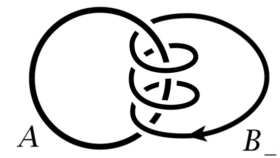


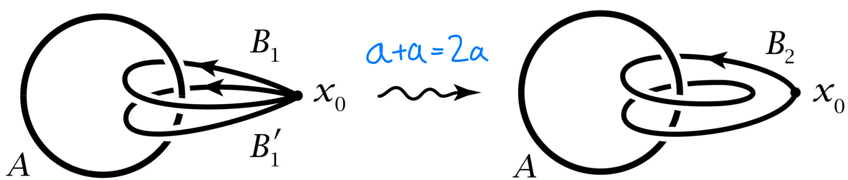
$\mathbb{R}^3 \setminus A \simeq$?

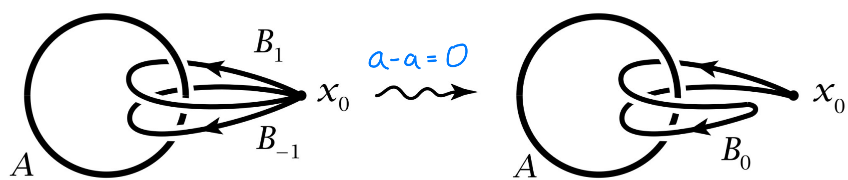
<u>The idea</u> What is the algebraic structure of loops B in the complement of a single loop A?

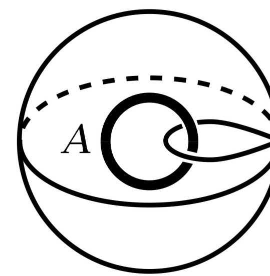








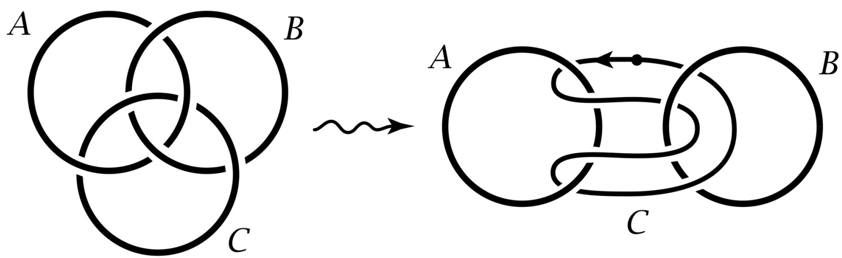




$$\mathbb{R}^3 \setminus A \simeq S^1 \vee S^2$$

$$\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$$

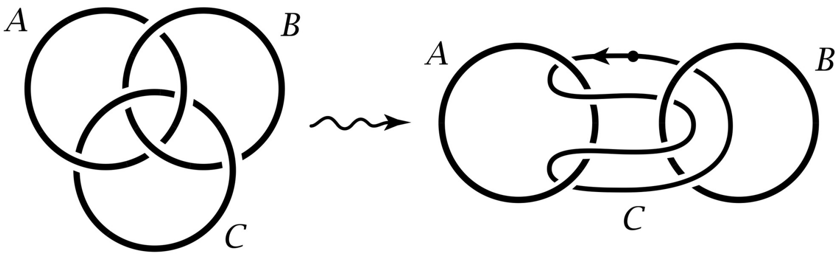
The idea What is the algebraic structure of loops C in the complement of two unlinked loops A and B?

