Chapter 3: Cohomology

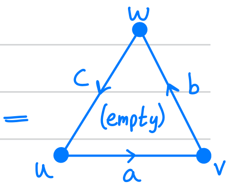
I recommend Hatcher's intro "The idea of cohomology", pages 186-189.

Section 3.1 Cohomology of spaces

(Later we return to the universal coefficient theorem.)

Running example

$X =$



For X a Δ-complex, recall simplicial homology:

$$\cdots \longrightarrow \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \longrightarrow \cdots \longrightarrow \underset{\substack{\parallel \\ 0}}{\Delta_2(X)} \xrightarrow{\partial_2} \underset{\substack{\cong \\ \mathbb{Z}^{\oplus 3}}}{\Delta_1(X)} \xrightarrow{\partial_1} \underset{\substack{\cong \\ \mathbb{Z}^{\oplus 3}}}{\Delta_0(X)} \longrightarrow 0.$$

Let G be an abelian group (think $G=\mathbb{Z}$).

Def The n-cochains with coefficients in G is the group $\Delta^n(X;G) := \mathrm{Hom}(\Delta_n(X), G)$.

Rmk For A, B abelian groups, $\mathrm{Hom}(A,B)$ is the group of homomorphisms $f: A \to B$.
Group structure: For $f, g \in \mathrm{Hom}(A,B)$, we have $(f+g): A \to B$ by $(f+g)(a) = f(a) + g(a)$ $\forall a \in A$.

Ex For X above, $\Delta^1(X;\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^{\times 3}$, with $f \in \Delta^1(X;\mathbb{Z})$ determined by $f(a), f(b), f(c) \in \mathbb{Z}$.
Indeed, $f(17a - 3c) = 17 f(a) - 3 f(c)$.

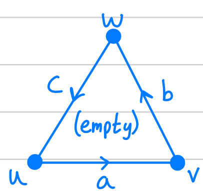
We have a <u>cochain complex</u>

$$\cdots \leftarrow \Delta^{n+1}(X;G) \xleftarrow{\delta^n} \Delta^n(X;G) \xleftarrow{\delta^{n-1}} \Delta^{n-1}(X;G) \leftarrow \cdots \xleftarrow{\delta^2} \underset{\substack{\| \\ 0}}{\Delta^2(X;G)} \xleftarrow{\delta^1} \underset{\substack{\cong \\ \mathbb{Z}^{\times 3}}}{\Delta^1(X;G)} \xleftarrow{\delta^0} \underset{\substack{\cong \\ \mathbb{Z}^{\times 3}}}{\Delta^0(X;G)} \leftarrow 0$$

$(G = \mathbb{Z})$

where the <u>coboundary map</u> $\delta^n : \Delta^n(X;G) \longrightarrow \Delta^{n+1}(X;G)$
is defined, for $f \in \Delta^n(X;G)$, by $\delta f = f\partial$.
$(\delta^n f = f\partial_{n+1})$

$$\begin{tikzcd} \Delta_{n+1}(X) \arrow[r, "\partial"] \arrow[rr, bend right, "\delta f := f\partial"'] & \Delta_n(X) \arrow[r, "f"] & G \end{tikzcd}$$

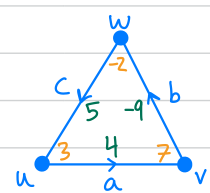
<u>Ex</u> $X =$ 

Let $f \in \Delta^0(X;\mathbb{Z})$ with $f(u) = 3$, $f(v) = 7$, $f(w) = -2$.
Then $\delta f \in \Delta^1(X;\mathbb{Z})$ satisfies

$\delta f(a) = f\partial(a) = f(v-u) = f(v) - f(u) = 7 - 3 = 4.$

$\delta f(b) = f\partial(b) = f(w-v) = -2 - 7 = -9.$

$\delta f(c) = f\partial(c) = f(u-w) = 3 - (-2) = 5.$



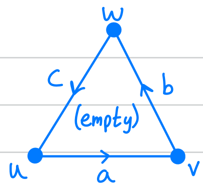
More generally, for $f \in \Delta^n(X;G)$ and $\sigma : \Delta^{n+1} \to X$ an $(n+1)$-simplex in the Δ-complex X, we have

$$\delta f(\sigma) = f\partial(\sigma) = f\left(\sum_{i=0}^{n+1} (-1)^i \sigma|_{[v_0, \dots, \hat{v_i}, \dots, v_{n+1}]}\right)$$
$$= \sum_{i=0}^{n+1} (-1)^i f(\sigma|_{[v_0, \dots, \hat{v_i}, \dots, v_{n+1}]}).$$

Note $\delta \circ \delta = 0$ since $\partial \circ \partial = 0$.
More explicitly, for $f \in \Delta^n(X;G)$, we have
$\delta^{n+1} \delta^n f = \delta^{n+1} f \partial_{n+1} = f \partial_{n+1} \partial_{n+2} = 0.$

<u>Def</u> The (simplicial) <u>cohomology group $H^n(X;G)$</u> of X with coefficients in G is $\mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1})$.

