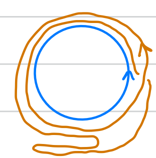
Section 2.2: Computations and Applications

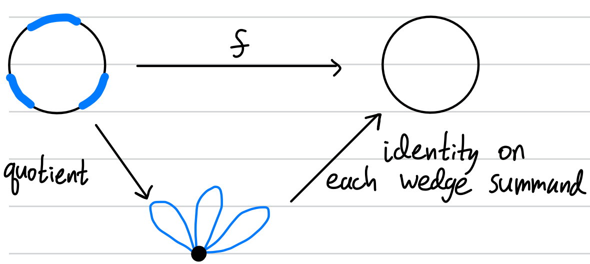
Degree

Def The *degree* $\deg(f)$ of a map $f: S^n \to S^n$ $(n \geq 1)$ is the integer d s.t. $f_*: \underset{\cong \atop \mathbb{Z}}{H_n(S^n)} \longrightarrow \underset{\cong \atop \mathbb{Z}}{H_n(S^n)}$ is of the form $f_*(\alpha) = d\alpha$.

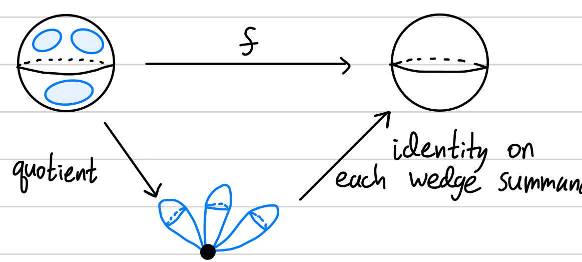
Ex $f: S^1 \to S^1$ with $\deg(f) = 2$

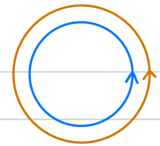


Ex $f: S^1 \to S^1$ with $\deg(f) = 3$



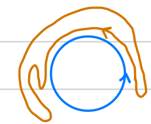
Ex $f: S^2 \to S^2$ with $\deg(f) = 3$





Basic properties

(a) $\deg(\mathbb{1}_{S^n}) = 1$ since $\mathbb{1}_* = \mathbb{1}_{H_n(S^n)}$.



(b) $\deg(f) = 0$ if f is not surjective.

Pf Pick $x_0 \in f(S^n)$.

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[rr, bend right, "f"'] & S^n - \{x_0\} \arrow[r, hook] & S^n \end{tikzcd}$$

Apply H_n to get

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "f_*"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}.$$

(c) If $f \simeq g$, then $\deg(f) = \deg(g)$ (since $f_* = g_*$).

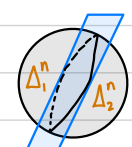
This is in fact an if-and-only-if (Corollary 4.25)!

(d) $\deg(fg) = \deg(f)\deg(g)$ since $(fg)_* = f_* g_*$.

As a consequence, $\deg(f) = \pm 1$ if f is a homotopy equivalence, since $fg \simeq \mathbb{1} \Rightarrow \deg(f)\deg(g) = \deg(fg) = \deg(\mathbb{1}) = 1$.

(e) $\deg(f) = -1$ if f is a reflection

Generator $\Delta_1^n - \Delta_2^n$ maps to $\Delta_2^n - \Delta_1^n$.



S^n

(f) The antipodal map $-\mathbb{1}: S^n \to S^n$ via $x \mapsto -x$ has degree $\deg(-\mathbb{1}) = (-1)^{n+1}$ since it is a composition of $n+1$ reflections.

(g) If $f: S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

Pf $f(x) \neq x$ implies $(1-t)f(x) + t(-x)$ misses the origin, so

$H(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$ is a homotopy from f to $-\mathbb{1}$.

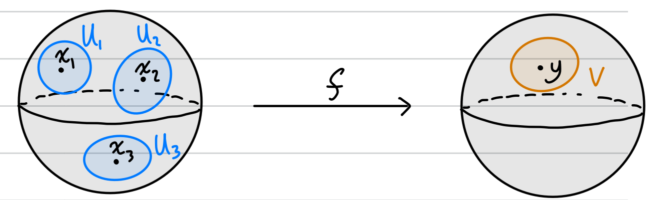
Thm 2.28 S^n has a continuous nonzero tangent vector field iff n odd.

Pf ($\Leftarrow$) For $n = 2k-1$ odd, consider the vector field $v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$.
($\Rightarrow$) Given this vector field v, $H: S^n \times [0,\pi] \longrightarrow S^n$ by $H(x,t) = \cos(t)x + \sin(t)\frac{v(x)}{\|v(x)\|}$ is a homotopy from $\mathbb{1}$ to $-\mathbb{1}$, meaning $1 = (-1)^{n+1}$, so n is odd.

Prop 2.29 For n even, $\mathbb{Z}_2$ is the only nontrivial group acting freely on S^n.
Pf Homemorphisms have degree ± 1, giving $G \xrightarrow{\deg} \{-1, 1\}$, a homomorphism by (d).
G acts freely $\Rightarrow$ $G \setminus \{id\} \to \{-1^{n+1}\} = \{-1\}$, so the kernel is trivial and $G \subset \mathbb{Z}_2$.