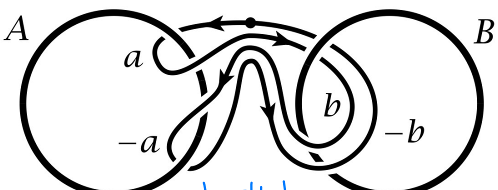


$aba^{-1}b^{-1} \neq 1$

$\mathbb{R}^3 \setminus (A \cup B) \simeq ?$

__The idea__ What is the algebraic structure of loops C in the complement of two unlinked loops A and B ?





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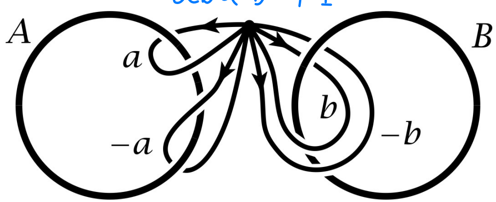
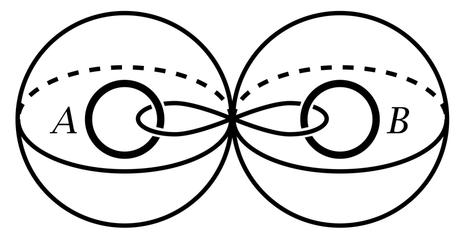
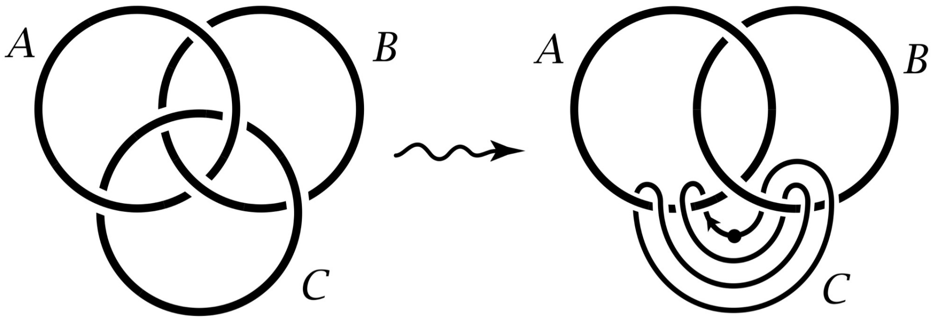


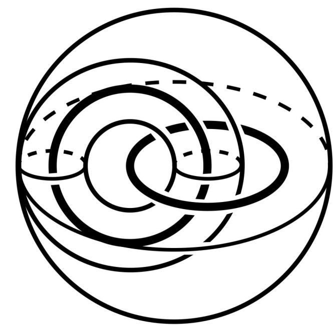


$$\mathbb{R}^3 \setminus (A \cup B) \simeq S^1 \vee S^1 \vee S^2 \vee S^2$$
$$\pi_1(S^1 \vee S^1 \vee S^2 \vee S^2) \cong \langle a, b \rangle \cong \mathbb{Z} * \mathbb{Z}$$

__The idea__ What is the algebraic structure of loops C in the complement of two linked loops A and B ?



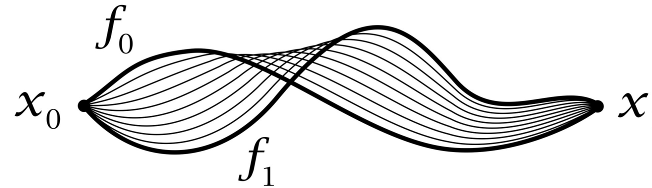
$aba^{-1}b^{-1} = 1$



$\mathbb{R}^3 \setminus (A \cup B) \simeq \text{torus} \vee S^2$

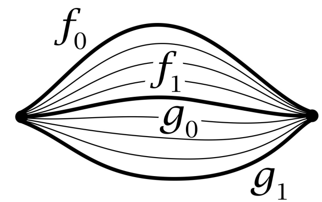
$\pi_1(\text{torus} \vee S^2) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

Section 1.1 Basic constructions



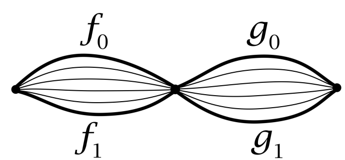
A *path* is a map $f: I \to X$

A *homotopy of paths* is a homotopy $f_t: I \to X$ rel $\{0,1\}$.
We say f_0 and f_1 are homotopic, denoted $f_0 \simeq f_1$,
or $[f_0] = [f_1]$ since this is an equivalence relation.

