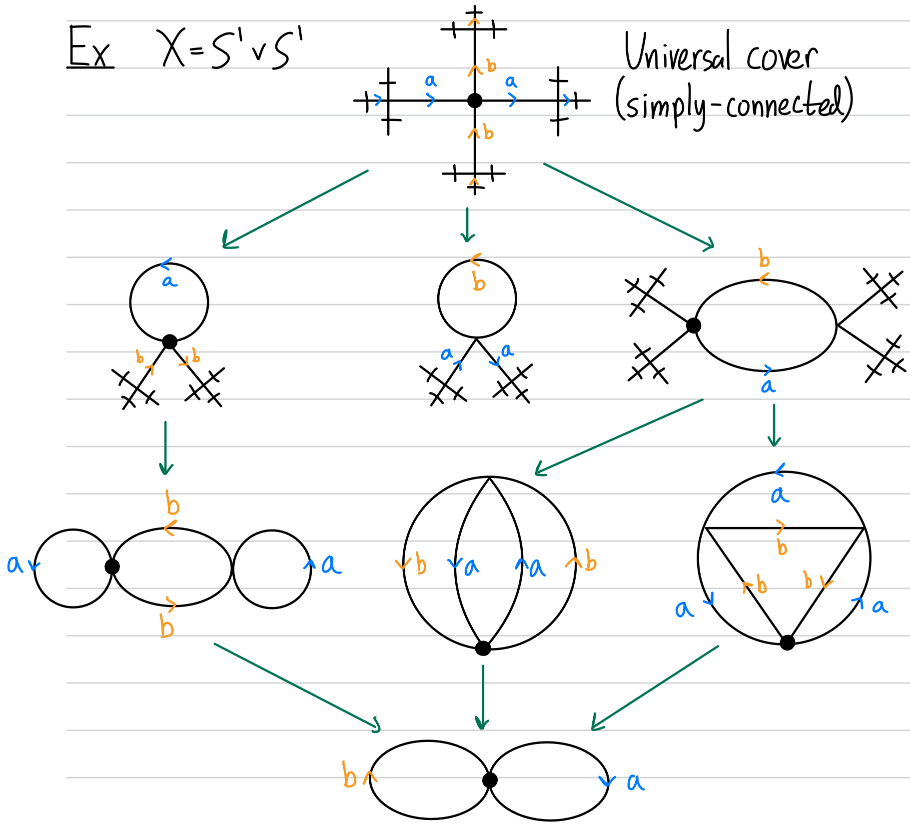
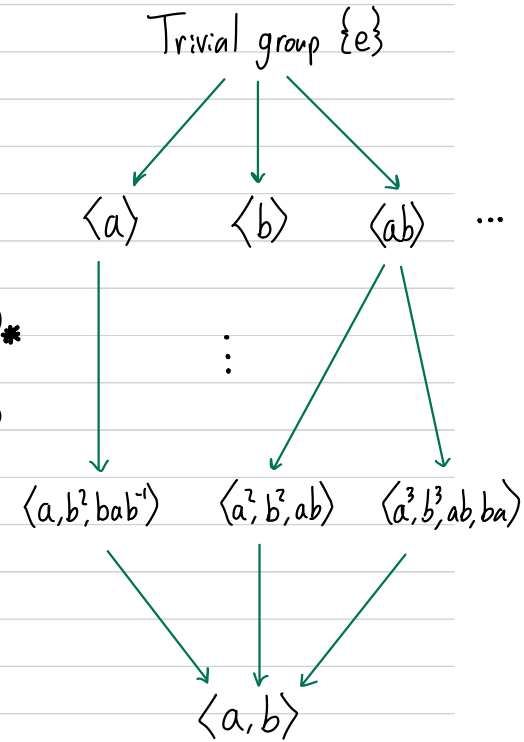
Section 1.3 Covering spaces

Ex $X = S^1 \vee S^1$

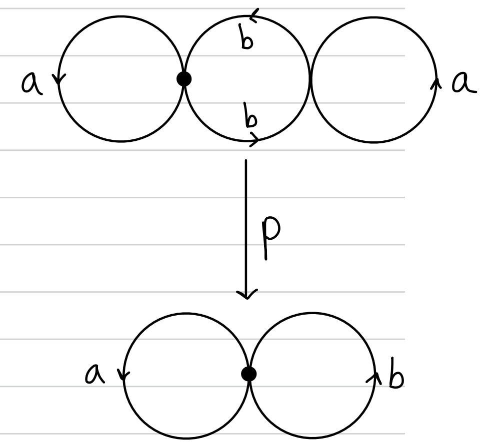


Apply p_* to π_1 $\rightsquigarrow$



Prop 1.30 Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a lift $\tilde{f}_0: Y \to \tilde{X}$ of f_0, there exists a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ that lifts f_t.