Local degree

Let $f: S^n \to S^n$. Suppose $f^{-1}(y) = \{x_1, \dots, x_m\}$ is finite for some $y \in S^n$. Choose disjoint neighborhoods $U_i \ni x_i$ and $V \ni y$ with $f(U_i) \subset V$ $\forall i$.



For all i we have:

$$\begin{tikzcd}
H_n(U_i, U_i - x_i) \arrow[rr, "f_*"] \arrow[d, "\cong \text{ by excision}"] & & H_n(V, V-y) \arrow[d, "\cong \text{ excision}"] \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - y) \\
H_n(S^n) \arrow[rr, "f_*"] \arrow[u, "\cong \text{ by LES of pair with } S^n - x_i \simeq *"] & & H_n(S^n) \arrow[u, "\cong \text{ LES pair}"']
\end{tikzcd}$$

Hence all six groups are isomorphic to $\mathbb{Z}$.

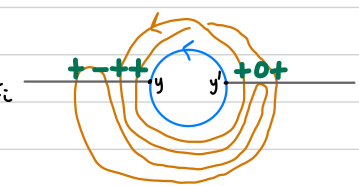
The top map is multiplication by an integer $\deg(f)|x_i$, the <u>local degree at x_i</u>.

<u>Prop 2.30</u> $\deg(f) = \sum_i \deg(f)|x_i$.

<u>Rmk</u> If f maps U_i homeomorphically onto V, then $\deg(f)|x_i = \pm 1$.

<u>Ex</u>

$\deg(f)$
$= \sum_i \deg(f)|x_i$
$= 1 + 1 - 1 + 1$
$= 2$



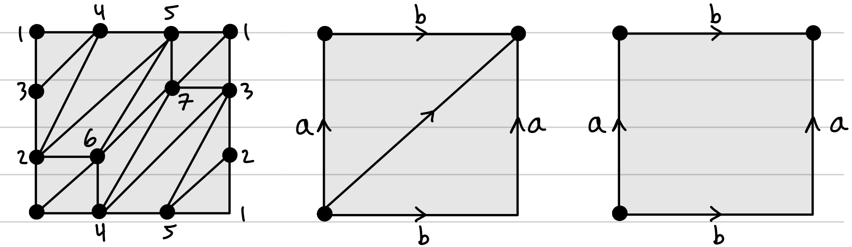
$\deg(f)$
$= \sum_i \deg(f)|x_i'$
$= 1 + 0 + 1$
$= 2$

Rmk $\deg(f)$ can be defined for any map $f: M \to N$ between orientable manifolds of the same dimension.

Rmk $\deg(Sf) = \deg(f)$, where $Sf: S^{n+1} \to S^{n+1}$ is the suspension of map $f: S^n \to S^n$.

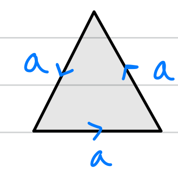
Cellular homology

For X a Δ-complex we defined $H_n^\Delta(X)$ and proved $H_n^\Delta(X) \cong H_n(X)$.



Simplicial complexes $\subsetneq$ Δ- complexes $\subsetneq$ CW complexes

For X a CW-complex we now define $H_n^{CW}(X)$ and prove $H_n^{CW}(X) \cong H_n(X)$.



$H_n^{CW}(X) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ is the homology of a chain complex

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} \underbrace{H_n(X^n, X^{n-1})}_{\cong\ \mathbb{Z}^{(\#\ n\text{-cells})}} \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

"homology squared"

We postpone a definition of d_n, and a verification that $d_n d_{n+1} = 0$.

Cellular boundary formula

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2})$$

$H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{(\#\ n\text{-cells})}$

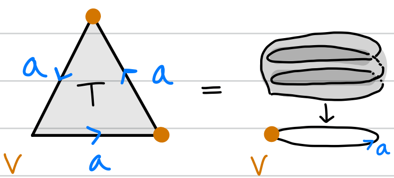
For e^n_α an n-cell in X, we have $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$, where $d_{\alpha\beta}$ is

- (roughly) the # of times the attaching map for e^n_α "wraps around" e^{n-1}_β

  (less roughly) the degree of the map

$$S^{n-1}_\alpha \xrightarrow[\text{attaching map of } e^n_\alpha]{} X^{n-1} \xrightarrow[\text{quotient collapsing } X^{n-1} - e^{n-1}_\beta \text{ to a point}]{} S^{n-1}_\beta \quad \left(\begin{smallmatrix}\text{for}\\ n>1\end{smallmatrix}\right).$$

Ex

$X =$ 

disk attached, wrapping 3 times around

Chain complex

$$\underset{\text{3-cells}}{0} \xrightarrow{d_3} \underset{\text{2-cells}}{\mathbb{Z}} \xrightarrow{d_2} \underset{\text{1-cells}}{\mathbb{Z}^3} \xrightarrow{d_1} \underset{\text{0-cells}}{\mathbb{Z}} \xrightarrow{d_0} 0$$