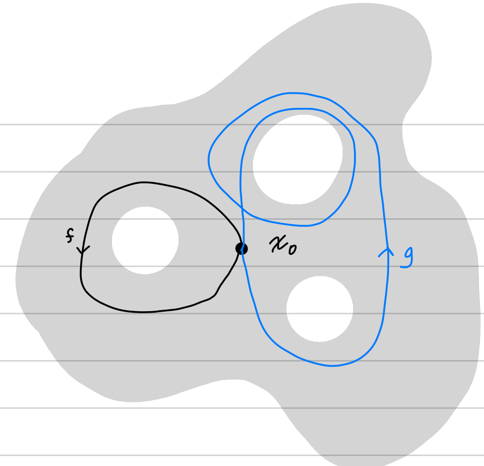


The *product* of paths $f, g: I \to X$
with $f(1) = g(0)$ is defined by
$$f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1. \end{cases}$$

The product respects homotopy classes.

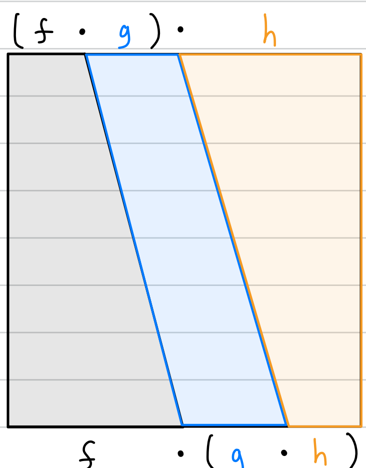


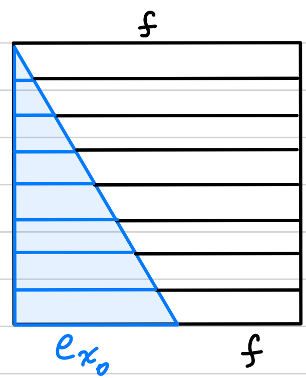
Path $f: I \to X$ is a <u>loop</u> if $f(0) = x_0 = f(1)$.
The set of all homotopy classes of loops based at x_0 is denoted $\pi_1(X, x_0)$.



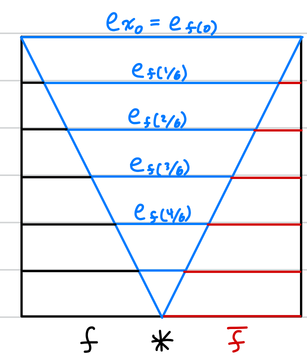
<u>Proposition 1.3</u> $\pi_1(X, x_0)$ is a group with respect to the product $[f]\cdot[g] = [f \cdot g]$.

<u>Pf</u> See how Hatcher uses reparametrizations to prove associativity, identity.





Inverses: $\bar{f}: I \to X$ by $\bar{f}(s) = f(1-s)$.



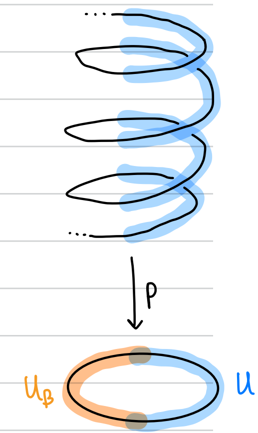
The fundamental group of the circle

We prove $\pi_1(S^1) \cong \mathbb{Z}$ using covering spaces.

Def (page 56) A covering space of X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$ such that

- there is an open cover $\{U_\alpha\}$ of X s.t. $\forall \alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ each mapped homeomorphically onto U_α by p.

Ex $p: \mathbb{R} \to S^1$ by $p(t) = (\cos 2\pi t, \sin 2\pi t)$



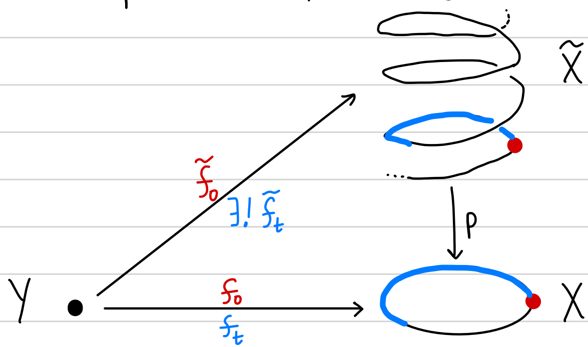
Def (page 60) A lift of a map $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ with $p\tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