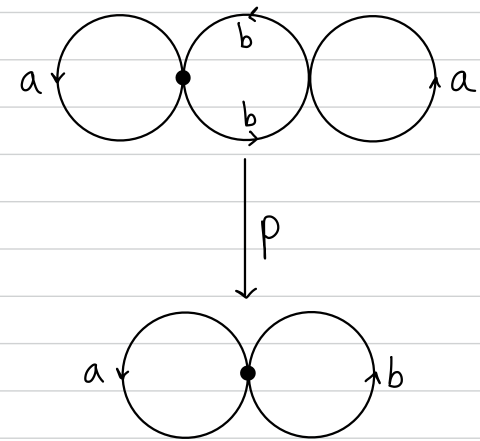
Prop 1.31 Let $p: \tilde{X} \to X$ be a covering space. Then $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. Also, $\mathrm{Image}(p_*)$ is all homotopy classes of loops in X that lift to loops (not paths) in $\tilde{X}$.



Pf If $[f] \in \ker(p_*)$, then pf is nullhomotipic in X. By Prop 1.30 we can lift to see f is nullhomotopic in $\tilde{X}$.

Clearly loops lifting to loops represent elements in $\mathrm{Image}(p_*)$. Conversely, $[g] \in \mathrm{Image}(p_*)$ implies $g \simeq g'$ with g' lifting to a loop, which by Prop 1.30 means g lifts to a loop.

Prop 1.32 Let $p: \tilde{X} \to X$ be a covering space with X and $\tilde{X}$ path-connected. The number of sheets $|p^{-1}(x_0)|$ is equal to the index $[\pi_1(X) : H]$, where $H = p_* \pi_1(\tilde{X})$.



Pf Define $\Phi : \{\text{cosets of } H\} \longrightarrow p^{-1}(x_0)$ by

$$H[g] \longmapsto \tilde{g}(1)$$

where $\tilde{g}$ is a lift of g starting at $\tilde{x}_0$.

Φ is well-defined since elements of H lift to loops.

Φ is surjective since $\tilde{X}$ is path-connected.

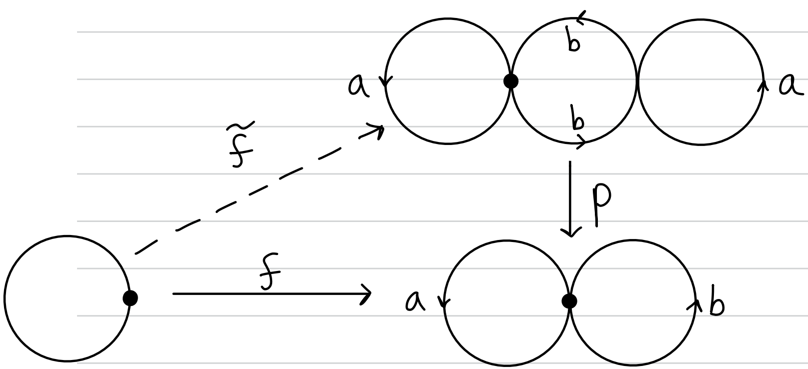
Φ is injective since $\Phi(H[g_1]) = \Phi(H[g_2])$ implies $g_1 \bar{g}_2$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$, so $[g_1][g_2]^{-1} \in H$ and $H[g_1] = H[g_2]$.

We care about lifts of general maps, not just of homotopies.

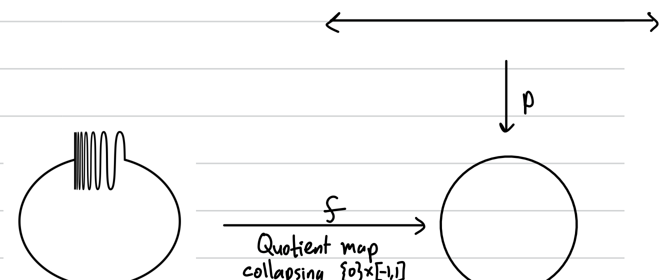
<u>Prop 1.33</u> (Lifting criterion) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space. Let $f:(Y,y_0)\to(X,x_0)$ be a map with Y connected and locally path-connected. Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

<u>Rmk</u> ($\Rightarrow$) is obvious since $f_* = p_*\tilde{f}_*$.



<u>Ex 1.3.7: Necessity of Y locally path-connected</u>



$\pi_1(Y,y_0)$ is trivial group, but no lift exists.

We care about lifts of general maps, not just of homotopies.

<u>Prop 1.33</u> (Lifting criterion) Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space.
Let $f:(Y,y_0)\to(X,x_0)$ be a map with Y connected and locally path-connected.
Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

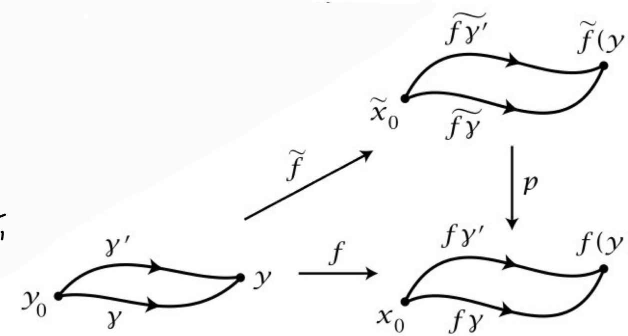
So our definition of $\tilde{f}$ will be well-defined.