$T \longmapsto 3a$

$a \longmapsto 0$

$v \longmapsto 0$

$$H_0^{CW}(X) = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$$
$$H_1^{CW}(X) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \cong \mathbb{Z}/3\mathbb{Z}$$
$$H_2^{CW}(X) = \operatorname{Ker} d_2 / \operatorname{Im} d_3 = 0$$

Cellular boundary formula

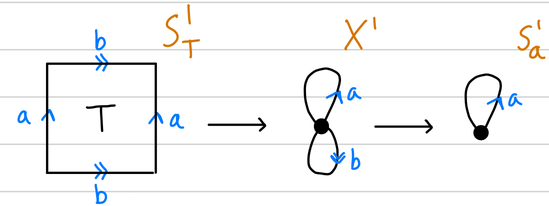
$$H_{n+1}(X^{n+1},X^n) \xrightarrow{d_{n+1}} \underset{\mathbb{Z}^{(\#\ n\text{-cells})}}{\underset{\cong}{H_n(X^n,X^{n-1})}} \xrightarrow{d_n} H_{n-1}(X^{n-1},X^{n-2})$$

For e^n_α an n-cell in X, we have $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$, where $d_{\alpha\beta}$ is

- (roughly) the # of times the attaching map for e^n_α "wraps around" e^{n-1}_β

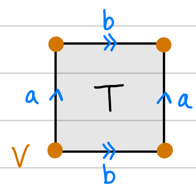
(less roughly) the degree of the map

$$S^{n-1}_\alpha \xrightarrow[\text{of } e^n_\alpha]{\text{attaching map}} X^{n-1} \xrightarrow[X^{n-1}-e^{n-1}_\beta \text{ to a point}]{\text{quotient collapsing}} S^{n-1}_\beta \quad \left(\text{for } n>1\right).$$



Ex

X = torus



To see $d_2(T)=0$, note $d_2(T) = d_{Ta}\,a + d_{Tb}\,b$, where T is attached along $aba^{-1}b^{-1}$, and so $d_{Ta} = 1-1 = 0$ and $d_{Tb} = 1-1 = 0$.

Chain complex

$$\overset{\text{3-cells}}{0} \xrightarrow{d_3} \overset{\text{2-cells}}{\mathbb{Z}} \xrightarrow{d_2} \overset{\text{1-cells}}{\mathbb{Z}^2} \xrightarrow{d_1} \overset{\text{0-cells}}{\mathbb{Z}} \xrightarrow{d_0} 0$$

$T \mapsto 0$

$a \mapsto 0$

$b \mapsto 0$

$V \mapsto 0$

$H^{CW}_0(X) = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$

$H^{CW}_1(X) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \cong \mathbb{Z}^2$

$H^{CW}_2(X) = \operatorname{Ker} d_2 / \operatorname{Im} d_3 = \mathbb{Z}$

Cellular boundary formula

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} \underset{\mathbb{Z}^{(\#\ n\text{-cells})}}{\underset{\shortparallel}{H_n(X^n, X^{n-1})}} \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2})$$

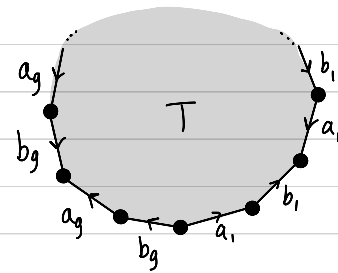
For e^n_α an n-cell in X, we have

$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$, where $d_{\alpha\beta}$ is

- (roughly) the # of times the attaching map for e^n_α "wraps around" e^{n-1}_β

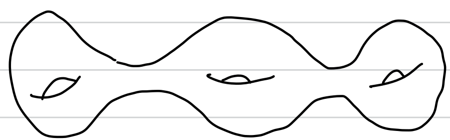
  (less roughly) the degree of the composition

$$S^{n-1}_\alpha \xrightarrow[\text{of } e^n_\alpha]{\text{attaching map}} X^{n-1} \xrightarrow[X^{n-1} - e^{n-1}_\beta \text{ to a point}]{\text{quotient collapsing}} S^{n-1}_\beta \quad (\text{for } n>1).$$



Ex

X = genus g torus



Chain complex

$$\overset{\text{3-cells}}{0} \xrightarrow{d_3} \overset{\text{2-cells}}{\mathbb{Z}} \xrightarrow{d_2} \overset{\text{1-cells}}{\mathbb{Z}^{2g}} \xrightarrow{d_1} \overset{\text{0-cells}}{\mathbb{Z}} \xrightarrow{d_0} 0$$

$T \mapsto 0$ (under d_2)

$a_1 \mapsto 0$, $b_1 \mapsto 0$, $\vdots$, $a_g \mapsto 0$, $b_g \mapsto 0$ (under d_1)