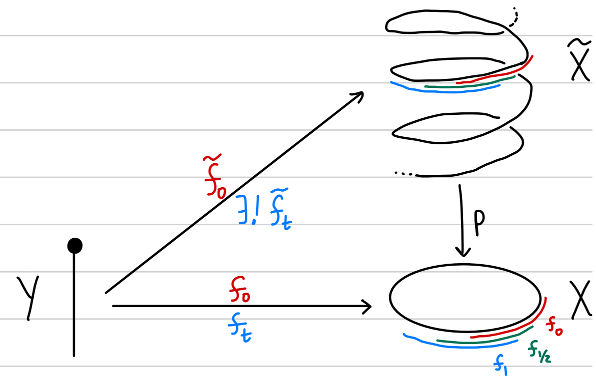
Prop 1.30 Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a lift $\tilde{f}_0: Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ that lifts f_t.

$$\left(\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "\{f_t\}"] \arrow[ur, dashed, "\exists!\ \{\tilde{f}_t\}"] & X \end{tikzcd}\right)$$

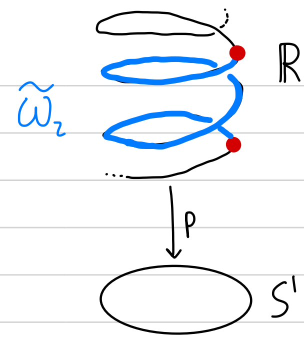
Ex $Y = pt = \{0\}$ (path lifting)



Ex $Y = I$ (homotopy lifting)



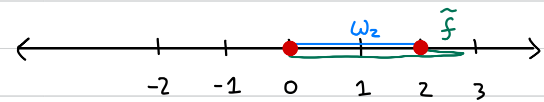
Pf idea Unique lifts over an open set exist by the homeomorphism property $(\bullet)$. Piecing these together takes a page in Hatcher.



Thm 1.7 $\Phi:\mathbb{Z}\to\pi_1(S^1)$ via $\Phi(n)=[\omega_n]$, where $\omega_n(s)=(\cos 2\pi ns,\ \sin 2\pi ns)$, is an isomorphism.

Pf Note $\tilde{\omega}_n: I\to\mathbb{R}$ via $\tilde{\omega}_n(s)=ns$ lifts $\omega_n: I\to S^1$ (since $p\tilde{\omega}_n=\omega_n$).

Note $\Phi(n)=[\omega_n]=[p\tilde{\omega}_n]=[p\tilde{f}]$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n (since $\tilde{f}\simeq\tilde{\omega}_n$ by a linear homotopy).



Φ is a homomorphism

Let $\tau_m:\mathbb{R}\to\mathbb{R}$ translate $\tau_m(x)=m+x$.

Note $\tilde{\omega}_m\cdot(\tau_m\tilde{\omega}_n)$ is a path in $\mathbb{R}$ from 0 to $m+n$

So $\Phi(m+n)=[p\,(\tilde{\omega}_m\cdot(\tau_m\tilde{\omega}_n))]=[\omega_m\cdot\omega_n]=\Phi(m)\cdot\Phi(n)$.

Φ is surjective

Let $f: I\to S^1$ be a loop based at $(1,0)$.

By Prop 1.30 (path lifting) $\exists!$ lift $\tilde{f}: I\to\mathbb{R}$ with $\tilde{f}(0)=0$.

Necessarily $\tilde{f}_1(1)=n$ for some $n\in\mathbb{Z}$, giving $\Phi(n)=[p\tilde{f}]=[f]$.

Φ is injective

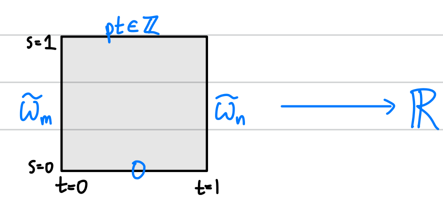
Suppose $\Phi(m)=\Phi(n)$, so $\overset{f_0}{\overset{\|}{\omega_m}} \underset{f_t}{\simeq} \overset{f_1}{\overset{\|}{\omega_n}}$.