So our definition of $\tilde{f}$ will be continuous.

<u>Pf</u> ($\Rightarrow$) is obvious since $f_* = p_*\tilde{f}_*$.

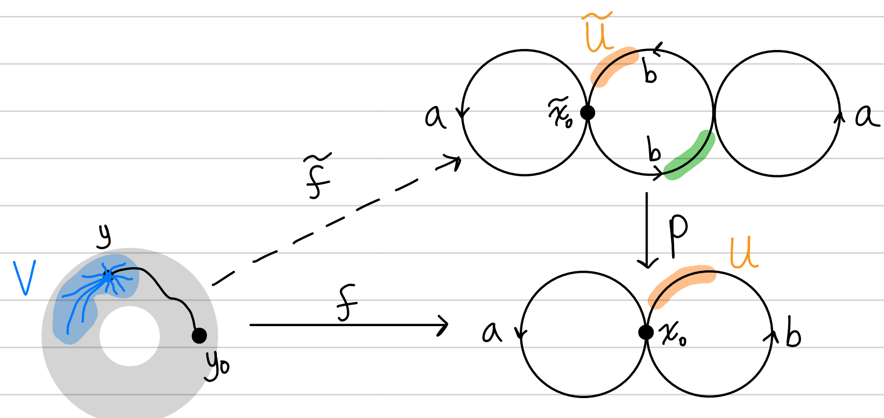
($\Leftarrow$) Y is path-connected since it is connected and locally path-connected.
For $y\in Y$, let γ be a path in Y from y_0 to y.
Path $f\gamma$ in X based at x_0 lifts uniquely to path $\widetilde{f\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$.
Define $\tilde{f}:Y\to\tilde{X}$ by $\tilde{f}(y)=\widetilde{f\gamma}(1)$.

<u>$\tilde{f}$ well-defined</u>: Given two such paths γ,γ', note
$[f\gamma'\cdot\overline{f\gamma}]\in f_*(\pi_1(Y,x_0))\subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$.
By Prop. 1.31, $f\gamma'\cdot\overline{f\gamma}$ lifts to a loop in $\tilde{X}$.
By uniqueness of path lifting, the first half of this loop is $\widetilde{f\gamma'}$
and the second half is $\widetilde{f\gamma}$ traversed backwards,
so $\widetilde{f\gamma'}(1)=\widetilde{f\gamma}(1)$.



$\tilde{f}$ continuous: Let $y \in Y$. Since $p: \tilde{X} \to X$ is a covering space,
let $f(y) \in U \underset{\text{open}}{\subseteq} X$ with $\tilde{f}(y) \in \tilde{U} \subset \tilde{X}$ and $p|_{\tilde{U}}: \tilde{U} \to U$ a homeomorphism.
Since $f^{-1}(U)$ is open in Y, choose a path-connected open set $y \in V \subset f^{-1}(U)$.
We will show $\tilde{f}|_V = (p|_U)^{-1} \circ f|_V$, hence $\tilde{f}$ is continuous at y.

Indeed, fix a path γ in Y from y_0 to y.
For each $y' \in V$, fix a path η in V from y to y'.
Each path $f\gamma \cdot f\eta$ in X has a lift $\widetilde{f\gamma} \cdot \widetilde{f\eta}$ with $\widetilde{f\eta} = (p|_U)^{-1} f\eta$ mapping to $\tilde{U}$.
Thus $\tilde{f}(V) \subset \tilde{U}$ and $\tilde{f}|_V = (p|_U)^{-1} \circ f|_V$.



We also have a unique lifting property.

<u>Prop 1.34</u> Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at a point and Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}_1, \tilde{f}_2"] \arrow[r, "f"'] & X \end{tikzcd}$$

<u>Pf</u> The main idea is to show $\{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is open and closed.

Classification of covering spaces

Thm 1.38 X path-connected, locally path-connected, semilocally simply-connected. Then

$$\left\{\begin{array}{c}\text{basepoint-preserving iso classes of path-connected} \\ \text{covering spaces } p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\} \xrightarrow{B} \left\{\begin{array}{c}\text{subgroups of} \\ \pi_1(X,x_0)\end{array}\right\}$$

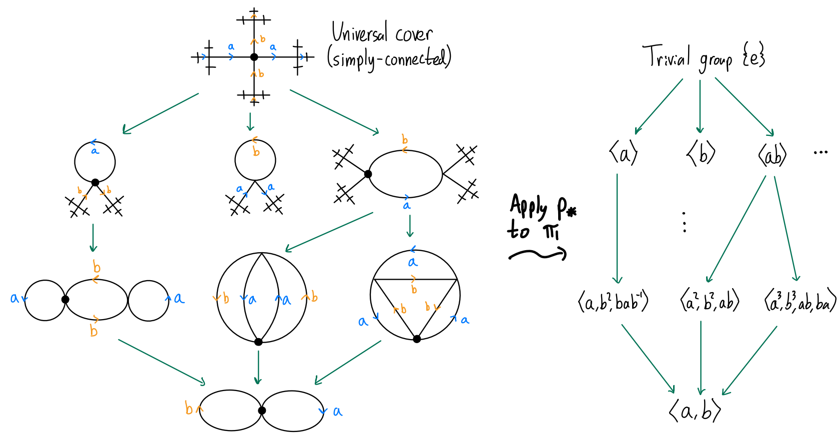
$$[p:(\tilde{X},\tilde{x}_0)\to(X,x_0)] \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

is a bijection.