For a cochain $f \in \Delta^n(X;G)$ to be a cocycle means $\delta^n f = f\partial_{n+1} = 0$, i.e. f vanishes on boundaries.

$\underline{\text{Ex}}$ $X =$ 

$$\begin{tikzcd}
\cdots & \Delta^2(X;G) \arrow[l, "\delta^2"'] \arrow[d, equal] & \Delta^1(X;G) \arrow[l, "\delta^1"'] \arrow[d, equal] & \Delta^0(X;G) \arrow[l, "\delta^0"'] \arrow[d, equal] & 0 \arrow[l, "\delta^{-1}"'] \\
& \mathrm{Hom}(\Delta_2(X), G) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\Delta_1(X), G) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\Delta_0(X), G) \arrow[d, phantom, "\cong"] & \\
& \mathrm{Hom}(0, G) \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}^{\oplus 3}, G) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z}^{\oplus 3}, G) \arrow[d, phantom, "\cong"] & \\
& 0 & \mathbb{Z}^{\times 3} & \mathbb{Z}^{\times 3} &
\end{tikzcd}$$

$$H^0(X;\mathbb{Z}) := \frac{\operatorname{Ker}\delta^0}{\operatorname{Im}\delta^{-1}} \cong \operatorname{Ker}\delta^0 \cong \mathbb{Z}$$

since if $f \in \Delta^0(X;\mathbb{Z})$ with $\delta f = 0$, then

$0 = \delta f a = f \partial a = f(v) - f(u) \Rightarrow f(v) = f(u)$

$0 = \delta f b = f \partial b = f(w) - f(v) \Rightarrow f(w) = f(v)$ (Seen spanning tree)

$0 = \delta f c = f \partial c = f(u) - f(w) \Rightarrow f(u) = f(w)$ (Not a new constraint.)

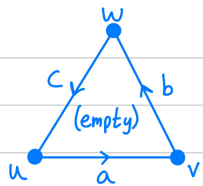
So f (with $f(u) = f(v) = f(w) = c$)

is determined by a single $c \in \mathbb{Z}$. (Constant on each connected component.)

$\underline{\text{More generally}}$ For X a Δ-complex,

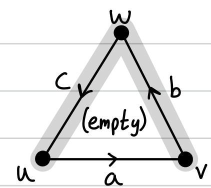
$H^0(X;\mathbb{Z}) \cong \mathbb{Z}^{\times(\#\text{ connected components of } X)}$

$H^0(X;G) \cong G^{\times(\#\text{ connected components of } X)}$.

Ex $X =$ 

$$\cdots \xleftarrow{\delta^2} \underset{\substack{\| \\ 0}}{\Delta^2(X;G)} \xleftarrow{\delta^1} \underset{\substack{\cong \\ \mathbb{Z}^{\times 3}}}{\Delta^1(X;G)} \xleftarrow{\delta^0} \underset{\substack{\cong \\ \mathbb{Z}^{\times 3}}}{\Delta^0(X;G)} \xleftarrow{\delta^{-1}} 0$$

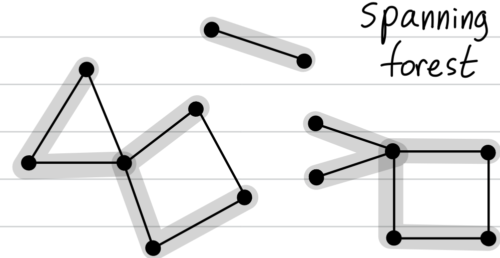
$$H^1(X;\mathbb{Z}) = \frac{\operatorname{Ker}\delta^1}{\operatorname{Im}\delta^0} = \frac{\Delta^1(X;\mathbb{Z})}{\operatorname{Im}\delta^0} \cong \mathbb{Z}.$$



Spanning tree

To understand this quotient, choose a spanning tree for X.
Let $g \in \Delta^1(X;\mathbb{Z})$.
We can find $f \in \Delta^0(X;\mathbb{Z})$ with $\delta^0 f(b) = g(b)$ and $\delta^0 f(c) = g(c)$ by choosing $f(u), f(v), f(w)$ so that $f(w) - f(v) = g(b)$, $f(u) - f(w) = g(c)$.
These choices determine $\delta^0 f(a) = f(v) - f(u)$, which need not be equal to $g(a) \in \mathbb{Z}$.

