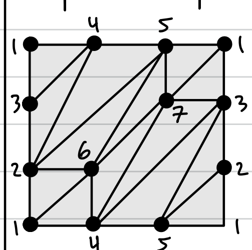
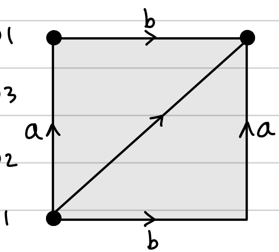
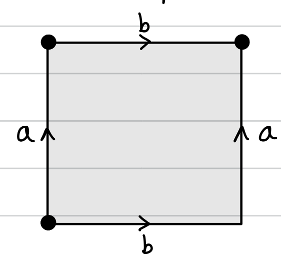
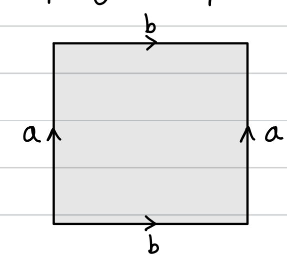
Chapter 2: Homology

Associates to each space X a group $H_n(X)$ measuring the n-dimensional holes.

Unlike the homotopy group $\pi_n(X)$, the homology group $H_n(X)$ is
- difficult to define, and
- easy to compute.

There are many homology theories, which all agree on nice spaces.

Homology Theory	Singular Homology $H_n^\Delta(X) = H_n(X)$	Cellular Homology $H_n(X)$	Singular Homology $H_n(X)$
Appropriate Spaces	Simplicial complexes $\subseteq$ Δ-complexes  	CW complexes 	Topological spaces 

A simplicial complex torus has at least 7 vertices, 21 edges, 14 triangles.

Let X, Y be simplicial complexes with $X \simeq Y$.
Homotopy invariance will be proven using singular homology!

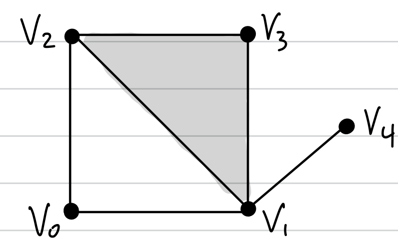
$$\begin{array}{ccc} H_n^\Delta(X) \cong H_n(X) & & \\ & \Vert\wr & \text{homotopy invariance} \\ H_n^\Delta(Y) \cong H_n(Y) & & \end{array}$$

Simplicial complexes

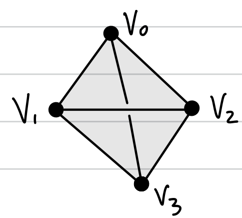
Def An <u>abstract simplicial complex</u> on a vertex set V is a collection K of finite subsets of V, including all singletons, such that $\sigma \in K$ and $\tau \subset \sigma$ implies $\tau \in K$.

Ex $V = \{v_0, v_1, v_2, v_3, v_4\}$

$$K = \left\{ \begin{array}{l} \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \\ \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \\ \{v_1, v_2, v_3\} \end{array} \right\}$$



Ex $V = \{v_0, v_1, v_2, v_3\}$
$K =$ the set of all subsets of V



Simplicial complexes are CW complexes, and are given the same topology (the weak topology).

Simplicial homology for simplicial complexes

Let X be a simplicial complex.

The chain group $\Delta_n(X)$ is the free abelian group on the set of oriented n-simplices in X.

(I.e., $\Delta_n(X)$ is the set of formal sums of n-simplices. Its elements are n-chains.)

Define the boundary operator $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ by

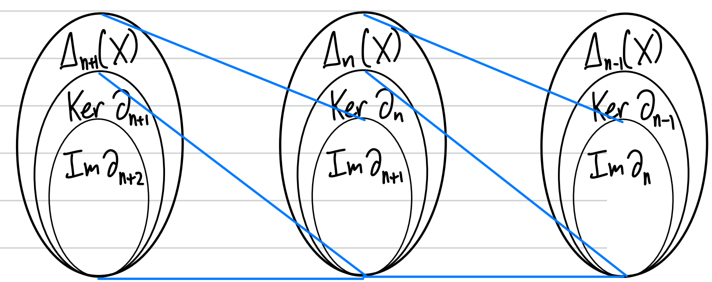
$\partial_n([x_0, x_1, \dots, x_n]) = \sum_{i=0}^{n} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_n]$ and extending linearly.

means x_i is removed

We will show $\partial_n \circ \partial_{n+1} = 0$.

Hence $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$

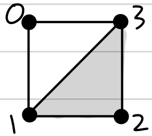
$$\Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X)$$



Def The n-dimensional simplicial homology group of X is $H_n^{\Delta}(X) = \dfrac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}$.

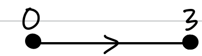
$\operatorname{Ker} \partial_n$ is the group of n-cycles.

$\operatorname{Im} \partial_{n+1}$ is the group of n-boundaries.

Ex $X =$ 

$\Delta_0(X) = \{a[0] + b[1] + c[2] + d[3] : a,b,c,d \in \mathbb{Z}\} \cong \mathbb{Z}^4.$

Group operation: $(a[0] + b[1] + c[2] + d[3])$
$+ (a'[0] + b'[1] + c'[2] + d'[3])$
$= (a+a')[0] + (b+b')[1] + (c+c')[2] + (d+d')[3]$

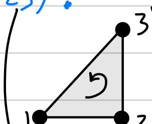
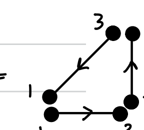
$\Delta_1(X) = \{a[0,1] + b[0,3] + c[1,2] + d[1,3] + e[2,3] : a,b,c,d,e \in \mathbb{Z}\} \cong \mathbb{Z}^5.$ 

We write a simplex as $[0,3]$ instead of $\{0,3\}$ to denote that it is oriented: $[0,3] = -[3,0]$.
Group operation: $(4[0,3] + [1,2]) + (-6[0,3] + [2,3]) = -2[0,3] + [1,2] + [2,3].$

$\Delta_2(X) = \{a[1,2,3] : a \in \mathbb{Z}\} \cong \mathbb{Z}.$

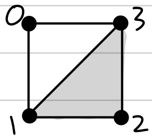
$[1,2,3] = [2,3,1] = [3,1,2]$ (differ from $[1,2,3]$ by an even permutation)
$-[1,2,3] = [1,3,2] = [2,1,3] = [3,2,1]$ (differ from $[1,2,3]$ by an odd permutation)

$\partial_1([0,3]) = (-1)^0[3] + (-1)^1[0] = [3] - [0].$ What is $\partial_1([0,3] + [3,2])$?