Rmk If you ignore basepoints, then you map to conjugacy classes of subgroups.

Def Covering spaces $p_1:\tilde{X}_1\to X$ and $p_2:\tilde{X}_2\to X$ are isomorphic if there is a homeomorphism $f:\tilde{X}_1\to\tilde{X}_2$ with $p_1 = p_2 f$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$



Prop 1.37 (B is well-defined and injective)

Let X be path-connected and locally path-connected.
Two connected covering spaces are basepoint-preserving isomorphic iff $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2,\tilde{x}_2))$.

$$\begin{tikzcd}
(\tilde{X}_1,\tilde{x}_1) \arrow[dr, "p_1"'] & & (\tilde{X}_2,\tilde{x}_2) \arrow[dl, "p_2"] \\
& (X,x_0) &
\end{tikzcd}$$

Pf ($\Rightarrow$) $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$ induce (or imply) $\subseteq$ and $\supseteq$.

($\Leftarrow$) By the lifting criterion (Prop 1.33)
$\subseteq$ gives a lift $\tilde{p}_1 : \tilde{X}_1 \to \tilde{X}_2$ (so $p_2\tilde{p}_1 = p_1$), and
$\supseteq$ gives a lift $\tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_1$ (so $p_1\tilde{p}_2 = p_2$).

$$\begin{tikzcd}
& \tilde{X}_2 \arrow[d, "p_2"] \\
\tilde{X}_1 \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$

Since these lifts compose to fix basepoints,
unique lifting (Prop 1.34) gives $\tilde{p}_2\tilde{p}_1 = \mathbb{1}_{\tilde{X}_1}$ and $\tilde{p}_1\tilde{p}_2 = \mathbb{1}_{\tilde{X}_2}$.

$$\begin{tikzcd}
& \tilde{X}_1 \arrow[d, "p_1"] \\
\tilde{X}_1 \arrow[ur, dotted, "\mathbb{1}_{\tilde{X}_1} = \tilde{p}_2\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
(\tilde{X}_1,\tilde{x}_1) \arrow[rr, bend left, "\tilde{p}_1"] \arrow[dr, "p_1"'] & & (\tilde{X}_2,\tilde{x}_2) \arrow[ll, bend left, "\tilde{p}_2"] \arrow[dl, "p_2"] \\
& (X,x_0) &
\end{tikzcd}$$

Note $p_1(\tilde{p}_2\tilde{p}_1) = (p_1\tilde{p}_2)\tilde{p}_1 = p_2\tilde{p}_1 = p_1$.

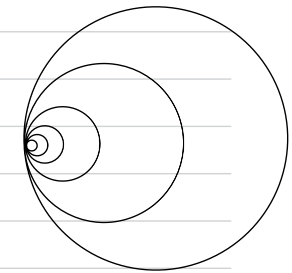
Classification of covering spaces

Thm 1.38 X path-connected, locally path-connected, semilocally simply-connected. Then

$$\left\{\begin{array}{c}\text{basepoint-preserving iso classes of path-connected} \\ \text{covering spaces } p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\} \xrightarrow{B} \left\{\begin{array}{c}\text{subgroups of} \\ \pi_1(X,x_0)\end{array}\right\}$$

$$[p:(\tilde{X},\tilde{x}_0)\to(X,x_0)] \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

is a bijection.



Def X is <u>semilocally simply-connected</u> (slsc) if $\forall x\in X$, $\exists$ open set $x\in V$ with $\pi_1(V)\to\pi_1(X)$ trivial.

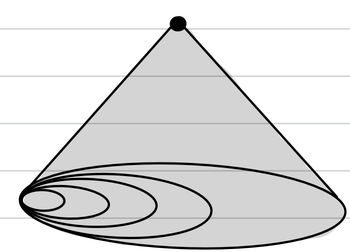
To see this condition is necessary, consider the universal cover, and V small enough to be evenly-covered.

Recall X is <u>locally simply-connected</u> (lsc) if it has a basis with simply-connected sets.

Note lsc $\Rightarrow$ slsc.

Ex The Hawaiian earrings are not slsc.

Ex The cone over the Hawaiian earrings is slsc but not lsc.