$V \mapsto 0$ (under d_0)

$H_0^{CW}(X) = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$

$H_1^{CW}(X) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \cong \mathbb{Z}^{2g}$

$H_2^{CW}(X) = \operatorname{Ker} d_2 / \operatorname{Im} d_3 = \mathbb{Z}$

<u>Cellular homology</u> For X a CW complex, <u>cellular homology</u> $H_n^{CW}(X) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ is the homology of the following chain complex, where $d_n = j_{n-1}\partial_n$ is the <u>cellular boundary map</u>.

$$
\begin{tikzcd}
 & & & & & 0 & & & & 0 & \\
 & & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & 0 \arrow[dr] & & H_{n-2}(X^{n-1}) \arrow[ur] & & \\
 & & & H_n(X^n) \arrow[ur, "i_{n+1}"] \arrow[dr, "j_n"] & & & & H_{n-2}(X^{n-2}) \arrow[ur, "i_{n-1}"] \arrow[dr, "j_{n-2}"] & & & \\
\cdots \arrow[rr] & & H_{n+1}(X^{n+1}, X^n) \arrow[rr, "d_{n+1}"] \arrow[ur, "\partial_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[rr] \arrow[ur, "\partial_{n-1}"] & & H_{n-2}(X^{n-2}, X^{n-3}) \arrow[rr] \arrow[dr, "\partial_{n-2}"] & & \cdots \\
 & H_{n+1}(X^{n+1}) \arrow[ur, "j_{n+1}"] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] \arrow[dr, "i_n"] & & & & H_{n-3}(X^{n-3}) \arrow[dr, "i_{n-2}"] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & H_{n-1}(X^n) \arrow[dr] & & & & {} \\
 & & & & & & & 0 & & &
\end{tikzcd}
$$

Each diagonal line is the LES for a pair (X^n, X^{n-1}).

Note $d_n d_{n+1} = j_{n-1}\partial_n j_n \partial_{n+1} = 0$ since $\partial_n j_n = 0$.

Lemma 2.34(c) implies $H_n(X^{n+1}) \cong H_n(X^{n+2}) \cong \ldots \cong H_n(X)$.

Pf for X finite follows from LES $\underbrace{H_{n+1}(X^{n+2}, X^{n+1})}_{=0} \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X^{n+2}) \longrightarrow \underbrace{H_n(X^{n+2}, X^{n+1})}_{=0}$ and induction.

Thm 2.35 For X a CW complex, $H_n^{CW}(X) \cong H_n(X)$.

Pf Note $H_n(X) \cong H_n(X^{n+1}) \underset{\substack{\uparrow \\ \text{First iso thm}}}{\cong} H_n(X^n)/\operatorname{Ker} i_{n+1} \underset{\substack{\uparrow \\ \text{Exactness}}}{=} H_n(X^n)/\operatorname{Im} \partial_{n+1}$.

We'll now show j_n induces an isomorphism $j_n : H_n(X^n)/\operatorname{Im}\partial_{n+1} \xrightarrow{\cong} \operatorname{Ker} d_n/\operatorname{Im} d_{n+1} =: H_n^{CW}(X)$.

Indeed, since j_n is injective it maps $\operatorname{Im}\partial_{n+1}$ isomorphically onto $\operatorname{Im} j_n\partial_{n+1} = \operatorname{Im} d_{n+1}$, and $H_n(X^n)$ isomorphically onto $\operatorname{Im} j_n = \operatorname{Ker}\partial_n \underset{\substack{\uparrow \\ \text{since } j_{n-1} \text{ injective}}}{=} \operatorname{Ker} j_{n-1}\partial_n = \operatorname{Ker} d_n$.

Immediate applications If a CW complex X

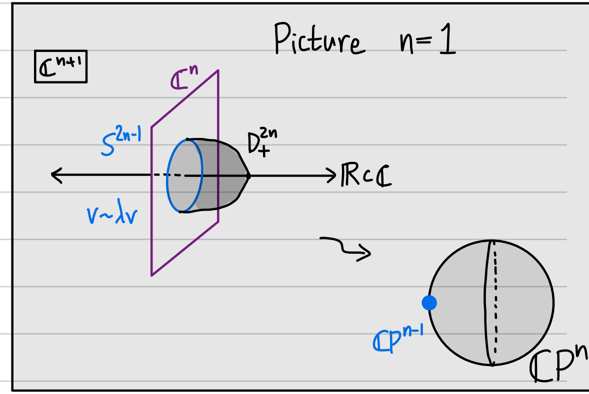
(i) has no n-cells, then $H_n(X) = 0$ (since $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{(\# n\text{-cells})} = 0$).

(ii) has k n-cells, then $H_n(X)$ is generated by $\leq k$ elements.

(iii) has no two cells in adjacent dimensions, then $H_n^{CW}(X) \cong \mathbb{Z}^{(\# n\text{-cells})}$ $\forall n$ (since $d_n = 0$ $\forall n$).