$\partial_2([1,2,3]) = (-1)^0[2,3] + (-1)^1[1,3] + (-1)^2[2,3] = [2,3] + [3,1] + [1,2].$ $\partial_2\Big($  $\Big) =$ 

$\partial_1\partial_2([1,2,3]) = \partial_1([2,3] + [3,1] + [1,2]) = \partial_1[2,3] + \partial_1[3,1] + \partial_1[1,2] = (\cancel{[3]} - \cancel{[2]}) + (\cancel{[1]} - \cancel{[3]}) + (\cancel{[2]} - \cancel{[1]}) = 0.$

$\partial_0([2]) = 0.$

Ex $X=$ 

$\operatorname{Ker}\partial_1 = \{a([0,1]+[1,3]+[3,0]) + b([1,2]+[2,3]+[3,1]) : a,b\in\mathbb{Z}\} \cong \mathbb{Z}^2.$

What about $[0,1]+[1,2]+[2,3]+[3,0]$?

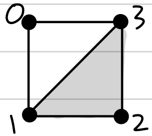
$\operatorname{Im}\partial_2 = \{b\,\partial_2([1,2,3]) : b\in\mathbb{Z}\} = \{b([1,2]+[2,3]+[3,1]) : b\in\mathbb{Z}\} \cong \mathbb{Z}.$

So $H_1^\Delta(X) = \operatorname{Ker}\partial_1/\operatorname{Im}\partial_2 \cong \{a([0,1]+[1,3]+[3,0]) : a\in\mathbb{Z}\} \cong \mathbb{Z}.$ X has one 1-dimensional hole.

$\operatorname{Ker}\partial_2 = 0$ since $\partial_2([1,2,3]) \neq 0$

$\operatorname{Im}\partial_3 = 0$ since $\Delta_3(X) = 0$

So $H_1^\Delta(X) = \operatorname{Ker}\partial_2/\operatorname{Im}\partial_3 = 0.$ X has no 2-dimensional holes.

Ex $X=$ 

$\operatorname{Ker}\partial_0 = \Delta_0 = \{a[0]+b[1]+c[2]+d[3] : a,b,c,d \in \mathbb{Z}\} \cong \mathbb{Z}^4.$

$\operatorname{Im}\partial_1 = \{a\partial_1[0,1]+b\partial_1[0,3]+c\partial_1[1,2]+d\partial_1[1,3]+e\partial_1[2,3] : a,b,c,d,e \in \mathbb{Z}\}$

$= \{a([1]-[0])+b([3]-[0])+c([2]-[1]) : a,b,c \in \mathbb{Z}\} \cong \mathbb{Z}^3$

since $\partial_1[1,3] = -\partial_1[0,1]+\partial_1[0,3]$

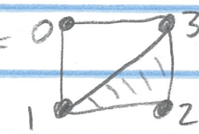
and $\partial_1[2,3] = -\partial_1[1,2]-\partial_1[0,1]+\partial_1[0,3].$

(Using three edges, we can walk from any vertex to any vertex.)

So $H_0^\Delta(X) = \operatorname{Ker}\partial_0/\operatorname{Im}\partial_1 \cong \{a[0]+b([1]-[0])+c([3]-[0])+d([2]-[1]) : a,b,c,d \in \mathbb{Z}\}/\operatorname{Im}\partial_1$

$\cong \{a[0] : a \in \mathbb{Z}\} \cong \mathbb{Z}.$ X has a single connected component.

In the prior example, how might we algorithmically compute $\operatorname{Ker}\partial_1 \cong \mathbb{Z}^2$ and $\operatorname{Im}\partial_1 \cong \mathbb{Z}^3$, while also finding generators?

$X =$ 

4×5 Matrix representing $\partial_1 : \Delta_1(X) \to \Delta_0(X)$

$$\begin{array}{c|ccccc} & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\ \hline [0] & -1 & -1 & 0 & 0 & 0 \\ [1] & 1 & 0 & -1 & -1 & 0 \\ [2] & 0 & 0 & 1 & 0 & -1 \\ [3] & 0 & 1 & 0 & 1 & 1 \end{array} \xrightarrow{\text{swap rows}} \begin{array}{c|ccccc} & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\ \hline [1] & 1 & 0 & -1 & -1 & 0 \\ [3] & 0 & 1 & 0 & 1 & 1 \\ [2] & 0 & 0 & 1 & 0 & -1 \\ [0] & -1 & -1 & 0 & 0 & 0 \end{array}$$

Add third row to first

$$\begin{array}{c|ccccc} & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\ \hline [1] & 1 & 0 & 0 & -1 & -1 \\ [3] & 0 & 1 & 0 & 1 & 1 \\ [2]-[1] & 0 & 0 & 1 & 0 & -1 \\ [0] & -1 & -1 & 0 & 0 & 0 \end{array} \xrightarrow{\text{Add first and third rows to fourth}} \begin{array}{c|ccccc} & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\ \hline [1]-[0] & 1 & 0 & 0 & -1 & -1 \\ [3]-[0] & 0 & 1 & 0 & 1 & 1 \\ [2]-[1] & 0 & 0 & 1 & 0 & -1 \\ [0] & 0 & 0 & 0 & 0 & 0 \end{array}$$