<u>More generally</u> For X a graph with k edges not in a spanning forest,
$H^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times k}$
$H^1(X;G) \cong G^{\times k}$.

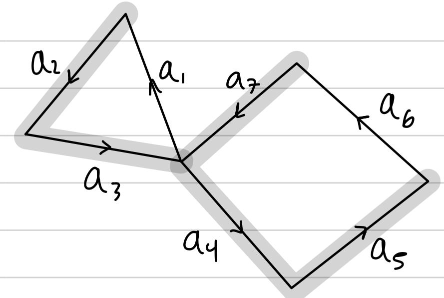


Spanning forest

$H^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times 3}$

Generators for cohomology

For X a graph, $H^1(X;\mathbb{Z})$ is generated by the cocycles assigning
- 1 to a single oriented edge not in a spanning forest
- 0 to every other edge.

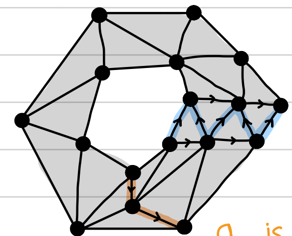
$X =$ 

Ex $H^1(X;\mathbb{Z})$ is generated by $f_{a_1}, f_{a_6} \in \operatorname{Ker}\delta^1 = \Delta^1(X;\mathbb{Z})$,
where $f_{a_1}: \Delta_1(X) \to \mathbb{Z}$ by $f_{a_1}(c_1a_1 + \dots + c_7a_7) = c_1$
and $f_{a_6}: \Delta_1(X) \to \mathbb{Z}$ by $f_{a_6}(c_1a_1 + \dots + c_7a_7) = c_6$.

Here $f_{a_6}(a_6) = 1$ (so $f_{a_6}(-a_6) = -1$).
We say f_{a_6} is the dual cochain to the edge a_6.
Notation $f_{a_6} = a_6^*$.

For X a Δ-complex with k-simplex σ,
the dual cochain σ^* is a cocycle (i.e. in $\operatorname{Ker}\delta^k$)
if σ is maximal (i.e. has no cofaces $\sigma \subseteq \tau$).

Ex X = 2-dimensional Δ-complex

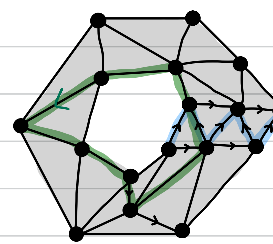


$H^1(X;\mathbb{Z})$ is generated by the cocycle f assigning 1 to each blue edge and 0 otherwise.

To see $f \in \operatorname{Ker} \delta^1$, note

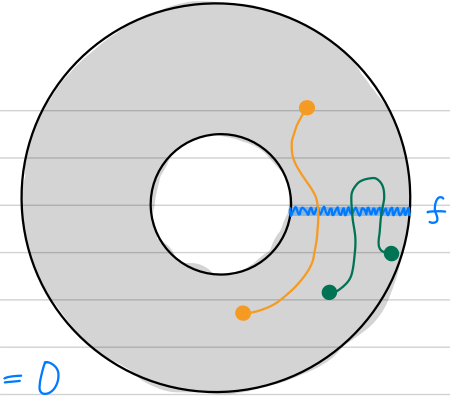
$$\delta^1 f(T) = \begin{cases} 1-1+0-0 & \text{if } T \text{ is a 2-simplex bordering two blue edges} \\ 0 & \text{otherwise.} \end{cases}$$

This generator $[f]$ for $H^1(X;\mathbb{Z})$ is dual to a generator $[z]$ for $H_1(X;\mathbb{Z})$.



Note $f(z) = 1$ even if we replace z with a homologous cycle, or g with a cohomologous cocycle.

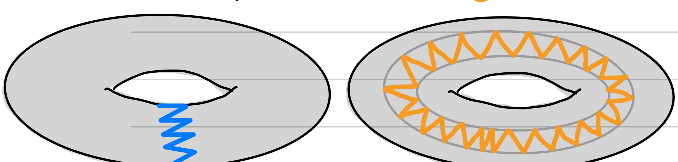
Rmk For singular $H^1(X;\mathbb{Z})$ where X is the annulus, a generating cocycle f assigns to each singular edge an "oriented count" of the # of times it crosses the blue line.



f(orange edge) $= 1$
f(green edge) $= 1-1=0$

Ex $X=$ torus.

