Exact sequences and excision

Recall a chain complex $C_\bullet$ is a sequence of abelian groups and homomorphisms
$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$ with $\partial_i \circ \partial_{i+1} = 0 \quad \forall i.$

Def A chain complex $\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$
is exact if $\mathrm{Im}\, \partial_{n+1} = \mathrm{Ker}\, \partial_n \quad \forall n.$

(So homology measures how far a chain complex is from being exact.)

Ex (i) $0 \to A \xrightarrow{\alpha} B$ is exact $\Leftrightarrow \mathrm{Ker}\, \alpha = 0$

$A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow \mathrm{Im}\, \alpha = B$

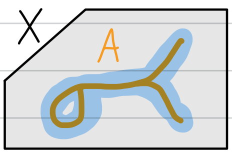
$0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow$ α is an isomorphism

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence (SES)

$\Leftrightarrow$ α is injective, β is surjective, and $\mathrm{Ker}\, \beta = \mathrm{Im}\, \alpha$.

In this case, $C \cong B/\mathrm{Ker}\,\beta = B/\mathrm{Im}\,\alpha \cong B/A$.

Def (X,A) is a good pair if $A \subset X$ is a nonempty closed subset that is a deformation retract of some neighborhood in X.



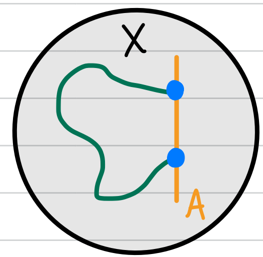
Ex X a CW complex;
A a subcomplex

Thm 2.13 For (X,A) a good pair, there is a long exact sequences (LES)

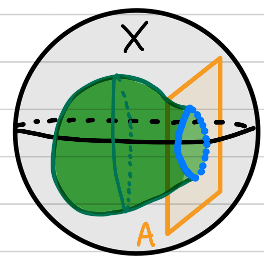
$$
\begin{array}{l}
\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \\
\tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{j_*} \tilde{H}_{n-1}(X/A) \xrightarrow{\partial} \cdots \\
\qquad \vdots \\
\cdots \to \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{j_*} \tilde{H}_0(X/A) \longrightarrow 0
\end{array}
$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \twoheadrightarrow X/A$ is the quotient map.

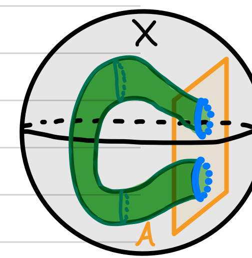
Rmk The idea behind $\partial : \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ is that an element $x \in \tilde{H}_n(X/A)$ is represented by $\alpha \in C_n(X)$ with $\partial\alpha$ a cycle in A whose homology class is $[\partial\alpha] =: \partial x \in \tilde{H}_{n-1}(A)$.



$X = D^2$
$\alpha \in C_1(X)$
$\partial\alpha \in H_0(A)$



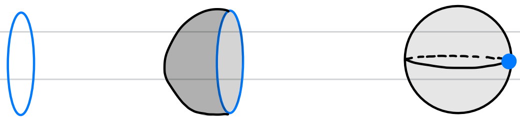
$X = D^3$
$\alpha \in C_2(X)$
$\partial\alpha \in H_1(A)$



Corollary 2.14 $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_k(S^n) = 0$ for $k \neq n$.

Pf Base case $n=0$: $S^0 = \bullet \quad \bullet$

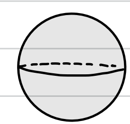
Inductive step: Assume true for S^{n-1}.



$$A = S^{n-1} \xhookrightarrow{i} X = D^n \xrightarrow{j} X/A \cong S^n$$

$$\begin{tikzcd}
 & \overset{0}{\cancel{\tilde{H}_n(D^n)}} \arrow[r, "j_*"] & \tilde{H}_n(S^n) \arrow[dll, "\partial"'] \\
\tilde{H}_{n-1}(S^{n-1}) \arrow[r, "i_*"] & \overset{0}{\cancel{\tilde{H}_{n-1}(D^n)}} &
\end{tikzcd}$$

$\Rightarrow \tilde{H}_n(S^n) \cong H_{n-1}(S^{n-1})$



Corollary 2.15 $\partial D^n = S^{n-1}$ is not a retract of D^n.