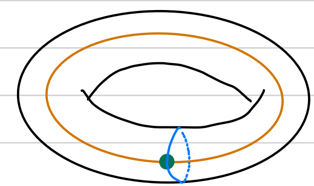
Ex of (iii) $\mathbb{C}P^n$ has a CW structure with one cell of each even dimension $2k \le 2n$. Hence

$$H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, 6, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$$



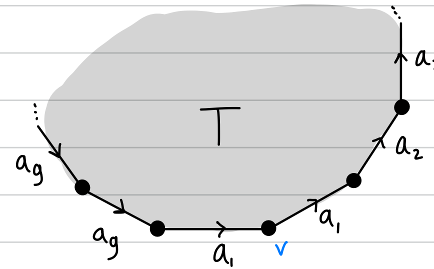
Ex of (iii) $S^n \times S^n$ has a CW structure with one 0-cell, two n-cells, and one 2n-cell. Hence for $n > 1$ we have

$$H_i(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & i = 0 \text{ or } 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$



(This is also true for $n=1$ but we had to consider boundary maps.)

<u>Ex 2.37</u> The nonorientable suface N_g of genus g has one 0-cell, g 1-cells, and one 2-cell attached by the word $a_1^2 a_2^2 \dots a_g^2$.



Chain complex

$$\begin{array}{ccccccccc} \text{3-cells} & & \text{2-cells} & & \text{1-cells} & & \text{0-cells} & & \\ 0 & \longrightarrow & \underset{T}{\mathbb{Z}} & \xrightarrow{d_2} & \mathbb{Z}^g & \xrightarrow{d_1} & \mathbb{Z} & \xrightarrow{d_0} & 0 \\ & & & & a_1 & \longmapsto & 0 & v \longmapsto & 0 \\ & & & & a_2 & \longmapsto & 0 & & \\ & & & & \vdots & & & & \\ & & & & a_g & \longmapsto & 0 & & \end{array}$$

We compute $d_2(T) = 2a_1 + 2a_2 + \dots + 2a_g$.

$H_0(N_g) = {}^{\text{Ker } d_0}/_{\text{Im } d_1} \cong \mathbb{Z}$

Ker d_1 has basis $\{a_1, \dots, a_{g-1}, a_g\}$ or $\{a_1, \dots, a_{g-1}, a_1 + \dots + a_g\}$
Im d_2 has basis $\{2a_1 + \dots + 2a_g\}$

$H_1(N_g) = {}^{\text{Ker } d_1}/_{\text{Im } d_2} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$

$H_2(N_g) = {}^{\text{Ker } d_2}/_{\text{Im } d_3} = 0$ since Ker $d_2 = 0$.

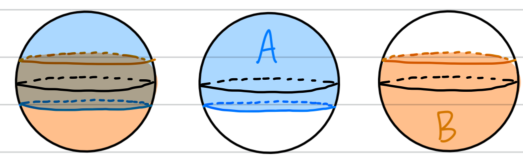
Mayer-Vietoris sequence

Thm (pg 149) For X a space and $A, B \subset X$ with $X = \operatorname{int} A \cup \operatorname{int} B$, there is a LES

$$\begin{tikzcd}
H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, out=-10, in=170] \\
H_{n-1}(A\cap B) \arrow[r] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \\
& \vdots & \\
H_0(A\cap B) \arrow[r] & H_0(A)\oplus H_0(B) \arrow[r] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

(also holds with $\widetilde{H}_0$ instead)

Ex 2.46 $X = S^2$ $\quad A = D^2$ $\quad B = D^2$ $\quad A \cap B \simeq S^1$



$$\begin{tikzcd}
H_2(A\cap B) \arrow[r] & \cancelto{0}{H_2(A)}\oplus \cancelto{0}{H_2(B)} \arrow[r] & H_2(S^2) \arrow[dll, out=-10, in=170, "\partial", "\cong"'] \\
H_1(A\cap B) \arrow[r] & \cancelto{0}{H_1(A)}\oplus \cancelto{0}{H_1(B)} &
\end{tikzcd}$$

So $H_2(S^2) \cong H_1(A\cap B) \cong H_1(S^1) \cong \mathbb{Z}$.

More generally, choosing $X = S^n$, $A = D^n$, $B = D^n$, $A \cap B \simeq S^{n-1}$ gives:
$H_{n-1}(S^{n-1}) \cong \mathbb{Z} \implies H_n(S^n) \cong \mathbb{Z}$.

Pf of Mayer-Vietoris LES

We have a SES of chain complexes

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_n(A\cap B) \arrow[r, "\varphi"] \arrow[d] & C_n(A)\oplus C_n(B) \arrow[r, "\psi"] \arrow[d] & C_n(A+B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A\cap B) \arrow[r, "\varphi"] \arrow[d] & C_{n-1}(A)\oplus C_{n-1}(B) \arrow[r, "\psi"] \arrow[d] & C_{n-1}(A+B) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Define $\varphi(x)=(x,-x)$ and $\psi(x,y)=x+y$.