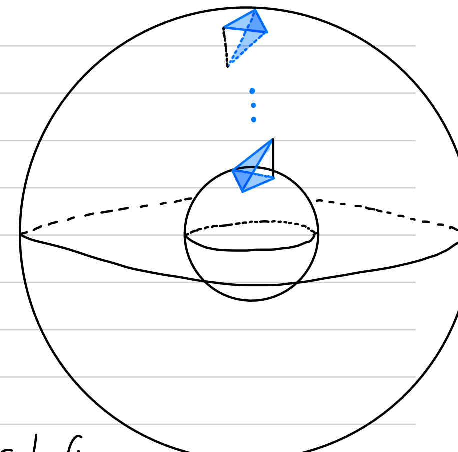
Simplicial $H^1(X;\mathbb{Z})$ generated by cocycles f and g.



Ex $X=S^2\times I$.

Simplicial $H^1(X;\mathbb{Z})\cong\mathbb{Z}$.

Generating cocycle $f\in\operatorname{Ker}\delta^2$

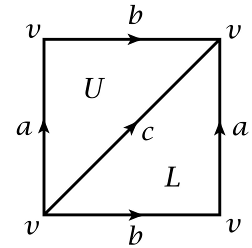


Each tetrahedron T satisfies

$$\delta^2 f(T)=\begin{cases}1-1+0-0 & \text{if } T \text{ borders two blue triangles}\\ 0-0+0-0 & \text{otherwise.}\end{cases}$$

Example simplicial cohomology computations

Ex Δ-complex X is a torus.



$$\partial^2 U = b - c + a$$
$$\partial^2 L = a - c + b$$

Cochain complex

3 cochains   2-cochains   1-cochains   0-cochains

$$\cdots \leftarrow \Delta^3(X;\mathbb{Z}) \xleftarrow{\delta^2} \Delta^2(X;\mathbb{Z}) \xleftarrow{\delta^1} \Delta^1(X;\mathbb{Z}) \xleftarrow{\delta^0} \Delta^0(X;\mathbb{Z}) \leftarrow 0$$

$\Delta^2(X;\mathbb{Z}) = \mathrm{Hom}(\Delta_2(X),\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{\oplus 2},\mathbb{Z}) \cong \mathbb{Z}^{\times 2}$ Gen. by U^*, L^*

$\Delta^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times 3}$ Gen. by a^*, b^*, c^*

$\Delta^0(X;\mathbb{Z}) \cong \mathbb{Z}$ Gen. by v^*

$$\delta^0 v^*(\lambda_1 a + \lambda_2 b + \lambda_3 c) = v^* \partial_1(\lambda_1 a + \lambda_2 b + \lambda_3 c)$$
$$= v^*(\lambda_1(v-v) + \lambda_2(v-v) + \lambda_3(v-v)) = v^*(0) = 0.$$

So $\delta^0 = 0$.

$U^* \in \Delta^2(X;\mathbb{Z})$ defined by $U^*(\lambda_1 U + \lambda_2 L) = \lambda_1$.

$$\delta^1 a^*(\lambda_1 U + \lambda_2 L) = a^* \partial_2(\lambda_1 U + \lambda_2 L)$$
$$= a^*(\lambda_1(b-c+a) + \lambda_2(a-c+b)) = \lambda_1 + \lambda_2$$

so $\delta^1 a^* = U^* + L^*$.

Similarly, $\delta^1 b^* = U^* + L^*$ and $\delta^1 c^* = -U^* - L^*$.

$\delta^2 = 0$ since $\Delta^3(X;\mathbb{Z}) = 0$.

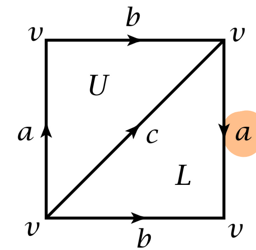
We compute $H^0(X;\mathbb{Z}) = \mathrm{Ker}\,\delta^0 \cong \mathbb{Z}$ Gen. by v^*.

$$H^1(X;\mathbb{Z}) = \frac{\mathrm{Ker}\,\delta^1}{\mathrm{Im}\,\delta^0} \cong \mathrm{Ker}\,\delta^1 \cong \mathbb{Z}^{\times 2}$$ Gen. by a^*+c^*, b^*+c^*.

$$H^2(X;\mathbb{Z}) = \frac{\mathrm{Ker}\,\delta^2}{\mathrm{Im}\,\delta^1} \cong \frac{\overbrace{\Delta^2(X;\mathbb{Z})}^{\text{Gen. } \{U^*, L^*\} \text{ or } \{U^*, U^*+L^*\}}}{\underbrace{\mathrm{Im}\,\delta^1}_{\text{Gen. } \{U^*+L^*\}}} \cong \mathbb{Z}$$ Gen. by $\{U^*\}$ (for example).