Prop 1.36 (B is surjective)
X path-connected, locally path-connected, semilocally simply-connected.
Then $\forall$ subgroups $H \subset \pi_1(X, x_0)$, $\exists$ covering space $p:(\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

Pf (1) Define the universal cover $p: \tilde{X} \to X$ with $\pi_1(\tilde{X})$ trivial.
(2) Define $\tilde{X}_H$ as a quotient of $\tilde{X}$.

(1) $\tilde{X} := \{[\gamma] \mid \gamma$ is a path in X starting at $x_0\}$.
$p: \tilde{X} \to X$ via $p([\gamma]) = \gamma(1)$.
The slsc hypothesis is used to define the topology on $\tilde{X}$ via a basis.
Can check this is a covering space.

To see that $\tilde{X}$ is path-connected, form a path $I \to \tilde{X}$ with $0 \mapsto [x_0]$ and $1 \mapsto [\gamma]$ via $t \mapsto [\gamma_t]$, where $\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t \le s \le 1. \end{cases}$

To see that $\tilde{X}$ is simply-connected, recall p_* injective. Let $[\gamma] \in \text{Image}(p_*)$.
$\text{Image}(p_*)$ is represented by loops lifting to loops.
Note $t \mapsto [\gamma_t]$ lifts γ, and for this to be a loop means $[x_0] = [\gamma_1] = [\gamma]$.

Before we define a topology on $\widetilde{X}$ we make a few preliminary observations. Let $\mathcal{U}$ be the collection of path-connected open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Note that if the map $\pi_1(U) \to \pi_1(X)$ is trivial for one choice of basepoint in U, it is trivial for all choices of basepoint since U is path-connected. A path-connected open subset $V \subset U \in \mathcal{U}$ is also in $\mathcal{U}$ since the composition $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ will also be trivial. It follows that $\mathcal{U}$ is a basis for the topology on X if X is locally path-connected and semilocally simply-connected.

Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{\, [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \,\}$$

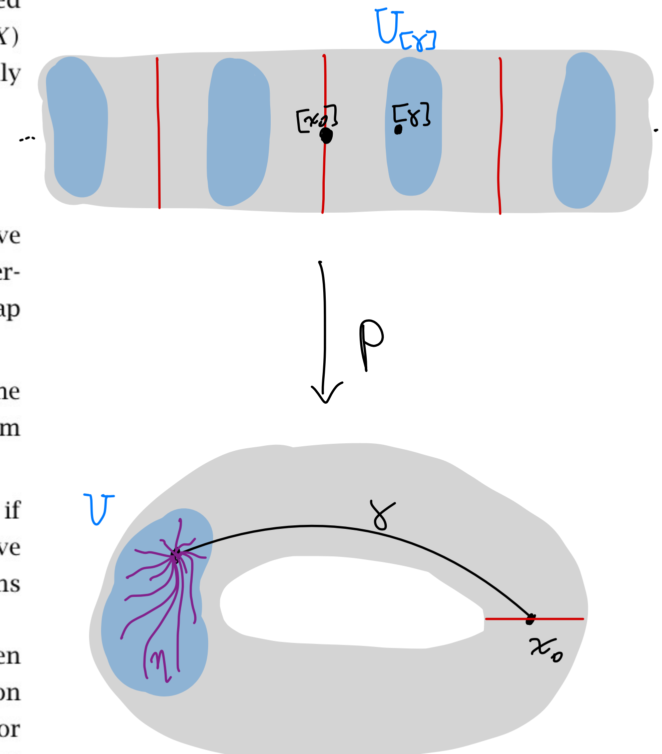
As the notation indicates, $U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$. Observe that $p : U_{[\gamma]} \to U$ is surjective since U is path-connected and injective since different choices of η joining $\gamma(1)$ to a fixed $x \in U$ are all homotopic in X, the map $\pi_1(U) \to \pi_1(X)$ being trivial. Another property is

(∗) $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. For if $\gamma' = \gamma \cdot \eta$ then elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ and hence lie in $U_{[\gamma]}$, while elements of $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \overline{\eta} \cdot \mu] = [\gamma' \cdot \overline{\eta} \cdot \mu]$ and hence lie in $U_{[\gamma']}$.

This can be used to show that the sets $U_{[\gamma]}$ form a basis for a topology on $\widetilde{X}$. For if we are given two such sets $U_{[\gamma]}, V_{[\gamma']}$ and an element $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$ by (∗). So if $W \in \mathcal{U}$ is contained in $U \cap V$ and contains $\gamma''(1)$ then $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$.

The bijection $p : U_{[\gamma]} \to U$ is a homeomorphism since it gives a bijection between the subsets $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U. Namely, in one direction we have $p(V_{[\gamma']}) = V$ and in the other direction we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V, since $V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$ and $V_{[\gamma']}$ maps onto V by the bijection p.

The preceding paragraph implies that $p : \widetilde{X} \to X$ is continuous. We can also deduce that this is a covering space since for fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$ partition $p^{-1}(U)$ because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$ by (∗).