Add col 1 − col 2 to col 4
Add col 1 − col 2 + col 3 to col 5

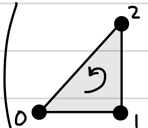
$$\begin{array}{c|ccccc} & [0,1] & [0,3] & [1,2] & \begin{matrix}[1,3]+[0,1]-[0,3] \\ \shortparallel \\ [0,1]+[1,3]+[3,0]\end{matrix} & \begin{matrix}[2,3]+[0,1]-[0,3]+[1,2] \\ \shortparallel \\ [0,1]+[1,2]+[2,3]+[3,0]\end{matrix} \\ \hline [1]-[0] & 1 & 0 & 0 & 0 & 0 \\ [3]-[0] & 0 & 1 & 0 & 0 & 0 \\ [2]-[1] & 0 & 0 & 1 & 0 & 0 \\ [0] & 0 & 0 & 0 & 0 & 0 \end{array}$$

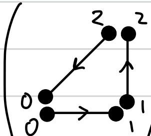
This matrix has rank 3 and nullity 2.
The first three rows give a generating set for $\operatorname{Im}(\partial_1)$.
The last two columns give a generating set for $\operatorname{Ker}(\partial_1)$.

Define the <u>boundary operator</u> $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ by

$\partial_n([x_0, x_1, \dots, x_n]) = \sum_{i=0}^{n} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_n]$ and extending linearly.

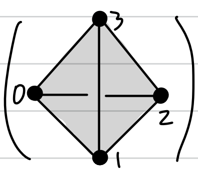
$\uparrow$ means x_i is removed

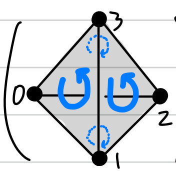
$\partial_1 \partial_2 \Big($ 

$\Big) = \partial_1 \Big($ 

$\Big) = 0.$

$$\partial_1\partial_2([0,1,2]) = \partial_1([1,2]-[0,2]+[0,1]) = ([\cancel{2}]-[\cancel{1}]) - ([\cancel{2}]-[\cancel{0}]) + ([\cancel{1}]-[\cancel{0}]) = 0.$$

$\partial_2 \partial_3 \Big($ 

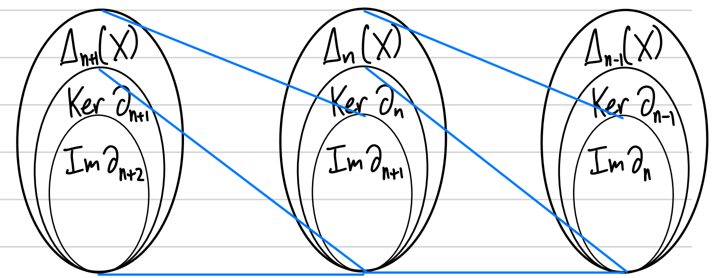
$\Big) = \partial_2 \Big($ 

$\Big) = 0.$

$$\partial_2\partial_3([0,1,2,3]) = \partial_2([1,2,3]-[0,2,3]+[0,1,3]-[0,1,2])$$
$$= ([\cancel{2,3}]-[\cancel{1,3}]+[\cancel{1,2}]) - ([\cancel{2,3}]-[\cancel{0,3}]+[\cancel{0,2}]) + ([\cancel{1,3}]-[\cancel{0,3}]+[\cancel{0,1}]) - ([\cancel{1,2}]-[\cancel{0,2}]+[\cancel{0,1}])$$
$$= 0.$$

Proof that $\partial_n \circ \partial_{n+1} = 0$

$$\Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X)$$



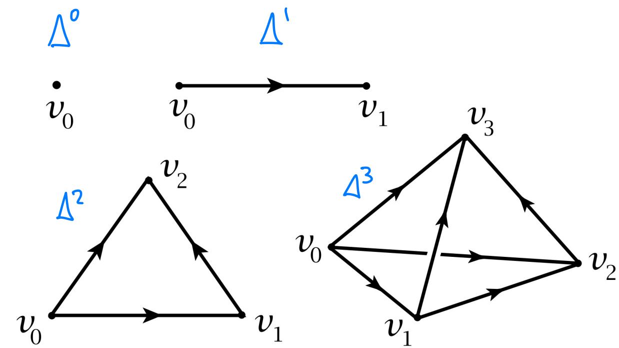
Since the boundary operators are linear, it suffices to show that applying $\partial_n \partial_{n+1}$ to a single $(n+1)$-simplex gives zero.

Note

$$\begin{aligned}
&\partial_n \left(\partial_{n+1}\left([x_0, x_1, \ldots x_{n+1}]\right)\right) \\
&= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}] \right) \\
&= \sum_{j<i} (-1)^j (-1)^i [x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{n+1}]) \\
&\quad + \sum_{i<j} (-1)^{j-1} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}])
\end{aligned}$$

$= 0$ by symmetry (canceling $(n-1)$-simplices in pairs).

<u>Simplicial homology for Δ-complexes</u>



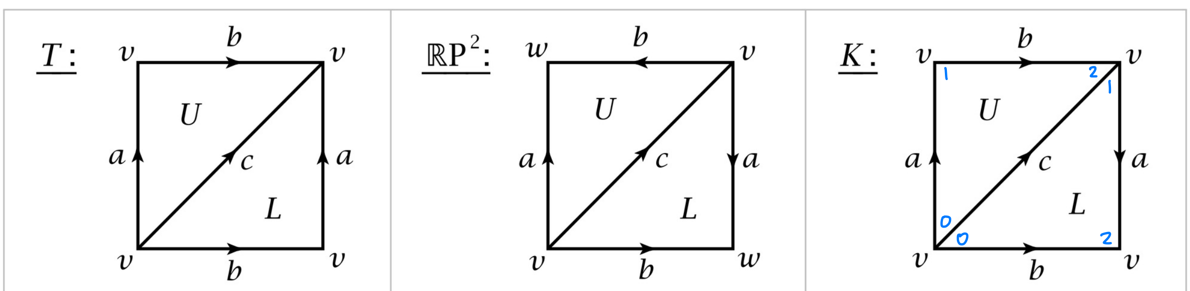
The n-simplex Δ^n has $n+1$ ordered vertices.