Example simplicial cohomology computations

Ex Δ-complex X is a Klein bottle.



$$\partial^2 U = b - c + a$$
$$\partial^2 L = a - b + c$$

Cochain complex

3 cochains | 2-cochains | 1-cochains | 0-cochains

$$\cdots \longleftarrow \Delta^3(X;\mathbb{Z}) \xleftarrow{\delta^2} \Delta^2(X;\mathbb{Z}) \xleftarrow{\delta^1} \Delta^1(X;\mathbb{Z}) \xleftarrow{\delta^0} \Delta^0(X;\mathbb{Z}) \longleftarrow 0$$

$$\Delta^2(X;\mathbb{Z}) = \mathrm{Hom}(\Delta_2(X),\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{\oplus 2},\mathbb{Z}) \cong \mathbb{Z}^{\times 2}$$

$$\Delta^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times 3}, \qquad \Delta^0(X;\mathbb{Z}) \cong \mathbb{Z}$$

Gen. by U^*, L^* | Gen. by a^*, b^*, c^* | Gen. by v^*

$$\delta^0 v^*(\lambda_1 a + \lambda_2 b + \lambda_3 c) = v^* \partial_1(\lambda_1 a + \lambda_2 b + \lambda_3 c)$$
$$= v^*\big(\lambda_1(v-v) + \lambda_2(v-v) + \lambda_3(v-v)\big) = v^*(0) = 0.$$

So $\delta^0 = 0$.

$$\delta^1 a^*(\lambda_1 U + \lambda_2 L) = a^* \partial_2(\lambda_1 U + \lambda_2 L)$$
$$= a^*\big(\lambda_1(b-c+a) + \lambda_2(a-b+c)\big) = \lambda_1 + \lambda_2$$

so $\delta^1 a^* = U^* + L^*$.

Similarly, $\delta^1 b^* = U^* - L^*$ and $\delta^1 c^* = -U^* + L^*$.

$\delta^2 = 0$ since $\Delta^3(X;\mathbb{Z}) = 0$.

We compute $H^0(X;\mathbb{Z}) = \mathrm{Ker}\,\delta^0 \cong \mathbb{Z}$ Gen. by v^*.

$$H^1(X;\mathbb{Z}) = \frac{\mathrm{Ker}\,\delta^1}{\mathrm{Im}\,\delta^0} \cong \mathrm{Ker}\,\delta^1 \cong \mathbb{Z}$$ Gen. by $b^* + c^*$.

Gen. $\{U^*, L^*\}$ or $\{U^*, U^*+L^*\}$

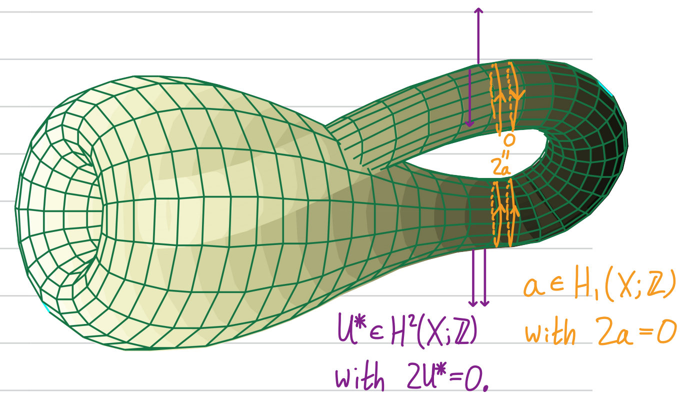
$$H^2(X;\mathbb{Z}) = \frac{\mathrm{Ker}\,\delta^2}{\mathrm{Im}\,\delta^1} \cong \frac{\Delta^2(X;\mathbb{Z})}{\mathrm{Im}\,\delta^1} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$ Gen. by U^*

Gen. $\{U^*-L^*, U^*+L^*\}$ or $\{2U^*, U^*+L^*\}$

This is the first example we've seen where cohomology is not isomorphic to homology, since for X = Klein bottle we had

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}} & i=1 \\ 0 & i\geq 2 \end{cases}$$

That torsion has "jumped" from $H_1(X;\mathbb{Z})$ to $H^1(X;\mathbb{Z})$ is related to Corollary 3.3.