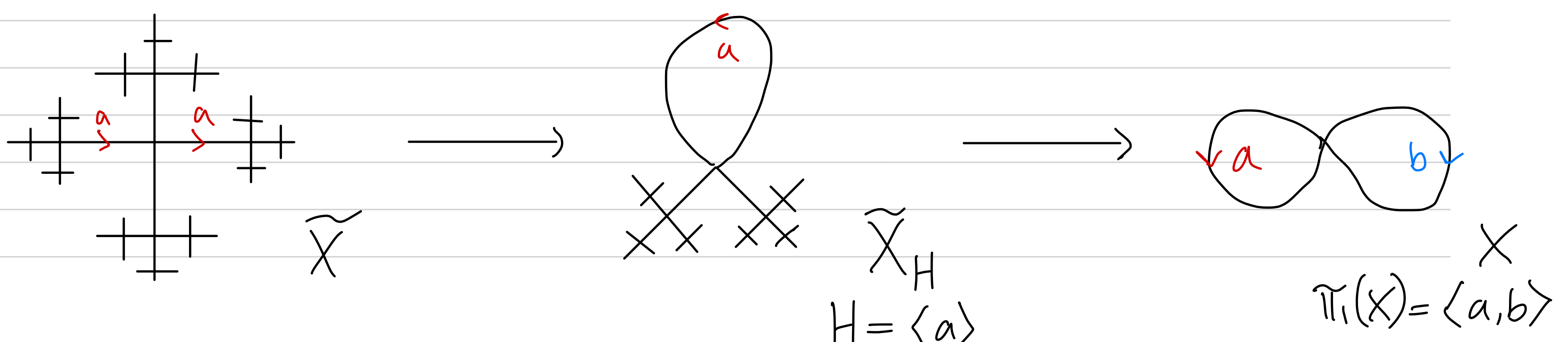


(2) For $[\gamma], [\gamma'] \in \tilde{X}$, define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$.
This is an equivalence relation since H is a subgroup

- reflexive: identity
- symmetric: inverses
- transitive: H closed under multiplication

Define $\tilde{X}_H$ to be the quotient space $\tilde{X}_H = \tilde{X}/\sim$.
Can check the map $\tilde{X}_H \to X$ induced from $[\gamma] \to \gamma(1)$ gives a covering space.

We claim $\pi_1(\tilde{X}_H, \tilde{x}_0) \to \pi_1(X, x_0)$ has image H
(where $\tilde{x}_0$ is the equivalence class of $[x_0]$).
Indeed, a loop γ in X lifts to a loop in $X_H \Leftrightarrow [\gamma] \sim [x_0] \Leftrightarrow [\gamma] \in H$.



Deck transformations and group actions

Let $p : \tilde{X} \to X$ be a covering space. The group of deck transformations is

$$G(\tilde{X}) = \left\{ \begin{array}{l} \text{covering space} \\ \text{isomorphisms} \end{array} \quad \begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \right\},$$

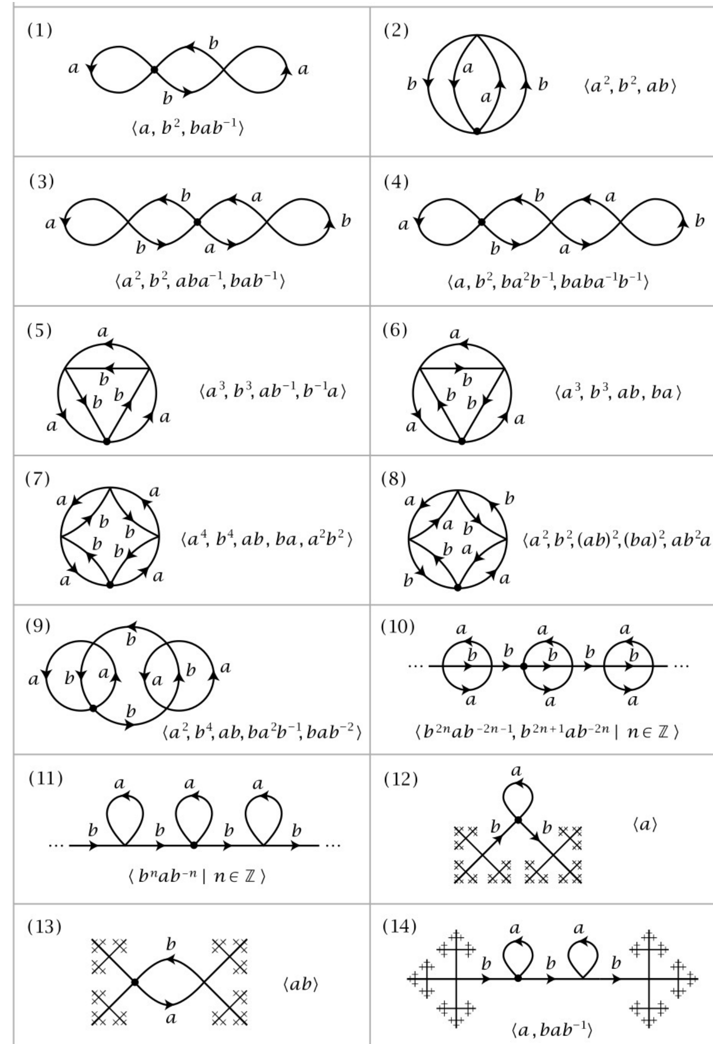
equipped with composition.