Recall $C_n(A+B)$ is the subgroup of $C_n(X)$ with all sums of chains in A and chains in B.

So ψ is surjective by definition.

And φ is injective.

$\operatorname{Ker}\psi \subset \operatorname{Im}\varphi$ since $\psi\varphi(x)=\psi(x,-x)=x-x=0$.

$\operatorname{Im}\varphi \subset \operatorname{Ker}\psi$ since $\psi(x,y)=0 \Rightarrow x=-y$

$\Rightarrow x,y$ are chains in $A\cap B$ $(x=-y\in C_n(A\cap B))$

$\Rightarrow (x,y)=(x,-x)=\varphi(x)$.

So we've verified exactness.

This SES of chain complexes now gives a LES of homology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A\cap B) \arrow[r, "\Phi"] & H_n(A)\oplus H_n(B) \arrow[r, "\Psi"] & H_n(A+B) \arrow[dll, "\partial"'] \\
 & H_{n-1}(A\cap B) \arrow[r, "\Phi"] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r, "\Psi"] & H_{n-1}(A+B) \arrow[r] & \cdots
\end{tikzcd}$$

Recall from Prop 2.21 (special case) that since $X=\operatorname{int}A\cup\operatorname{int}B$, the inclusion $C_n(A+B)\to C_n(X)$ is a chain homotopy equivalence, inducing $H_n(A+B)\cong H_n(X)$ $\forall n$.

Not standard notation

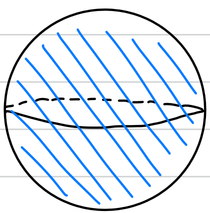
This gives the LES

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A\cap B) \arrow[r, "\Phi"] & H_n(A)\oplus H_n(B) \arrow[r, "\Psi"] & H_n(X) \arrow[dll, "\partial"'] \\
 & H_{n-1}(A\cap B) \arrow[r, "\Phi"] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r, "\Psi"] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

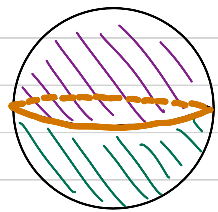
Rmk The connecting homomorphism $\partial : H_n(X) \to H_{n-1}(A \cap B)$
can be made explicit.

different!

Let $[z] \in H_n(X)$, where z is a cycle in X $(\partial z = 0)$.
By Prop 2.21 (special case) we can write $z = x + y$ for
x a chain in A and y a chain in B.
Note $\partial z = 0 \Rightarrow \partial(x+y) = 0 \Rightarrow \partial x = -\partial y$.
We have $\partial[z] = [\partial x] = [-\partial y] \in H_{n-1}(A \cap B)$.



$z \in \operatorname{Ker}(\partial_2^X)$

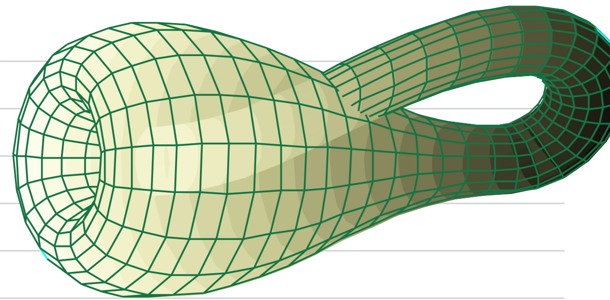


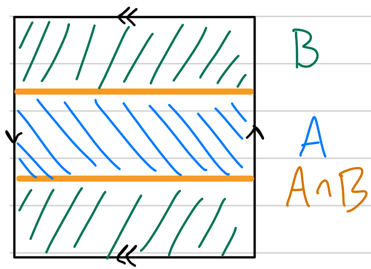
$x \in C_2(A)$

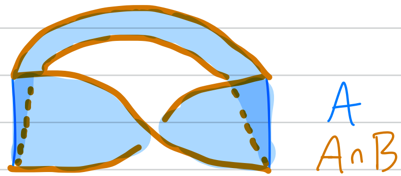
$\partial x = -\partial y \in \operatorname{Ker}(\partial_1^{A \cap B})$

$y \in C_2(B)$

Ex 2.47 The Klein bottle K is the union of two Möbius bands glued together along their boundary circle.







Note $A\cap B$ "wraps twice" around A and around B.

Mayer-Vietoris gives a LES

$$\overset{\cong 0}{H_2(A)\oplus H_2(B)} \longrightarrow H_2(K) \xrightarrow{\partial}$$

$$\overset{\cong\mathbb{Z}}{H_1(A\cap B)} \xrightarrow{\Phi} \overset{\cong\mathbb{Z}}{H_1(A)} \oplus \overset{\cong\mathbb{Z}}{H_1(B)} \xrightarrow{\Psi} \overset{\cong\mathbb{Z}}{H_1(K)} \to$$

$$1 \longmapsto (2,-2)$$

$$\underset{= 0}{\tilde{H}_0(A\cap B)}$$

Φ injective $\Rightarrow \partial = 0 \Rightarrow H_2(K) = 0$.

$$H_1(K) \cong \mathrm{Im}\,\Psi \cong \frac{H_1(A)\oplus H_1(B)}{\mathrm{Ker}\,\Psi} \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}\,\Phi} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

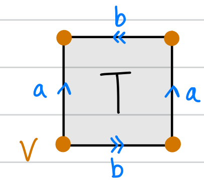
basis $\{(1,-1),(1,0)\}$

basis $\{(2,-2)\}$

Homology with coefficients

So far we have been doing homology with $\mathbb{Z}$ coefficients: $H_n(X) = H_n(X;\mathbb{Z})$.
This can be generalized to homology $H_n(X;G)$ with G coefficients, for G any abelian group.