More generally, for M a closed connected nonorientable n-manifold, we have

$$H^n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } M \text{ orientable} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } M \text{ nonorientable.} \end{cases}$$

Singular cohomology

For X a topological space, recall singular homology:

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0.$$

Let G be an abelian group (think $G=\mathbb{Z}$).

Def The n-cochains with coefficients in G is the group $C^n(X;G) := \mathrm{Hom}(C_n(X), G)$.

We have a cochain complex

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta^n} C^n(X;G) \xleftarrow{\delta^{n-1}} C^{n-1}(X;G) \leftarrow \cdots \xleftarrow{\delta^2} C^2(X;G) \xleftarrow{\delta^1} C^1(X;G) \xleftarrow{\delta^0} C^0(X;G) \longleftarrow 0$$

where the coboundary map $\delta^n : C^n(X;G) \longrightarrow C^{n+1}(X;G)$ is defined, for $f \in C^n(X;G)$, by $\delta^n f = f\partial_{n+1}$.

Note $\delta\circ\delta = 0$ since $\partial\circ\partial = 0$.

Def The (singular) cohomology group $H^n(X;G)$ of X with coefficients in G is $\mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1})$.

The main features of singular and simplicial homology extend to cohomology, even though maps reverse directions.

- Reduced cohomology: Apply $\mathrm{Hom}(-,G)$ to the augmented chain complex

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$\textstyle\sum c_i v_i \longmapsto \sum c_i$$

$\tilde{H}^n(X;G) \cong H^n(X,G)$ for $n>0$.

$\tilde{H}^n(X;G) \times G \cong H^n(X,G)$.

- Relative cohomology and the LES of a pair (X,A):

Apply $\mathrm{Hom}(-;G)$ to the SES

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$

to get the SES

$$0 \longleftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} \underset{\substack{\shortparallel \\ \mathrm{Hom}(C_n(X,A),G)}}{C^n(X,A;G)} \longleftarrow 0.$$

Check surjectivity!

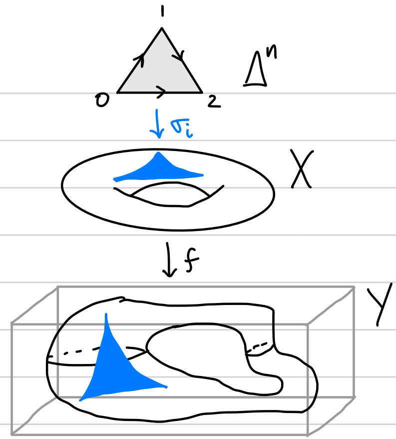
This is in fact a SES of cochain complexes (i^* and j^* commute with δ).

The snake lemma gives a LES

$$\cdots \longleftarrow H^{n+1}(X,A;G) \overset{\delta}{\longleftarrow} H^n(A;G) \xleftarrow{i^*} H^n(X;G) \xleftarrow{j^*} H^n(X,A;G) \overset{\delta}{\longleftarrow} H^{n-1}(A;G) \longleftarrow \cdots$$

- **Induced homomorphisms:**

  A map of spaces $f: X \to Y$
  induces $f^\#: C^n(Y;G) \longrightarrow C^n(X;G)$
  $\qquad\qquad\quad \shortparallel$
  $\quad\;\; \mathrm{Hom}(C_n(Y), G)$



$$\begin{tikzcd} C_n(X) \arrow[r, "f_\#"] \arrow[drr, dashed] & C_n(Y) \arrow[dr] & \\ & & G \end{tikzcd}$$

Since $f_\#: C_\bullet(X) \longrightarrow C_\bullet(Y)$ is a chain map $(f_\# \partial = \partial f_\#)$,

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"] & C_n(Y) \arrow[r] & \cdots \end{tikzcd}$$

$\Downarrow$
Apply $\mathrm{Hom}(-, G)$
$\Downarrow$

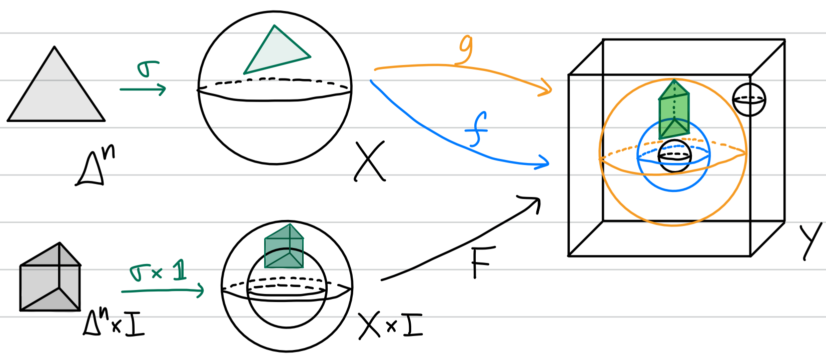
$f^\#: C^\bullet(Y) \to C^\bullet(X)$ is a cochain map $(\delta f^\# = f^\# \delta)$.