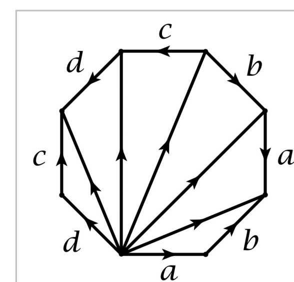
Let $\mathring{\Delta}^n = \Delta^n \setminus \partial \Delta^n$ be the open n-simplex.

<u>Def</u> A <u>Δ-complex structure</u> on space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ such that

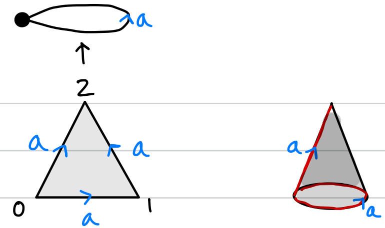
- $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and their images partition X.
- Restricting $\sigma_\alpha : \Delta^n \to X$ to a face gives some $\sigma_\beta : \Delta^{n-1} \to X$, where $\Delta^{n-1} \subseteq \Delta^n$ is order preserving.
- $A \subset X$ is open $\Leftrightarrow$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n $\forall \alpha$.

The edge orders determine the vertex order in any simplex: vertex i has i entering edges in that simplex.

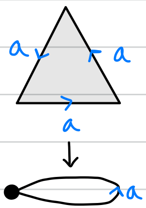




Ex The dunce hat is a Δ-complex.

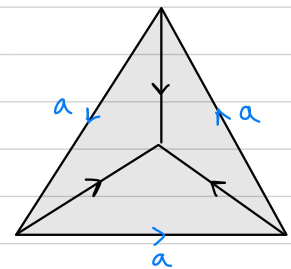


Ex



is not a Δ-complex since the restricted face maps are not order-preserving.

Ex

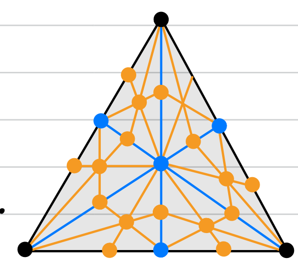


is a Δ-complex.

Ex Every simplicial complex X is a Δ-complex, with one $\sigma_\alpha : \Delta^n \to X$ for each n-simplex.

Simplicial complexes $\subseteq$ Δ-complexes $\subseteq$ CW complexes

Aside Ex 23 on Hatcher page 133 shows the 2nd barycentric subdivision of a Δ-complex is a simplicial complex.



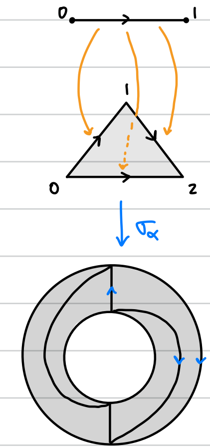
Aside Thm 2C.5 shows every CW complex is homotopy equivalent to a simplicial complex.

Simplicial homology for Δ-complexes

Let X be a Δ-complex.

The chain group $\Delta_n(X)$ is the free abelian group on the set of $\sigma_\alpha : \Delta^n \to X$.

$\Delta_n(X) = \{\sum_\alpha n_\alpha \sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X,\ n_\alpha \in \mathbb{Z},$ finitely many n_α nonzero$\}$.



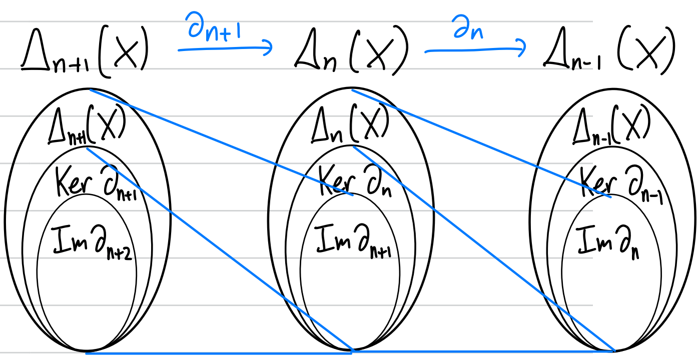
Define the boundary operator $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ by

$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[x_0, \dots, \hat{x}_i, \dots, x_n]}$ and extending linearly.

Lemma 2.1 $\partial_n \circ \partial_{n+1} = 0$.

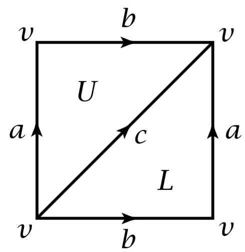
Pf Let $\sigma_\alpha : \Delta^{n+1} \to X$. Note

$$
\begin{aligned}
&\partial_n \partial_{n+1}(\sigma_\alpha) = \\
&= \partial_n \Big(\sum_{i=0}^{n+1} (-1)^i \sigma_\alpha|_{[x_0, \dots, \hat{x}_i, \dots, x_{n+1}]} \Big) \\
&= \sum_{j<i} (-1)^j (-1)^i \sigma_\alpha|_{[x_0, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_{n+1}]} \\
&\quad + \sum_{i<j} (-1)^{j-1} (-1)^i \sigma_\alpha|_{[x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}]} \\
&= 0.
\end{aligned}
$$



Def The n-dimensional simplicial homology group of X is $H_n^\Delta(X) = \dfrac{\mathrm{Ker}\,\partial_n}{\mathrm{Im}\,\partial_{n+1}}$.