By Prop 1.30 (homotopy lifting) $\exists$ lift $\tilde{f}_t: I\to\mathbb{R}$.

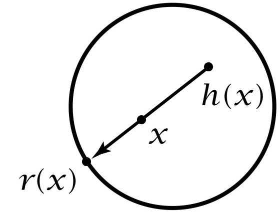
Necessarily $\tilde{f}_t(0)=0$ $\forall t$ and $\tilde{f}_t(1)=pt\in\mathbb{Z}$ $\forall t$.

By uniqueness of path lifting, $\tilde{f}_0=\tilde{\omega}_m$ and $\tilde{f}_1=\tilde{\omega}_n$.

Hence $m=\tilde{\omega}_m(1)=\tilde{\omega}_n(1)=n$.



Applications of $\pi_1(S^1) \cong \mathbb{Z}$



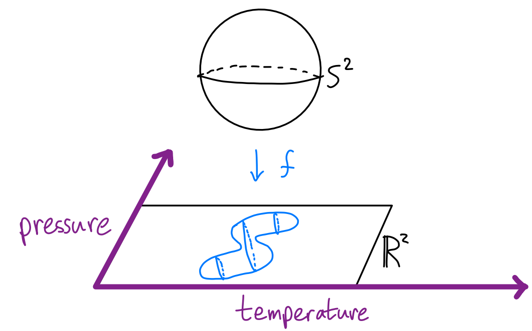
Thm 1.8 Every nonconstant polynomial has a root in $\mathbb{C}$.

Thm 1.9 $n=2$ case of Brouwer fixed point theorem (Cor 2.15):

"Every map $h: D^n \to D^n$ has a fixed point: $h(x)=x$."

Thm 1.10 $n=2$ case of Borsuk-Ulam theorem (Cor 2B.7):

"Every map $f: S^n \to \mathbb{R}^n$ identifies some antipodal pair: $f(x)=f(-x)$."



Cor 1.11 $n=2$ case of:

"If S^n is the union of $n+1$ closed sets,
then some set contains an antipodal pair $\{x,-x\}$."

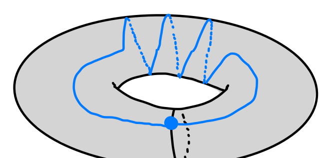
Cor 1.16 $n=2$ case of Invariance of Dimension (Thm 2.26):

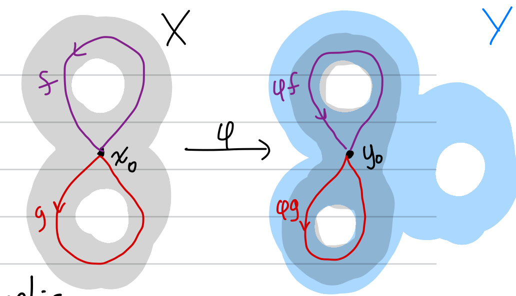
"$\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$."

Prop 1.12 $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Ex 1.13 $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

$\pi_1((S^1)^n) \cong \mathbb{Z}^n$





Induced homomorphisms

Def A map $\varphi : (X, x_0) \to (Y, y_0)$ induces a group homomorphism

$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $\varphi_*([f]) = [\varphi f]$.

$\varphi_*([f]\cdot[g]) = [\varphi(f\cdot g)] = [\varphi f \cdot \varphi g] = \varphi_*([f])\cdot \varphi_*([g])$

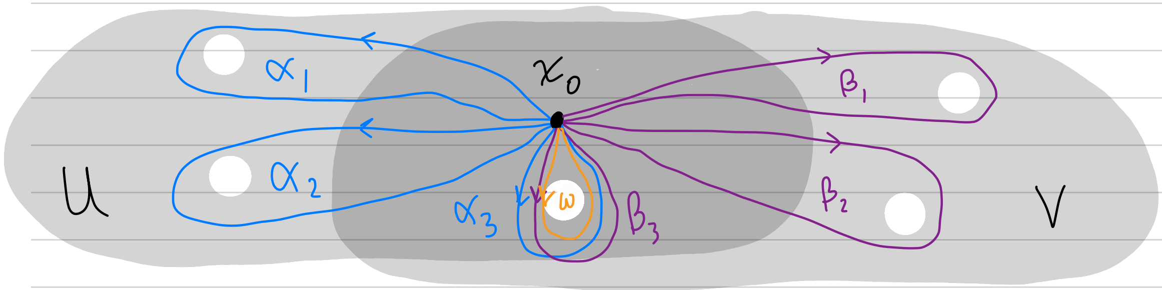
well-defined since $f_0 \simeq f_1$ via $f_t \Rightarrow \varphi f_0 \simeq \varphi f_1$ via φf_t