$$\begin{tikzcd} \cdots & C^{n+1}(X;G) \arrow[l] \arrow[dr, phantom, "\circlearrowleft"] & C^n(X;G) \arrow[l, "\delta^n_X"'] & \cdots \arrow[l] \\ \cdots & C^{n+1}(Y;G) \arrow[l] \arrow[u, "f^\#"] & C^n(Y;G) \arrow[l, "\delta^n_Y"'] \arrow[u, "f^\#"] & \cdots \arrow[l] \end{tikzcd}$$

So $f^\#$ maps $\mathrm{Ker}\,\delta^n_Y$ to $\mathrm{Ker}\,\delta^n_X$ and $\mathrm{Im}\,\delta^{n-1}_Y$ to $\mathrm{Im}\,\delta^{n-1}_X$,
hence inducing $f^*: H^n(Y;G) \longrightarrow H^n(X;G)$.

- Contravariant functor from spaces to groups.

- Homotopy invariance: If $f \simeq g : X \to Y$, then $f^* = g^* : H^n(Y) \to H^n(X)$.

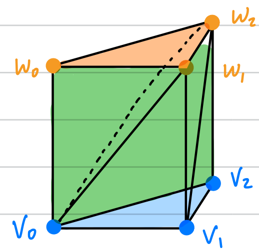


We defined a chain homotopy

$$\partial^Y_{n+1} P_n + P_{n-1}\partial^X_n = g_\# - f_\#$$

$$\partial P = \underset{\text{top}}{g_\#} - \underset{\text{bottom}}{f_\#} - \underset{\text{sides}}{P\partial}$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[dl, "P_n"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_n(Y) \arrow[r, "\partial^Y_n"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$



Apply $\mathrm{Hom}(-, G)$, distributes over $+$:

Get a cochain homotopy

$$P^{n+1}\delta^n_Y + \delta^{n-1}_X P^n = g^\# - f^\#.$$

$$\begin{tikzcd}
C^{n+1}(X;G) & C^n(X;G) \arrow[l, "\delta^n_X"'] & C^{n-1}(X;G) \arrow[l, "\delta^{n-1}_X"'] \\
C^{n+1}(Y;G) \arrow[ur, "P^{n+1}"] & C^n(Y;G) \arrow[l, "\delta^n_Y"] \arrow[u, shift left, "f^\#"] \arrow[u, shift right, "g^\#"'] \arrow[ur, "P^n"] & C^{n-1}(Y;G) \arrow[l, "\delta^{n-1}_Y"]
\end{tikzcd}$$

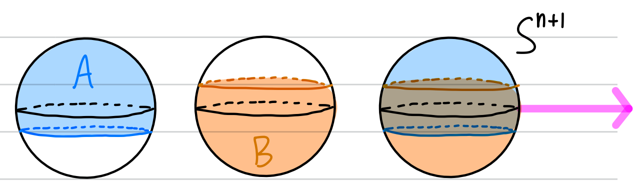
On cocycles, $g^\#$ and $f^\#$ differ by a coboundary, hence inducing $g^* = f^*$.

- Excision
- Axioms for cohomology
- Singular, simplicial, and cellular cohomology

- Mayer-Vietoris LES. $X = \mathrm{int}(A) \cup \mathrm{int}(B)$

$$\cdots \longleftarrow H^{n+1}(X;G) \xleftarrow{\ \delta\ } H^n(A\cap B;G) \xleftarrow{\ \Phi\ } H^n(A;G)\oplus H^n(B;G) \xleftarrow{\ \Psi\ } H^n(X;G) \xleftarrow{\ \delta\ } H^{n-1}(A\cap B;G) \longleftarrow \cdots$$

<u>Ex</u> $X = S^{n+1}$ $A = D^{n+1}$ $B = D^{n+1}$ $A \cap B \simeq S^n$



The universal coefficient theorem (for cohomology)

- Cohomology groups are determined algebraically by homology groups.
- It is subtle! Derived functors (Ext).
- Ring (cup) product structure on cohomology not determined by homology.