<u>Ex 2.3</u> X = torus



$\partial_0(v) = 0$

$\partial_1(a) = v - v = 0$
$\partial_1(b) = v - v = 0$
$\partial_1(c) = v - v = 0$

$$\begin{array}{ccccccccc} \Delta_3(X) & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) & \xrightarrow{\partial_0} & 0 \\ \cong & & \cong & & \cong & & \cong & & \\ 0 & & \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z} & & \\ & & U, L & & a, b, c & & v & & \end{array}$$

$\partial_2(U) = b - c + a = a + b - c$
$\partial_2(L) = a - c + b = a + b - c$

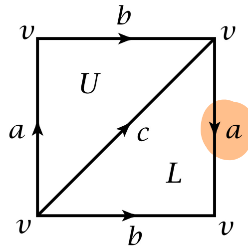
$\operatorname{Ker}\partial_0$ has basis $\{v\}$ and $\operatorname{Im}\partial_1 = 0 \Rightarrow H_0^\Delta(X) \cong \operatorname{Ker}\partial_0 / \operatorname{Im}\partial_1 \cong \mathbb{Z}$.

$\operatorname{Ker}\partial_1$ has basis $\{a, b, c\}$ or $\{a, b, a+b-c\}$
$\operatorname{Im}\partial_2$ has basis $\{a+b-c\}$ $\Rightarrow H_1^\Delta(X) \cong \dfrac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} \cong \mathbb{Z}^2$.

$\operatorname{Ker}\partial_2$ has basis $U - L$
$\operatorname{Im}\partial_3 = 0$ $\Rightarrow H_2^\Delta(X) \cong \dfrac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3} \cong \mathbb{Z}$.

<u>Pf</u>: $\partial_2(pU + qL) = p(a+b-c) + q(a+b-c) = 0 \iff q = -p$.

Ex $X =$ Klein bottle



$\partial_0(v) = 0$

$\partial_1(a) = v - v = 0$

$\partial_1(b) = v - v = 0$

$\partial_1(c) = v - v = 0$

$$\begin{array}{ccccccccc} \Delta_3(X) & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) & \xrightarrow{\partial_0} & 0 \\ \cong & & \cong & & \cong & & \cong & & \\ 0 & & \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z} & & \\ & & U, L & & a, b, c & & v & & \end{array}$$

$\partial_2(U) = b - c + a = a + b - c$

$\partial_2(L) = a - b + c = a - b + c$

$\operatorname{Ker}\partial_0$ has basis $\{v\}$ and $\operatorname{Im}\partial_1 = 0 \Rightarrow H_0^\Delta(X) \cong \operatorname{Ker}\partial_0 / \operatorname{Im}\partial_1 \cong \mathbb{Z}$.

$\operatorname{Ker}\partial_1$ has basis $\{a, b, c\}$ or $\{a, b, a+b-c\}$

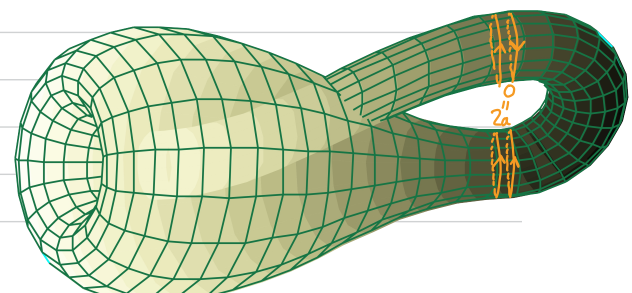
$\operatorname{Im}\partial_2$ has basis $\{a+b-c, a-b+c\}$ or $\{a+b-c, 2a\}$

$$\Rightarrow H_1^\Delta(X) \cong \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

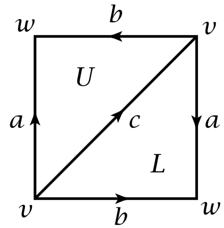
$\operatorname{Ker}\partial_2 = 0$

$\operatorname{Im}\partial_3 = 0$

$$\Rightarrow H_2^\Delta(X) \cong \frac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3} \cong 0.$$



Ex 2.4 $\quad X = \mathbb{R}P^2$



$$\partial_0(v) = 0$$
$$\partial_0(w) = 0$$

$$\partial_1(a) = w - v$$
$$\partial_1(b) = w - v$$
$$\partial_1(c) = v - v = 0$$

$$\partial_2(U) = b - a + c = -a + b + c$$
$$\partial_2(L) = a - b + c$$

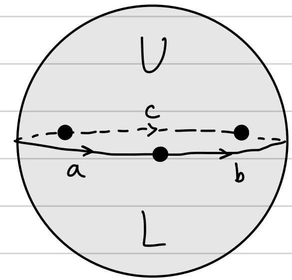
$$\begin{array}{ccccccccc}
\Delta_3(X) & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) & \xrightarrow{\partial_0} & 0 \\
\cong & & \cong & & \cong & & \cong & & \\
0 & & \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z}^2 & & \\
 & & U, L & & a, b, c & & v, w & &
\end{array}$$

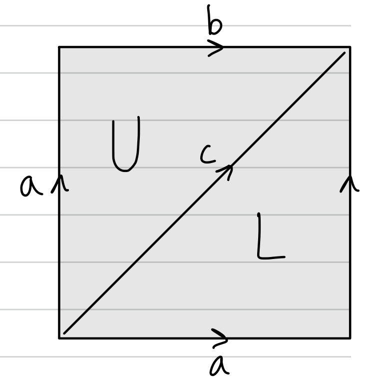
$\operatorname{Ker}\partial_0$ has basis $\{v, w\}$ or $\{v, w-v\}$
$\operatorname{Im}\partial_1$ has basis $\{w-v\}$
$$\Rightarrow H_0^\Delta(X) \cong \frac{\operatorname{Ker}\partial_0}{\operatorname{Im}\partial_1} \cong \mathbb{Z}.$$