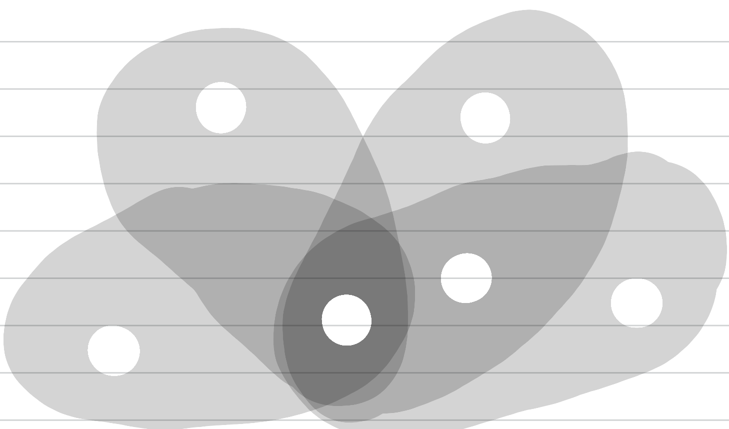
Functor π_1 is a functor (see §2.3) since

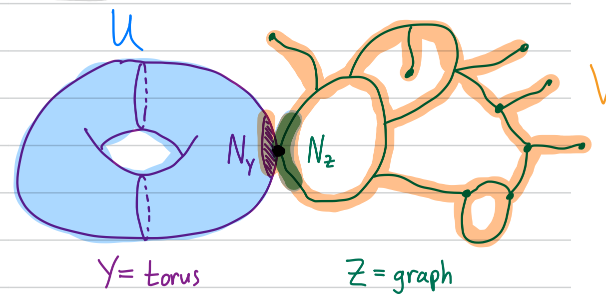
- $(\varphi\psi)_* = \varphi_* \psi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$.
- $\mathbb{1}_* = \mathbb{1}$, i.e., $\mathbb{1} : X \to X$ induces $\mathbb{1} : \pi_1(X, x_0) \to \pi_1(X, x_0)$.

Prop 1.18 If $\varphi : X \to Y$ is a homotopy equivalence, then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.

Section 1.2 Van Kampen's theorem (arbitrary unions)







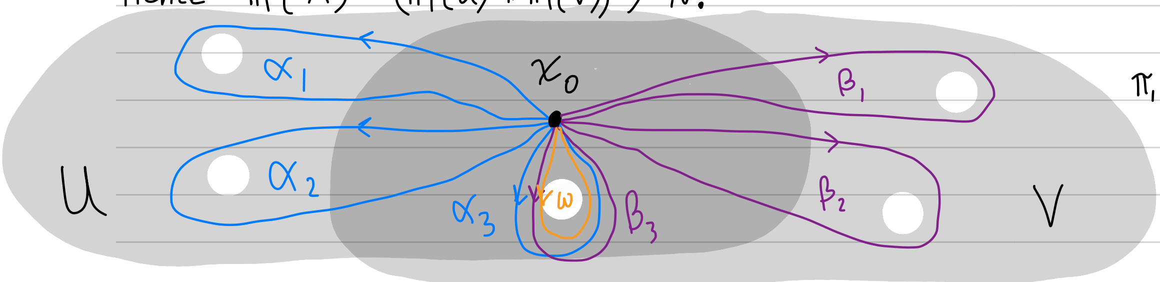
RECALL The Seifert-van Kampen theorem (Two set version, from Munkres!)