Pf $\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[rr, bend right, "\mathbb{1}"'] & D^n \arrow[r, "r"] & S^{n-1} \end{tikzcd}$ would induce $\begin{tikzcd} \underset{\cong\,\mathbb{Z}}{H_{n-1}(S^{n-1})} \arrow[r] \arrow[rr, bend right, "\mathbb{1}"'] & \underset{=\,0}{H_{n-1}(D^n)} \arrow[r] & \underset{\cong\,\mathbb{Z}}{H_{n-1}(S^{n-1})} \end{tikzcd}$

The proof of Thm 2.13 is a long & important story.

- For $A \subset X$, define <u>relative homology groups</u> $H_n(X,A)$.
- For $A \subset X$ arbitrary, prove there is a LES

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to \cdots$$

- For (X,A) a good pair, we have $H_n(X,A) \cong \tilde{H}_n(X/A)$ $\forall n$ by Prop 2.22.

Relative homology groups

Let $A \subset X$ be spaces. By ignoring structure (chains in A) we can sometimes go further.

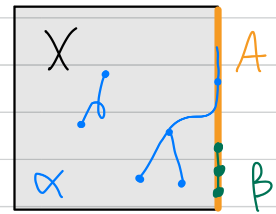
Let $C_n(X,A) = C_n(X)/C_n(A)$.
Note $\partial: C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, hence inducing $\partial: C_n(X,A) \longrightarrow C_{n-1}(X,A)$.
We get a chain complex $(\partial^2 = 0)$

$$\cdots \to C_{n+1}(X,A) \longrightarrow C_n(X,A) \to C_{n-1}(X,A) \to \cdots$$

Def The relative homology $H_n(X,A)$ is the n-th homology of the above chain complex.

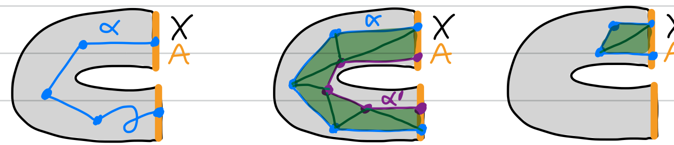
- Elements of $H_n(X,A)$ are represented by relative n-cycles: $\alpha \in C_n(X)$ with $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial in $H_n(X,A)$ iff it is a relative boundary: $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Ex



$\alpha \in C_1(X)$
$\beta \in C_1(A) \subset C_1(X)$
$\beta = 0$ in $C_1(X,A)$
$\alpha + \beta = \alpha$ in $C_1(X,A)$

Ex Relative homology is "modulo A"



$[\alpha] \in H_1(X,A)$ $\quad$ $[\alpha] = [\alpha'] \in H_1(X,A)$ $\quad$ $[\alpha] = 0 \in H_1(X,A)$

<u>Theorem (No number Pg 115 or 117)</u> For $A \subset X$ any pair of spaces, there is a LES

$$\begin{tikzcd}
\cdots \arrow[dr] & & \\
& H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X,A) \arrow[dll, "\partial"'] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r, "j_*"] & H_{n-1}(X,A) \arrow[dll, "\partial"'] \\
& \cdots & \vdots & \cdots \\
& H_0(A) \arrow[r, "i_*"] & H_0(X) \arrow[r, "j_*"] & H_0(X,A) \arrow[r] & 0
\end{tikzcd}$$

<u>Pf</u> The commutative diagram

$$\begin{tikzcd}
& \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "\partial"] & C_n(X) \arrow[r] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
& & & &
\end{tikzcd}$$

is called a <u>short exact sequence (SES) of chain complexes</u>
Since each row is exact and each column is a chain complex.

The theorem now follows from a more general fact:

Thm 2.16 (snake lemma) Let

$$
\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i"] & B_{n-2} \arrow[r, "j"] & C_{n-2} \arrow[r] & 0
\end{tikzcd}
$$

$b \mapsto c$, $b \mapsto \partial b$, $c \mapsto 0$, $a \mapsto \partial b$

be any SES of chain complexes.
Then we get a LES of homology groups:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial"'] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r, "\partial"] & \cdots
\end{tikzcd}
$$

Pf i_* and j_* are well-defined since i and j are chain maps.

Defining $\partial : H_n(C) \to H_{n-1}(A)$ takes work!
See diagram, where $\partial[c] := a$.