$\operatorname{Ker}\partial_1$ has basis $\{a-b, c\}$ or $\{a-b+c, c\}$
$\operatorname{Im}\partial_2$ has basis $\{-a+b+c, a-b+c\}$ or $\{a-b+c, 2c\}$
$$\Rightarrow H_1^\Delta(X) \cong \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} \cong \mathbb{Z}/2.$$

$\operatorname{Ker}\partial_2 = 0$
$\operatorname{Im}\partial_3 = 0$
$$\Rightarrow H_2^\Delta(X) \cong \frac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3} \cong 0.$$

Ex 2.4 n-sphere S^n





One Δ-complex structure consists of two n-simplices, U and L, glued together along their common boundary.

$$\left.\begin{array}{l}\operatorname{Ker} \partial_n \text{ has basis } \{U-L\} \\ \operatorname{Im} \partial_{n+1} = 0\end{array}\right. \implies H_n^\Delta(S^n) \cong \mathbb{Z}.$$

More generally,

$$H_i^\Delta(S^n) \cong \begin{cases} \mathbb{Z} & i=0 \\ 0 & 1 \le i \le n-1 \\ \mathbb{Z} & i=n \\ 0 & i \ge n+1 \end{cases}$$

← Cellular homology will give easily

Question Is $H_n^\Delta(X)$ independent of the Δ-complex structure on X?
Question Does $X \simeq Y$ imply $H_n^\Delta(X) \cong H_n^\Delta(Y)$?

Answers "Yes" and "yes", as we will show using singular homology.

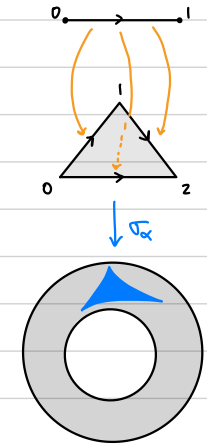
Singular homology

Let X be a topological space.

A <u>singular n-simplex</u> is a map $\sigma : \Delta^n \to X$.

The <u>chain group</u> $C_n(X)$ is the free abelian group on the set of singular n-simplices.

$C_n(X) = \{ \sum_\alpha n_\alpha \sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X,\ n_\alpha \in \mathbb{Z}, \text{ finitely many } n_\alpha \text{ nonzero} \}$.



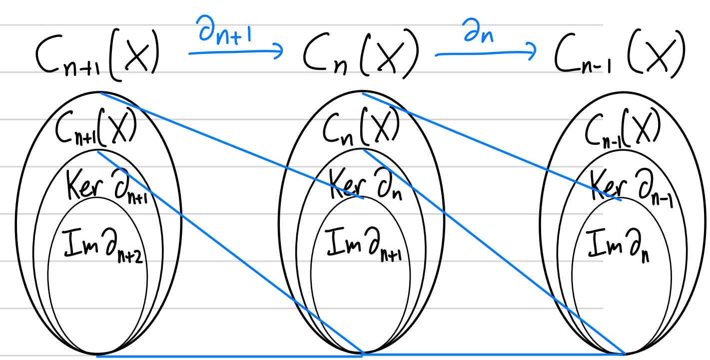
Define the <u>boundary operator</u> $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ by

$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[x_0,\dots,\hat{x}_i,\dots,x_n]}$ and extending linearly.

<u>Lemma 2.1</u> $\partial_n \circ \partial_{n+1} = 0$.

<u>Pf</u> Let $\sigma_\alpha : \Delta^{n+1} \to X$. Note

$$\begin{aligned}
\partial_n \partial_{n+1}(\sigma_\alpha) &= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i \sigma_\alpha|_{[x_0,\dots,\hat{x}_i,\dots,x_{n+1}]} \right) \\
&= \sum_{j<i} (-1)^j (-1)^i \sigma_\alpha|_{[x_0,\dots,\hat{x}_j,\dots,\hat{x}_i,\dots,x_{n+1}]} \\
&\quad + \sum_{i<j} (-1)^{j-1} (-1)^i \sigma_\alpha|_{[x_0,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}]} \\
&= 0.
\end{aligned}$$



<u>Def</u> The n-dimensional singular homology group of X is $H_n(X) = \dfrac{\text{Ker}\, \partial_n}{\text{Im}\, \partial_{n+1}}$.

homeomorphic

Advantage: Clearly $X \cong Y \Rightarrow H_n(X) \cong H_n(Y)$.
Also, we will show $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y)$.

homotopy equivalent

- Disadvantage: Typically $C_n(X)$ is infinite dimensional for all n.

- Let X be a Δ-complex. Thm 2.27 will show $H_n(X) \cong H_n^\Delta(X)$.
Hence $H_n(X)$ is finitely generated if X is a finite Δ-complex, which a priori is not clear.

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots$$

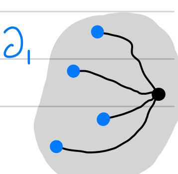
- Prop 2.6 $\quad H_n\left(\coprod_\alpha X_\alpha\right) \cong \bigoplus_\alpha H_n(X_\alpha)$.

- Prop 2.7 $\quad H_0(X) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$,
where A is the set of path-components of X.

Pf Sketch $\quad H_0(X) = C_0(X)/\operatorname{Im}\partial_1$. For $X \neq \emptyset$, homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$ via $\sum n_\alpha \sigma_\alpha \longmapsto \sum n_\alpha$ is surjective.
For X path-connected, Hatcher shows $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$, giving $H_0(X) = C_0(X)/\operatorname{Im}\partial_1 = C_0(X)/\operatorname{Ker}\varepsilon \cong \mathbb{Z}$.

$\operatorname{Im}\partial_1 \subset \operatorname{Ker}\varepsilon$ clear.

$\operatorname{Ker}\varepsilon \subset \operatorname{Im}\partial_1$



- <u>Prop 2.8</u> $H_n(pt) = 0$ for $n \geq 1$.