Ex Cellular homology of the Klein bottle K



$G=\mathbb{Z}$ coefficients

$$\begin{array}{ccccccccc} & & \text{2-cells} & & \text{1-cells} & & \text{0-cells} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

$T \longmapsto 2b$

$a \longmapsto 0$

$b \longmapsto 0$

$v \longmapsto 0$

$$H_i(K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus \mathbb{Z}/2 & i=1 \\ 0 & \text{otherwise} \end{cases}$$

$G=\mathbb{Z}/2\mathbb{Z}$ coefficients

$$\begin{array}{ccccccccc} & & \text{2-cells} & & \text{1-cells} & & \text{0-cells} & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^2 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

$T \longmapsto 0$

$a \longmapsto 0$

$b \longmapsto 0$

$v \longmapsto 0$

$$H_i(K;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i=0 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i=1 \\ \mathbb{Z}/2\mathbb{Z} & i=2 \\ 0 & \text{otherwise} \end{cases}$$

Note $H_i(K;\mathbb{Z}/2\mathbb{Z}) \cong H_i(\text{torus};\mathbb{Z}/2\mathbb{Z})$.

In singular homology with coefficients, the chain groups
$C_n(X) = C_n(X;\mathbb{Z}) = \{\sum_\alpha n_\alpha \sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X,\ n_\alpha \in \mathbb{Z},$ finitely many n_α nonzero$\}$
are replaced by

$$C_n(X;G) = \{\textstyle\sum_\alpha n_\alpha \sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X,\ n_\alpha \in G, \text{ finitely many } n_\alpha \text{ nonzero}\}$$

$$\cdots \longrightarrow C_{n+1}(X;G) \xrightarrow{\partial_{n+1}} C_n(X;G) \xrightarrow{\partial_n} C_{n-1}(X;G) \longrightarrow \cdots$$

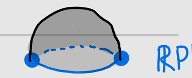
The boundary operator formula $\partial : C_n(X;G) \to C_{n-1}(X;G)$
remains unchanged $\partial(\sum_i n_i \sigma_i) = \sum_i n_i \partial\sigma_i$ with $\partial\sigma_i = \sum_{j=0}^n (-1)^j \sigma_i|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}$
except with $n_i \in G$ instead of $n_i \in \mathbb{Z}$.

The most common coefficients are $G = \mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

$H_n(X;\mathbb{Z}/m\mathbb{Z})$ is sometimes easier to compute than $H_n(X;\mathbb{Z})$,
for example with the Klein bottle.

Rmk Homology with coefficients can sometimes tell us more.

$\mathbb{R}P^2 = e^0 \cup e^1 \cup e^2$



Is the quotient map $q: \mathbb{R}P^2 \to \mathbb{R}P^2/(\mathbb{R}P^2)^{(1)} \cong S^2$ nullhomotopic?
(homotopic to a constant map)

$\mathbb{Z}$ coefficients $\quad \tilde{H}_n(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & \text{o.w.} \end{cases} \qquad \tilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z} & n=2 \\ 0 & \text{o.w.} \end{cases}$

No map $q_*: \tilde{H}_n(\mathbb{R}P^2) \to \tilde{H}_n(S^n)$ can be nonzero. Not sure.

$\mathbb{Z}/2\mathbb{Z}$ coefficients $\quad \tilde{H}_n(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1,2 \\ 0 & \text{o.w.} \end{cases} \qquad \tilde{H}_n(S^2; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=2 \\ 0 & \text{o.w.} \end{cases}$

We have a LES of the (good) pair $(\mathbb{R}P^2, (\mathbb{R}P^2)^{(1)})$

$$\cdots \to \underbrace{H_2((\mathbb{R}P^2)^{(1)}; \mathbb{Z}/2\mathbb{Z})}_{= 0} \to \underbrace{H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})}_{\cong \mathbb{Z}/2\mathbb{Z}} \xrightarrow[\text{injective}]{q_*} H_2\left(\frac{\mathbb{R}P^2}{(\mathbb{R}P^2)^{(1)}}; \mathbb{Z}/2\mathbb{Z}\right) \to \cdots$$

The induced map q_* on H_2 with $\mathbb{Z}/2\mathbb{Z}$ coefficients is nonzero, hence q is not nullhomotopic.