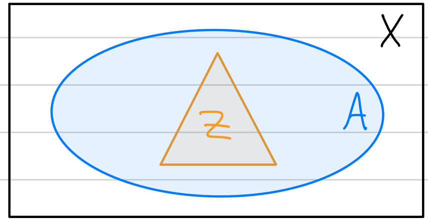
$\partial a = 0$ since $i(\partial a) = \partial i a = \partial\partial b = 0$ and i is injective.

- Also $\partial : H_n(C) \to H_{n-1}(A)$ is well-defined up to
  - the choice of b
    $j(b) = j(b') \Rightarrow b - b' \in \operatorname{Ker} j = \operatorname{Im} i$
    $\Rightarrow b - b' \in i(a_n)$ for $a_n \in A_n$
  - the choice of c.
    Say $c' \in C_{n+1}$ with $j(b') = c'$.
    Then $c + \partial c' = c + \partial j b' = c + j\partial b' = j(b + \partial b')$.
    Note $\partial(b + \partial b') = \partial b + \partial\partial b' = \partial b$.

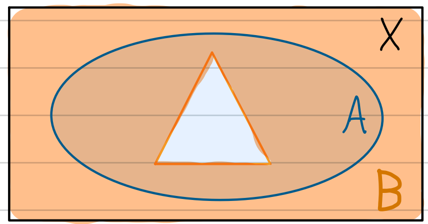
- Exactness follows by checking:
  (i) $\operatorname{Im} i_* \subset \operatorname{Ker} j_*$ — $ji = 0 \Rightarrow j_* i_* = 0$.
  (ii) $\operatorname{Im} j_* \subset \operatorname{Ker} \partial$ — When defining $\partial[c]$ in this case we have b a cycle, hence $\partial b = 0$.
  (iii) $\operatorname{Im} \partial \subset \operatorname{Ker} i_*$ — $i_*\partial[c] = i_*[a] = [\partial b] = 0$.
  (iv) $\operatorname{Ker} j_* \subset \operatorname{Im} i_*$
  (v) $\operatorname{Ker} \partial \subset \operatorname{Im} j_*$
  (vi) $\operatorname{Ker} i_* \subset \operatorname{Im} \partial$
  $\Big\}$ HW

Thm 2.20 (Excision Theorem)

If $Z \subset A \subset X$ with $\mathrm{cl}\, Z \subset \mathrm{int}\, A$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$ $\forall n$.



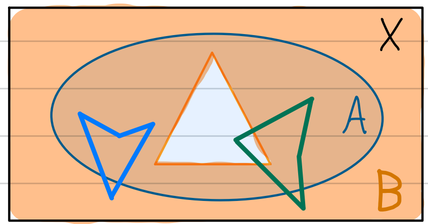
Equivalently, for $A, B \subset X$ with $X = \mathrm{int}\, A \cup \mathrm{int}\, B$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \longrightarrow H_n(X,A)$ $\forall n$.



(Translation: $B = X - Z$, so $X - \mathrm{int}\, B = \mathrm{cl}\, Z$.
So $\mathrm{cl}\, Z \subset \mathrm{int}\, A \iff X = \mathrm{int}\, A \cup \mathrm{int}\, B$.)

The proof uses the following machinery.

Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each $\sigma_i : \Delta^n \to X$ has image contained in A or in B.



$\sigma \in C_1(A+B)$ $\sigma' \notin C_1(A+B)$

Prop 2.21 (Special case) Let $A, B \subset X$ with $X = \operatorname{int} A \cup \operatorname{int} B$.
The inclusion $\iota : C_n(A+B) \to C_n(X)$ is a chain homotopy equivalence.
($\exists\, \rho : C_n(X) \to C_n(A+B)$ with $\iota\rho$ chain homotopic to $\mathbb{1}_{C_n(X)}$
and $\rho\iota$ chain homotopic to $\mathbb{1}_{C_n(A+B)}$.)
Hence ι induces isomorphisms $H_n(A+B) \cong H_n(X)$ $\forall n$.

$$\begin{tikzcd} C_\bullet(A+B) \arrow[r, bend left, "\iota"] & C_\bullet(X) \arrow[l, bend left, "\rho"] \end{tikzcd}$$