<u>Pf</u>

$$\cdots \xrightarrow[0]{\partial_5} \underset{\mathbb{Z}}{\underset{\parallel}{C_4(X)}} \xrightarrow[\cong]{\partial_4} \underset{\mathbb{Z}}{\underset{\parallel}{C_3(X)}} \xrightarrow[0]{\partial_3} \underset{\mathbb{Z}}{\underset{\parallel}{C_2(X)}} \xrightarrow[\cong]{\partial_2} \underset{\mathbb{Z}}{\underset{\parallel}{C_1(X)}} \xrightarrow[0]{\partial_1} \underset{\mathbb{Z}}{\underset{\parallel}{C_0(X)}} \xrightarrow{\partial_0} 0$$

For $X \neq \emptyset$, the <u>reduced homology</u> $\tilde{H}_n(X)$
satisfies $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$
and $H_n(X) \cong \tilde{H}_n(X)$ $\forall n \geq 1$.
It is defined as the homology of the chain complex

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$\sum n_\alpha \sigma_\alpha \longmapsto \sum n_\alpha$$

Homotopy Invariance of Singular Homology

Corollary 2.11 $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y)$.

The proof will use algebraic formalisms of chain complexes, chain maps, and chain homotopies.

Def A chain complex $C_\bullet$ is a sequence of abelian groups and homomorphisms $\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$ with $\partial_i \circ \partial_{i+1} = 0 \ \forall i$.

A chain map is a collection of homomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

with $f_{i-1}\partial_i = \partial'_i f_i \ \forall i$.

$$\begin{tikzcd} C_\bullet \arrow[d, "f"] \\ C'_\bullet \end{tikzcd}$$

Prop 2.9 A chain map induces homomorphisms $H_n(C_\bullet) \to H_n(C'_\bullet)$ $\forall n$.

Pf f_n maps $\operatorname{Ker}\partial_n$ to $\operatorname{Ker}\partial'_n$ since $\partial_n\alpha = 0$ implies $\partial'_n(f_n\alpha) = f_{n-1}(\partial_n\alpha) = f_{n-1}(0) = 0$.

f_n maps $\operatorname{Im}\partial_{n+1}$ to $\operatorname{Im}\partial'_{n+1}$ since $f_n(\partial_{n+1}\beta) = \partial'_{n+1}(f_{n+1}\beta)$.

So we get an induced homomorphism $$H_n(C_\bullet) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} \xrightarrow{f_*} \frac{\operatorname{Ker}\partial'_n}{\operatorname{Im}\partial'_{n+1}} = H_n(C'_\bullet)$$

defined by $\alpha + \operatorname{Im}\partial_{n+1} \longmapsto f_n(\alpha) + \operatorname{Im}\partial'_{n+1}$ for $\alpha \in \operatorname{Ker}\partial_n$.

$[\alpha] \longmapsto [f_n\alpha]$

Two chain maps $f, g : C_\bullet \to C'_\bullet$ are <u>chain homotopic</u> if there are homomorphisms $P : C_\bullet \to C'_{\bullet+1}$ with $\partial' P + P\partial = f - g$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

<u>Prop 2.12</u> Chain homotopic chain maps induce the same homomorphism on homology.

<u>Pf</u> If $\alpha \in \operatorname{Ker} \partial_n$, then

$f_n\alpha - g_n\alpha = \partial'_{n+1} P_n \alpha - P_{n-1}\partial_n \alpha = \partial'_{n+1} P_n \alpha$, so

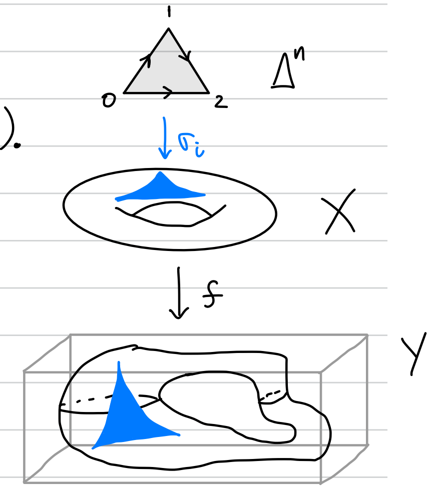
$f_n \alpha + \operatorname{Im} \partial'_{n+1} = g_n \alpha + \operatorname{Im} \partial'_{n+1}$.

For $[\alpha] \in H_n(C_\bullet)$, $[f_n\alpha] = [g_n\alpha]$.

<u>Aside</u> Chain complexes and maps form a "model category", meaning you can do homotopy theory on them! Other model categories include topological spaces, simplicial sets, and spectra.

Back to spaces!

A map $f: X \to Y$ of spaces induces
$f_\#: C_n(X) \to C_n(Y)$ $\forall n$ defined by
$f_\#\left(\sum_\alpha n_\alpha \sigma_\alpha\right) = \sum_\alpha n_\alpha f\sigma_\alpha$ (where $\sigma_\alpha: \Delta^n \to X$).



Note $f_\#\partial = \partial f_\#$ since

$$\begin{aligned} f_\# \partial\sigma &= f_\#\left(\textstyle\sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) \\ &= \textstyle\sum_i (-1)^i f\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \\ &= \partial f_\#\sigma. \end{aligned}$$

So $f_\#: C_\bullet(X) \to C_\bullet(Y)$ is a chain map.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_n(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Prop 2.9 gives an induced map $f_*: H_n(X) \to H_n(Y)$ $\forall n$, defined by

$$f_*(\alpha + \operatorname{Im} \partial^X_{n+1}) = f_\#\alpha + \operatorname{Im} \partial^Y_{n+1} \quad \text{for } \alpha \in \operatorname{Ker} \partial^X_n.$$

$$f_*([\alpha]) = [f_\#\alpha]$$

Homology is a functor from spaces to abelian groups.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathbb{1}_X"'] & X \end{tikzcd}$$

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"] \arrow[rr, bend right, "(gf)_* = g_* f_*"'] & H_n(Y) \arrow[r, "g_*"] & H_n(Z) \end{tikzcd} \qquad \begin{tikzcd} H_n(X) \arrow[r, "(\mathbb{1}_X)_* = \mathbb{1}_{H_n(X)}"'] & H_n(X) \end{tikzcd}$$

Indeed, for $[\alpha] \in H_n(X)$, i.e. $\alpha \in \operatorname{Ker} \partial_n^X$, we have

$$\begin{aligned} g_* f_* [\alpha] &= g_* [f_\# \alpha] \\ &= [g_\# f_\# \alpha] \\ &= [(gf)_\# \alpha] \\ &= (gf)_* [\alpha] \end{aligned}$$

Homotopy Invariance of Singular Homology

Thm 2.10 If two maps $f, g : X \to Y$ of spaces are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.