Ex (7) $G(\tilde{X}) \cong \mathbb{Z}/4$ (8) $G(\tilde{X}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

A covering space is normal if $\forall x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, $\exists\, h \in G(\tilde{X})$ with $h(\tilde{x}) = h(\tilde{x}')$.

(Maximal symmetry)

Ex (1), (2), (5)-(8), (11) normal.



Deck transformations and group actions

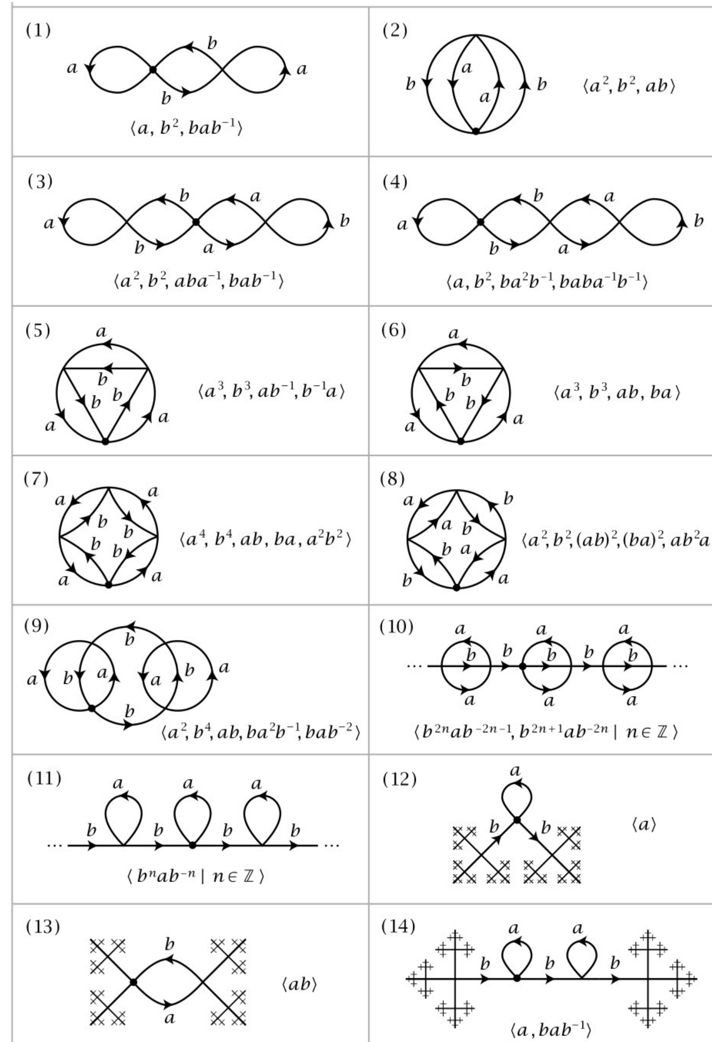
Prop 1.39 Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a p.c. covering space of the p.c., l.p.c. space X. Let $H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

(a) $p:\tilde{X}\to X$ normal $\Longleftrightarrow$ H normal in $\pi_1(X,x_0)$.

(b) $G(\tilde{X})\cong N(H)/H$, where the normalizer of H is $N(H)=\{g\in\pi_1(X,x_0)\mid g^{-1}Hg=H\}$.

- $G(\tilde{X})\cong\pi_1(X,x_0)/H$ if $p:\tilde{X}\to X$ normal.
- $G(\tilde{X})\cong\pi_1(X,x_0)$ for $\tilde{X}$ the universal cover.

Ex (5) a normal covering space.
$H=\langle a^3, b^3, ab^{-1}, b^{-1}a\rangle$.
Note $b^{-1}a^3b \in b^{-1}Hb = H$.
$\|$
$(b^{-1}a)\,a^3\,(a^{-1}b)$



Deck transformations and group actions

Prop 1.39 Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a p.c. covering space of the p.c., l.p.c. space X. Let $H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

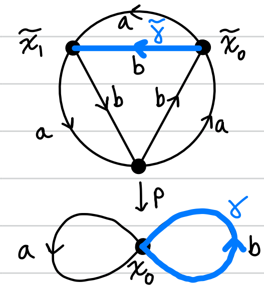
(a) $p:\tilde{X}\to X$ normal $\iff$ H normal in $\pi_1(X,x_0)$.

(b) $G(\tilde{X})\cong N(H)/H$, where the normalizer of H is $N(H)=\{g\in\pi_1(X,x_0)\mid g^{-1}Hg=H\}$.

- $G(\tilde{X})\cong\pi_1(X,x_0)/H$ if $p:\tilde{X}\to X$ normal.
- $G(\tilde{X})\cong\pi_1(X,x_0)$ for $\tilde{X}$ the universal cover.