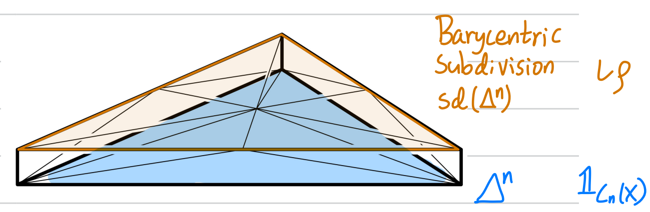
$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] & C_n(X) \arrow[r] \arrow[dl, "D_n"'] \arrow[d, shift right, "\iota\rho"'] \arrow[d, shift left, "\mathbb{1}_{C_n(X)}"] & C_{n-1}(X) \arrow[r] \arrow[dl, "D_{n-1}"'] & \cdots \\ \cdots \arrow[r] & C_{n+1}(X) \arrow[r] & C_n(X) \arrow[r] & C_{n-1}(X) \arrow[r] & \cdots \end{tikzcd}$$

$$\partial D_n + D_{n-1}\partial = \mathbb{1}_{C_n(X)} - \iota\rho$$

Here $\rho\iota = \mathbb{1}_{C_n(A+B)}$.

Rmks

- The proof is $4\frac{1}{2}$ pages in Hatcher.
- $H_n(A+B)$ is non-standard notation.
- Note ρ must split chains in X into smaller pieces.
- D_n is constructed via stacked iterations of barycentric subdivision



Pf of Thm 2.20 (Excision)

Let $X = \operatorname{int} A \cup \operatorname{int} B$.
By Prop 2.21 we have $\iota : C_n(A+B) \to C_n(X)$ and $\rho : C_n(X) \to C_n(A+B)$ with $\rho\iota = \mathbb{1}_{C_n(A+B)}$ and $\partial D + D\partial = \mathbb{1}_{C_n(X)} - \iota\rho$.

These maps all take chains in A to chains in A, giving

chain homotopy equivalence

isomorphism

$$\begin{tikzcd}
C_{n+1}(X)/C_{n+1}(A) \arrow[r, "\partial"] & C_n(X)/C_n(A) \arrow[r, "\partial"] & C_{n-1}(X)/C_{n-1}(A) \\
C_{n+1}(A+B)/C_{n+1}(A) \arrow[u, "\iota"'] \arrow[r, "\partial"] & C_n(A+B)/C_n(A) \arrow[u, "\iota"'] \arrow[r, "\partial"] & C_{n-1}(A+B)/C_{n-1}(A) \arrow[u, "\iota"'] \\
C_{n+1}(B)/C_{n+1}(A\cap B) \arrow[u, "\cong"'] \arrow[r, "\partial"] & C_n(B)/C_n(A\cap B) \arrow[u, "\cong"'] \arrow[r, "\partial"] & C_{n-1}(B)/C_{n-1}(A\cap B) \arrow[u, "\cong"']
\end{tikzcd}$$

- ι is still a chain homotopy equivalence (D still induces a chain homotopy).
  $C_n(B) \hookrightarrow C_n(A+B)$ induces a chain isomorphism $C_n(B)/C_n(A\cap B) \xrightarrow{\cong} C_n(A+B)/C_n(A)$: both quotient groups are free with basis n-chains in B not contained in A.

  2nd iso thm of groups $\dfrac{H}{H\cap N} \cong \dfrac{HN}{N}$

- Hence $H_n(X, A) \cong H_n(A+B, A) \cong H_n(B, A\cap B)$.

From the LES in <u>Theorem $\left(\substack{\text{No number} \\ \text{Pg 115 or 117}}\right)$</u>

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} \cdots$$

we prove the LES in <u>Thm 2.13</u>

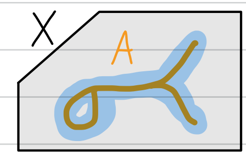
$$\cdots \xrightarrow{\partial} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \cdots$$

when (X,A) is a <u>good pair</u> (nonempty A is a deformation retract of some neighborhood in X).