Corollary 2.11 $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y)$.

Pf of 2.11

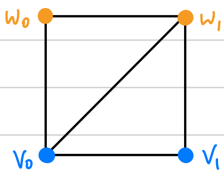
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} H_n(X) \arrow[r, bend left, "f_*"] & H_n(Y) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

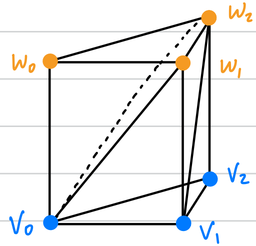
$gf \simeq \mathbb{1}_X \Rightarrow g_* f_* = (\mathbb{1}_X)_* = \mathbb{1}_{H_n(X)}$
$fg \simeq \mathbb{1}_Y \Rightarrow f_* g_* = \mathbb{1}_{H_n(Y)}$.

Pf of Thm 2.10 We will build a chain homotopy $P : C_\bullet(X) \to C_{\bullet+1}(Y)$ with $\partial P + P\partial = g_\# - f_\#$.

Pf of Thm 2.10

$\Delta^n \times \{1\} = [w_0, \ldots, w_n]$
$\Delta^n \times \{0\} = [v_0, \ldots, v_n]$

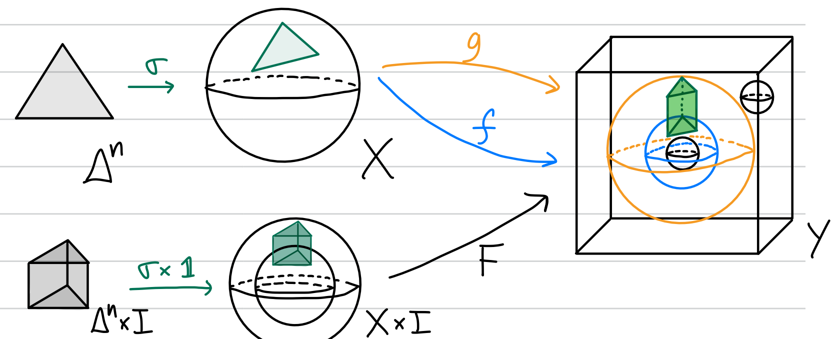




Subdivide $\Delta^n \times I$ into $(n+1)$-dimensional simplices

$[v_0, \ldots v_n, w_n]$
$[v_0, \ldots v_{n-1}, w_{n-1}, w_n]$
$\vdots$
$[v_0, \ldots v_i, w_i, \ldots w_n]$
$\vdots$
$[v_0, w_0, \ldots w_n]$.

Let $F: X \times I \to Y$ be a homotopy with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$.
Given a singular simplex $\sigma: \Delta^n \to X$, note
$F \circ (\sigma \times \mathbb{1}) : \Delta^n \times I \longrightarrow X \times I \longrightarrow Y$



Define the prism operators $P_n : C_n(X) \to C_{n+1}(Y)$ by
$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, F \circ (\sigma \times \mathbb{1})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$.

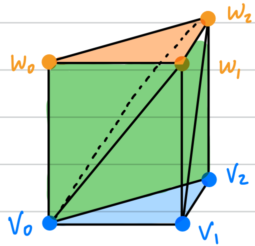
We will show this gives a chain homotopy, and hence $f_* = g_* : H_n(X) \to H_n(Y)$ by Prop 2.12.

Recall a chain homotopy satisfies

$$\partial^Y_{n+1} P_n + P_{n-1}\partial^X_n = g_\# - f_\#$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[dl, "P_n"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_n(Y) \arrow[r, "\partial^Y_n"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

i.e. $\partial P = \underbrace{g_\#}_{\text{top}} - \underbrace{f_\#}_{\text{bottom}} - \underbrace{P\partial}_{\text{sides}}$



We calculate

$$\partial P(\sigma) = \partial \sum_{i=0}^n (-1)^i\, F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,v_i,w_i,\dots,w_n]}$$

$$= \sum_{j\le i} (-1)^i(-1)^j\, F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}$$

$$+ \sum_{i\le j} (-1)^i(-1)^{j+1}\, F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

The terms with $i=j$ cancel

$[v_0,\dots,v_{i-1},\cancel{v_i},w_i,\dots,w_n]$ first sum, $j=i$

$[v_0,\dots,v_{i-1},\cancel{w_{i-1}},w_i,\dots,w_n]$ second sum

canceling interior n-simplices

except for

$$F\circ(\sigma\times\mathbb{1})\big|_{[\hat{v}_0,w_0,\dots,w_n]} = g\circ\sigma = g_\#\sigma \quad \text{(top)}$$

and

$$-F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,v_n,\hat{w}_n]} = -f\circ\sigma = -f_\#\sigma \quad \text{(bottom)}$$

The terms with $i\neq j$ are exactly $-P\partial(\sigma)$ since

$$P\partial(\sigma) = P\Big(\sum_{j=0}^n (-1)^j\, \sigma\big|_{[x_0,\dots,\hat{x}_j,\dots,x_n]}\Big)$$

$$= \sum_{i<j} (-1)^i(-1)^j\, F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

$$+ \sum_{j<i} (-1)^{i-1}(-1)^j\, F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} \quad \text{(sides)}$$