Thm (Seifert-van Kampen) Let $X = U \cup V$ with U, V open in X, with $U, V, U \cap V$ path-connected, and $x_0 \in U \cap V$. Then the homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective, and its kernel N is the least normal subgroup containing all words of the form $i_U(w)^{-1} i_V(w)$ for $w \in \pi_1(U \cap V, x_0)$.

Hence $\pi_1(X) \cong (\pi_1(U) * \pi_1(V)) / N$.



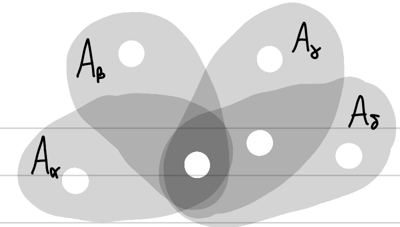
$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "j_U"] & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "j_V"'] &
\end{tikzcd}$$

$\alpha_3 \in \pi_1(U)$, $\exists w \mapsto \alpha_3$, $w \mapsto \beta_3$, $\beta_3 \in \pi_1(V)$

$\pi_1(U) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \qquad \pi_1(V) = \langle \beta_1, \beta_2, \beta_3 \rangle \qquad \pi_1(U \cap V) = \langle w \rangle$

$$\begin{aligned}
\pi_1(X) &\cong \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid i_U(w)^{-1} i_V(w) \rangle \\
&= \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \mid \alpha_3^{-1} \beta_3 \rangle \\
&\cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2, w \rangle
\end{aligned}$$

Note $i_U(w) = \alpha_3$ and $i_V(w) = \beta_3$



Section 1.2 Van Kampen's theorem (arbitrary unions)

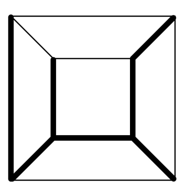
Thm 1.20 Let $X = \bigcup_\alpha A_\alpha$ s.t. each A_α is open, path-connected, and contains x_0.

- If each $A_\alpha \cap A_\beta$ is path-connected, then $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ (defined using the j_α) is surjective.
  If furthermore each $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then $\ker(\Phi)$ is the normal subgroup generated by all elements $i_{\alpha\beta}(w)\, i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$.

(For $g_\alpha, \tilde{g}_\alpha \in \pi_1(A_\alpha)$, $g_\beta \in \pi_1(A_\beta)$, $g_\gamma \in \pi_1(A_\gamma)$, $\Phi(g_\alpha g_\beta^3 \tilde{g}_\alpha g_\gamma^{-2}) = j_\alpha(g_\alpha)\, j_\beta(g_\beta)^3\, j_\alpha(\tilde{g}_\alpha)\, j_\gamma(g_\gamma)^{-2}$.)

$$\begin{tikzcd} & \pi_1(A_\alpha) \arrow[dr, "j_\alpha"] & \\ \pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X) \\ & \pi_1(A_\beta) \arrow[ur, "j_\beta"'] & \end{tikzcd}$$

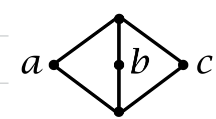
Ex Write X as union of 5 open sets each containing the bold tree and one extra edge.



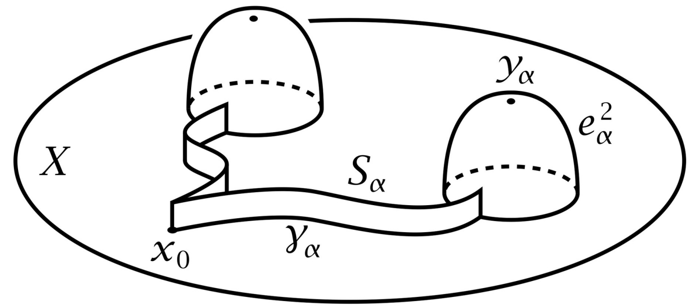
Double, triple intersections path-connected.
Double intersections contractible $\Rightarrow \ker(\Phi)$ trivial.

So Φ gives an isomorphism $\pi_1(X) \cong *_{i=1}^5 \pi_i(A_i) = *_{i=1}^5 \mathbb{Z}$.

Non-Ex To see the triple intersection assumption is necessary, consider $A_\alpha = X \setminus \{a\}$, $A_\beta = X \setminus \{b\}$, $A_\gamma = X \setminus \{c\}$.