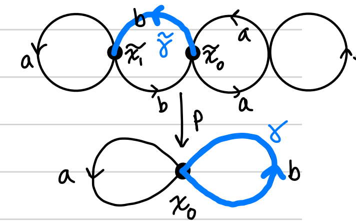
Pf (a) Let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1\in p^{-1}(x_0)$.

Note $p_*(\pi_1(\tilde{X},\tilde{x}_1))=[\gamma]^{-1}H[\gamma]$ for $[\gamma]\in\pi_1(X,x_0)$ with $\gamma=p\circ\tilde{\gamma}$.



$H=\langle a^3,b^3,ab^{-1},b^{-1}a\rangle$

$[\gamma]\in N(H)$



$H=\langle a^2,b^2,aba^{-1},bab^{-1}\rangle$

$[\gamma]\notin N(H)$. $b^{-1}(bab^{-1})b=a\notin H$

$p_*(\pi_1(\tilde{X},\tilde{x}_1))\neq H$.

No deck transformation $\tilde{x}_1\mapsto\tilde{x}_0$.

So $[\gamma]\in N(H)$ iff $p_*(\pi_1(\tilde{X},\tilde{x}_1))=p_*(\pi_1(\tilde{X},\tilde{x}_0))$, which by the lifting criterion is equivalent to deck transformations taking $\tilde{x}_1$ to $\tilde{x}_0$ and vice-versa.

Hence $p:\tilde{X}\to X$ is normal
$\iff$ such deck transformations exist $\forall\tilde{x}_1\in p^{-1}(x_0)$
$\iff$ $N(H)=\pi_1(X,x_0)$
$\iff$ H is normal in $\pi_1(X,x_0)$.

Deck transformations and group actions

A <u>group action</u> on a set Y is a function $G \times Y \to Y$, denoted $(g,y) \mapsto g.y$, satisfying

- $id.\, y = y \quad \forall y \in Y$
- $g'.(g.y) = (g'g).y \quad \forall g, g' \in G,\ \forall y \in Y.$

That is, it is a homomorphism from G to the group of permutations of Y.

A <u>group action</u> on a space Y is a homomorphism from G to the group of homeomorphisms of Y.

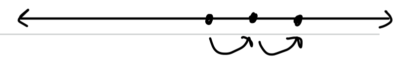
The <u>orbit space</u> Y/G is the quotient space $Y/\sim$, where $y \sim g.y \quad \forall y \in Y$ and $g \in G$.

<u>Ex</u> The group $G(\tilde{X})$ of deck transformations acts on the covering space $\tilde{X}$ by

$$G(\tilde{X}) \times \tilde{X} \to \tilde{X}$$
$$(h, \tilde{x}) \mapsto h(\tilde{x})$$

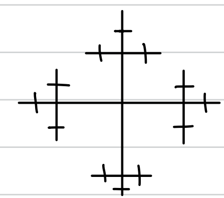
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & \circlearrowleft & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

<u>Ex</u> $\mathbb{Z}$ acts on $\mathbb{R}$



<u>Ex</u> $\mathbb{Z}^n$ acts on $\mathbb{R}^n$

<u>Ex</u> $\langle a, b \rangle$ acts on



So does $\mathbb{Z}/4$, via rotations, but not freely.

For a normal covering space $\tilde{X} \to X$, the orbit space $\tilde{X}/G(\tilde{X})$ is homeomorphic to X.

Deck transformations and group actions

Proposition 1.40. *If an action of a group G on a space Y satisfies $(*)$, then:*

(a) *The quotient map $p : Y \to Y/G$, $p(y) = Gy$, is a normal covering space.*

(b) *G is the group of deck transformations of this covering space $Y \to Y/G$ if Y is path-connected.*

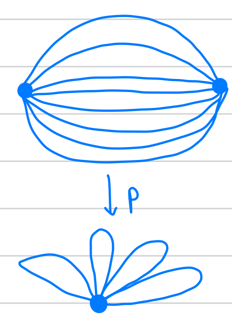
(c) *G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$ if Y is path-connected and locally path-connected.*

$(*)$ Each $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.

Section 1.A Graphs and free groups

Theorem 1A.4 Every subgroup of a free group is free.

Pf Let $F = \langle g_\alpha \rangle_{\alpha \in A}$ be a free group.
Let G be a subgroup.
Let $X = \bigvee_{\alpha \in A} S^1$. Note $\pi_1(X) \cong F$.
By Prop 1.36, $\exists$ covering space $p: \tilde{X} \to X$ with $p_*(\pi_1(\tilde{X})) = G$;
hence $\pi_1(\tilde{X}) \cong G$ since p_* is injective.
Lemma 1A.3 says any covering space of a graph is a graph.
And the fundamental group of a graph is a free group.
So $G \cong \pi_1(\tilde{X})$ is free.



$F = \langle g_1, g_2, g_3, g_4 \rangle$