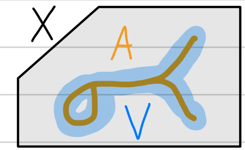
This follows from Prop 2.22 (and noting that $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ is exact).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\oplus & & \oplus & & \oplus & & \oplus \\
\tilde{H}_1(X/A) & \xrightarrow{\partial} & \tilde{H}_0(A) & \longrightarrow & \tilde{H}_0(X) & \longrightarrow & \tilde{H}_0(X/A) \\
\text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\
H_1(X,A) & \xrightarrow{\partial} & H_0(A) & \longrightarrow & H_0(X) & \longrightarrow & H_0(X,A)
\end{array}$$

<u>Prop 2.22</u> For (X,A) a good pair, the quotient map $(X,A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X,A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ $\forall n$.



Prop 2.22 For (X,A) a good pair, the quotient map $(X,A) \to (X/A, A/A)$ induces isomorphisms $q_*: H_n(X,A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ $\forall n$.



Pf Let V be a neighborhood that deformation retracts onto A.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "q_*"'] \arrow[dr, phantom, "\curvearrowright"] & H_n(X,V) \arrow[d, "q_*"] \arrow[dr, phantom, "\curvearrowleft"] & H_n(X-A, V-A) \arrow[l, "\cong"', "\text{excision}"] \arrow[d, "q_*"', "\cong"] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\cong"', "\text{excision}"]
\end{tikzcd}$$

- The right horizontal maps are $\cong$ by excision.
- The rightmost vertical map is $\cong$ since $q: X \to X/A$ is a homeomorphism on the complement of A.
- It's plausible the left horizontal maps are $\cong$ since V deformation retracts to A.

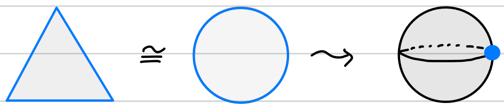
Proof uses LES of a triple

$$\cdots \to \underset{\overset{\parallel}{0}}{H_n(V,A)} \to H_n(X,A) \xrightarrow{\cong} H_n(X,V) \to \underset{\overset{\parallel}{0}}{H_{n-1}(V,A)} \to \cdots$$

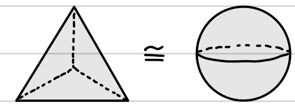
- Hence the leftmost vertical map is $\cong$ by commutativity.

<u>Ex 2.23</u> $H_n(\Delta^k, \partial\Delta^k) \cong H_n(D^k, S^{k-1}) \cong \tilde{H}_n(D^k/S^{k-1}) \cong \tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{otherwise.} \end{cases}$

<u>$k=2$</u>



<u>$k=3$</u>



The equivalence of simplicial and singular homology

The Five Lemma Consider the following commutative diagram of abelian groups and exact rows.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[r, "k"] \arrow[d, "\gamma"'] & D \arrow[r, "\ell"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

$$\begin{tikzcd}
b \arrow[d, mapsto] & c \arrow[r, mapsto] & d \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
b' & c' \arrow[r, mapsto] & k'c' \arrow[r, mapsto] & 0
\end{tikzcd}$$

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

<u>Pf</u> We'll show

(i) β, δ surjective and ε injective $\Rightarrow \gamma$ surjective

(ii) β, δ injective and α surjective $\Rightarrow \gamma$ injective.

For (i), let $c' \in C'$.

Construct $c \in C$ via "diagram chasing".

$k'(c' - \gamma c) = k'c' - k'\gamma c = k'c' - \delta k c = k'c' - \delta d = 0$

$\Rightarrow c' - \gamma c = j'b'$ for some $b' \in B'$.

β surjective $\Rightarrow b' = \beta b$ for some $b \in B$.

Note $\gamma(c + jb) = \gamma c + \gamma j b = \gamma c + j'\beta b = \gamma c + j'b' = c'$.

So γ is surjective.

<u>The equivalence of simplicial and singular homology</u>

<u>The Five Lemma</u> Consider the following commutative diagram of abelian groups and exact rows.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[r, "k"] \arrow[d, "\gamma"'] & D \arrow[r, "\ell"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[d, mapsto] & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
a' \arrow[r, mapsto] & \beta b & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