(To get the right answer $\mathbb{Z} * \mathbb{Z}$, use only A_α and A_β.)



Applications to cell complexes

Y obtained from path-connected X by attaching 2-cells e^2_α via $\varphi_\alpha : S^1 \to X$.
Fix $x_0 \in X$. Choose paths γ_α to image(φ_α).
Let $N \subseteq \pi_1(X, x_0)$ be normal subgroup generated by all $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$.

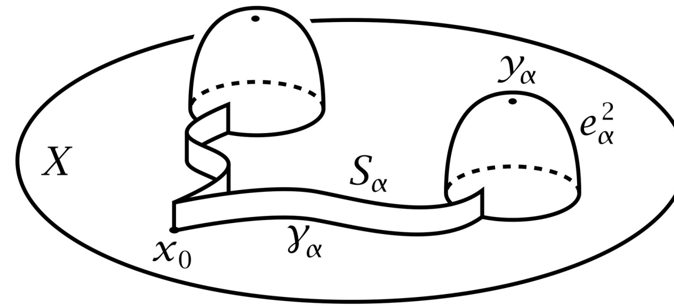
Prop 1.26

(a) $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ with kernel N, so $\pi_1(Y) \cong \pi_1(X)/N$.

(b) If instead Y were obtained by attaching n-cells for some $n > 2$, then $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.

(c) For X a path-connected CW complex, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \cong \pi_1(X)$.

Rmk Choice of path γ_α doesn't matter, since a different path η_α gives a conjugate element
$\eta_\alpha \varphi_\alpha \overline{\eta_\alpha} = (\eta_\alpha \overline{\gamma_\alpha}) \gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha} (\gamma_\alpha \overline{\eta_\alpha})$.



Pf (a)
Let $Z = Y \cup \{\text{rectangular strips}\} \simeq Y$.
Choose $y_\alpha \in e^2_\alpha$.
Note $A = Z - \bigcup_\alpha \{y_\alpha\} \simeq X$ and $B = Z - X \simeq *$
are open path-connected sets with union Z.
Note $A \cap B \simeq \bigvee_\alpha S^1$ with $\pi_1(A \cap B)$ generated
(loosely speaking) by $[\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha]$.

Van Kampen's says $\pi_1(Y) \cong \pi_1(Z)$ is isomorphic to the quotient of
$\pi_1(A) \cong \pi_1(X)$ by the normal subgroup generated by the image of $\pi_1(A \cap B) \to \pi_1(A)$,
which corresponds to N.

(b) The only difference with the above proof is $A \cap B \simeq \bigvee_\alpha S^{n-1}$, with $n > 2$.
So $\pi_1(A \cap B)$ is trivial and van Kampen's gives $\pi_1(Y) \cong \pi_1(Z) \underset{\text{van Kampen}}{\cong} \pi_1(A) \cong \pi_1(X)$.

(c) If X is finite-dimensional ($X = X^n$ for some n), then (c) follows from (b) and induction.
(Add on 3-cells, then 4-cells, etc.)

Otherwise, let $f : I \to X$ be a loop based at $x_0 \in X^2$. $\mathrm{Im}(f)$ is compact and hence lives in a finite subcomplex of X by Proposition A.1, and hence in X^n for some n. Since $\pi_1(X^2) \to \pi_1(X^n)$ is surjective by (b), f is homotopic to a loop in X^2. So $\pi_1(X^2) \to \pi_1(X)$ is surjective.

To see it is also injective, suppose f is a loop in X^2 which is nullhomotopic in X via a nullhomotopy $F : I \times I \to X$. $\mathrm{Im}(F)$ is compact, hence lies in X^n for some $n \geq 2$. Since $\pi_1(X^2) \to \pi_1(X^n)$ is injective by (b), it follows that f is nullhomotopic in X^2.

Corollary For every group G there is a 2-dimensional CW complex X_G with $\pi_1(X_G) \cong G$

Pf Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$, which exists since every group is a quotient of a free group.

Construct X_G from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β via loops specified by the words r_β.

Ex $G = \mathbb{Z}/n\mathbb{Z}$