Universal coefficient theorem for homology

Thm 3A.3 If C is a chain complex of free abelian groups, then there are natural SES's

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$$

for all n and for all abelian groups G.

Corollary 3A.6

(a) $H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$.

So for $H_n(X;\mathbb{Z})$ finitely generated,
$\dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{Z})$.

(b) If $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated and p is prime, then $H_n(X;\mathbb{Z}/p\mathbb{Z})$ consists of one $\mathbb{Z}/p\mathbb{Z}$ summand for each

(i) $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$

(ii) $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$

(iii) $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$.

Ex We saw

$$H_n(\text{Klein bottle};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & n \geq 2. \end{cases}$$

Corollary 3A.6 then implies

$$H_n(\text{Klein bottle};\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \\ 0 & n \geq 3. \end{cases}$$

<u>Rmk</u> Sometimes you can "go backwards" and get $H_n(X;\mathbb{Z})$ from $H_n(X;\mathbb{Q})$ and $H_n(X;\mathbb{Z}/p\mathbb{Z})$ $\forall$ primes p:

<u>Cor 3A.7</u>

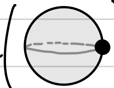
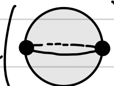
(a) $H_n(X;\mathbb{Z})=0 \iff H_n(X;\mathbb{Q})=0$ and
$H_n(X;\mathbb{Z}/p\mathbb{Z})=0$ $\forall$ primes p.

(b) A map $f:X\to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients $\iff$ it does so for homology with $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$ coefficients $\forall$ primes p.

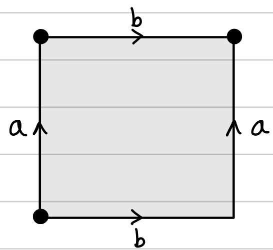
Euler characteristic

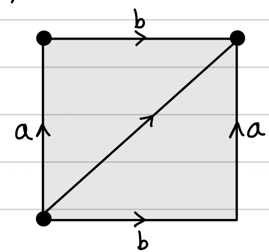
Def For X a finite CW complex, the Euler characteristic is $\chi(X) = \sum_n (-1)^n c_n$, where c_n is the # of n-cells in X.

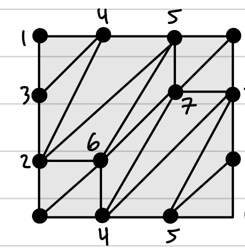
Ex $X = S^2$

$\chi\Big($  $\Big) = 1 - 0 + 1 = 2$ $\qquad$ $\chi\Big($  $\Big) = 2 - 2 + 2 = 2$

Ex $\chi(\text{torus}) = 0$.

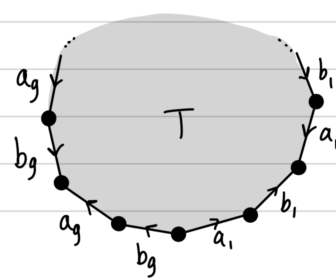






$\chi(\text{torus}) = 1 - 2 + 1$ $\quad = 1 - 3 + 2$ $\quad = 7 - 21 + 14.$

Ex For M_g the torus of genus g,
$\chi(M_g) = 1 - 2g - 1 = 2 - 2g.$



Thm 2.44 For X a finite CW complex,

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X).$$

Hence the Euler characteristic is independent of CW structure, and also a homotopy invariant.

Rmk The rank of a finitely generated abelian group is the number of $\mathbb{Z}$ summands when the group is expressed as a direct sum of cyclic groups.

Ex $\operatorname{rank}\left(\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^5\right) = 5.$

Rmk If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence (SES) of finitely generated abelian groups, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$ since

$C = \operatorname{im}\beta \cong B/\ker\beta = B/\operatorname{im}\alpha$ with α injective

$\Rightarrow \operatorname{rank} C = \operatorname{rank} B - \operatorname{rank} A.$

Pf of Thm 2.44

(Algebra Step) Let

$$0 \longrightarrow C_k \longrightarrow C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

be any chain complex of finitely generated abelian groups.
We have SES's

$$0 \longrightarrow \ker d_n \longrightarrow C_n \longrightarrow \operatorname{im} d_n \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} d_{n+1} \longrightarrow \ker d_n \longrightarrow H_n \longrightarrow 0.$$

Hence $\operatorname{rank} C_n = \operatorname{rank} \ker d_n + \operatorname{rank} \operatorname{im} d_n$
and $\operatorname{rank} \ker d_n = \operatorname{rank} \operatorname{im} d_{n+1} + \operatorname{rank} H_n$.
Substitution gives $\operatorname{rank} C_n = \operatorname{rank} \operatorname{im} d_{n+1} + \operatorname{rank} H_n + \operatorname{rank} \operatorname{im} d_n$.
Cancellation in the alternating sum gives $\sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n$.

(Topology Step)
Let $C_n = H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{(\#\, n\text{-cells})} = \mathbb{Z}^{c_n}$.
Hence $\chi(X) = \sum_n (-1)^n c_n = \sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n(X)$.