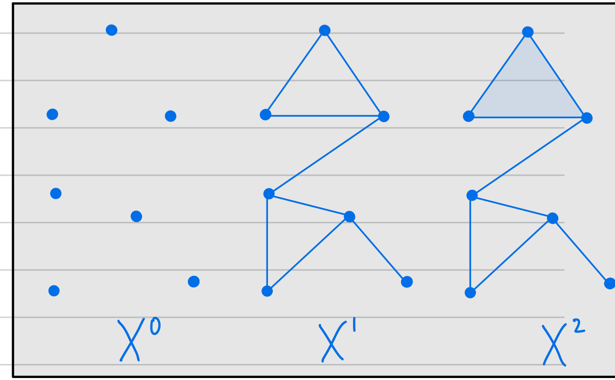
<u>Pf</u> We'll show

(i) β, δ surjective and ε injective $\Rightarrow \gamma$ surjective

(ii) β, δ injective and α surjective $\Rightarrow \gamma$ injective.

For (ii), let $c \in C$ with $\gamma(c) = 0$.
Construct $a \in A$ via "diagram chasing".
$\beta(ia - b) = \beta ia - \beta b = i'\alpha a - \beta b = i'a' - \beta b = 0$
$\Rightarrow ia - b = 0$ since β injective.
Thus $ia = b$ and so $c = jb = jia = 0$ since $ji = 0$.
So γ is injective.

Thm 2.27 The homomorphisms $H_n^\Delta(X,A) \to H_n(X,A)$ from simplicial to singular homology are isomorphisms $\forall n$ when (X,A) is a Δ-complex pair.



Rmk Taking $A=\emptyset$ gives $H_n^\Delta(X) \cong H_n(X)$.

Pf We first do the case when X is finite-dimensional and $A=\emptyset$. Let X^k be the k-skeleton of X.

We have a commutative diagram with exact rows.

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "\alpha"'] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "\beta"'] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "\gamma"'] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "\delta"'] & H_{n-1}^\Delta(X^{k-1}) \arrow[r] \arrow[d, "\varepsilon"'] & {} \\
{} \arrow[r] & H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \arrow[r] & {}
\end{tikzcd}$$

We may assume β, ε are isomorphisms by induction on k.

$$\cdots \to \underset{\substack{\| \\ 0}}{\Delta_{n+1}(X^n, X^{n-1})} \xrightarrow{\partial} \Delta_n(X^n, X^{n-1}) \xrightarrow{\partial} \underset{\substack{\| \\ 0}}{\Delta_{n-1}(X^n, X^{n-1})} \to \cdots$$

Note $H_n^\Delta(X^k, X^{k-1}) \cong \Delta_n(X^k, X^{k-1}) = \begin{cases} \mathbb{Z}^{\#k\text{-simplices}} & n=k \\ 0 & \text{otherwise} \end{cases}$

For singular homology, recall (Ex 2.23) that

$H_n(\Delta^k, \partial\Delta^k) = \begin{cases} \mathbb{Z} & n=k \quad (\text{generated by } \Delta^k \to \Delta^k) \\ 0 & n \neq k \end{cases}$

The map $\Phi : \coprod_\alpha (\Delta^k_\alpha, \partial\Delta^k_\alpha) \longrightarrow (X^k, X^{k-1})$ via attaching maps induces a homeomorphism

$$\vee_\alpha S^k = \vee_\alpha (\Delta^k_\alpha / \partial\Delta^k_\alpha) = \coprod_\alpha \Delta^k_\alpha \Big/ \coprod_\alpha \partial\Delta^k_\alpha \xrightarrow[\text{from } \Phi]{\cong} X^k / X^{k-1}$$

and hence an isomorphism on H_n.

So $H_n(X^k, X^{k-1}) \cong \begin{cases} \mathbb{Z}^{\#k\text{-simplices}} & n=k \\ 0 & n \neq k \end{cases}$

and also α, δ are isomorphisms.

By the five lemma, γ is an isomorphism.

- The case when X is infinite-dimensional requires more work.

- Now assume $A \neq \emptyset$. We have

$$
\begin{tikzcd}
\rightarrow H_n^\Delta(A) \arrow[r] \arrow[d, "\cong"] & H_n^\Delta(X) \arrow[r] \arrow[d, "\cong"] & H_n^\Delta(X,A) \arrow[r] \arrow[d] & H_{n-1}^\Delta(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}^\Delta(X) \rightarrow \arrow[d, "\cong"] \\
\rightarrow H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \rightarrow
\end{tikzcd}
$$

Applying the five lemma again finishes the proof of Thm 2.27.