Ex Applying $\mathrm{Hom}(-,\mathbb{Z})$ to the chain complex C

$$\underset{C_3}{0 \longrightarrow \mathbb{Z}} \xrightarrow{0} \underset{C_2}{\mathbb{Z}} \xrightarrow{2} \underset{C_1}{\mathbb{Z}} \xrightarrow{0} \underset{C_0}{\mathbb{Z}} \longrightarrow 0$$

$$H_n(C) = \begin{cases} \mathbb{Z} & n = 0,3 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{tikzcd} C_2 \arrow[r, "2"] \arrow[dr, dashed, "1 \mapsto 2"'] & C_1 \arrow[d, "1 \mapsto 1"] \\ & \mathbb{Z} \end{tikzcd}$$

gives the cochain complex

$$\underset{C^3}{0 \longleftarrow \mathbb{Z}} \xleftarrow{0} \underset{C^2}{\mathbb{Z}} \xleftarrow{2} \underset{C^1}{\mathbb{Z}} \xleftarrow{0} \underset{C^0}{\mathbb{Z}} \longleftarrow 0$$

$$H^n(C;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,3 \\ \mathbb{Z}/2\mathbb{Z} & n = 2 \\ 0 & \text{o.w.} \end{cases}$$

In general, $H^n(C;\mathbb{Z}) \not\cong H_n(C)$ and $H^n(C;\mathbb{Z}) \not\cong \mathrm{Hom}(H_n(C),\mathbb{Z})$.

Thm 3.2 Universal coefficient theorem (for cohomology)

For a chain complex C of free abelian groups, the cohomology $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C_n,G)$ is determined by the split SES

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C),G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G).$$

- $\mathrm{Ext}(A,G)=0$ if A is free.
  If $H_{n-1}(C)$ is free, then h is an isomorphism.

- If group A is finitely generated, then $\mathrm{Hom}(A,\mathbb{Z}) \cong$ free part of A and $\mathrm{Ext}(A,\mathbb{Z}) \cong$ torsion part of A, giving:

$A \cong \mathbb{Z}^r \oplus \left(\bigoplus_{i=1}^m \mathbb{Z}/m_i\mathbb{Z}\right)$

$\mathbb{Z}^r$

$\bigoplus_{i=1}^m \mathbb{Z}/m_i\mathbb{Z}$

Cor 3.3 If a chain complex C of free abelian groups has finitely generated homology H_n and H_{n-1}, with torsion subgroups $T_n \subseteq H_n$ and $T_{n-1} \subseteq H_{n-1}$, then $H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$.

Let $Z_n := \operatorname{Ker} \partial_n \subseteq C_n$ and $B_n := \operatorname{Im} \partial_{n+1} \subseteq C_n$.

Define $h : H^n(C;G) \to \operatorname{Hom}(H_n(C), G)$ as follows.

Let $[\varphi] \in H^n(C;G)$.

So $\varphi : C_n \to G$ with $0 = \delta\varphi = \varphi\partial$, so φ vanishes on B_n.

The restriction $\varphi_0 : Z_n \to G$ induces $\overline{\varphi_0} : \underset{\substack{\| \\ H_n(C)}}{Z_n/B_n} \to G$, i.e. $\overline{\varphi_0} \in \operatorname{Hom}(H_n(C), G)$.

$$\begin{array}{ll} \varphi : C_n \longrightarrow G & \\ & \Big\} \text{restrict} \\ \varphi_0 : Z_n \longrightarrow G & \\ & \Big\} \text{quotient} \\ \overline{\varphi_0} : H_n(C) \longrightarrow G & \end{array}$$

If $\varphi \in \operatorname{Im} \delta$, say $\varphi = \delta\psi = \psi\partial$, then φ is zero on Z_n, so $\overline{\varphi_0} = 0$.

Hence
$$\begin{aligned} h : H^n(C;G) &\longrightarrow \operatorname{Hom}(H_n(C), G) \\ [\varphi] &\longmapsto \overline{\varphi_0} \end{aligned}$$
is well-defined.

(Also a homomorphism.)

To see h is surjective, note the SES

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, hook] & C_n \arrow[r, "\partial"] \arrow[l, bend left, dashed, "p"] & B_{n-1} \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$

splits since B_{n-1} is free (abelian), as a subgroup of C_{n-1}.
Hence $\exists$ projection $p: C_n \to Z_n$ restricting to identity on Z_n.
Extend $\varphi_0 : Z_n \to G$ vanishing on B_n to $\varphi_0 p : C_n \to G$ vanishing on B_n.
This extends homomorphisms $H_n(C) \to G$ to elements of $\operatorname{Ker} \delta$.

Get $\operatorname{Hom}(H_n(C), G) \longrightarrow \operatorname{Ker} \delta \longrightarrow \frac{\operatorname{Ker} \delta^n}{\operatorname{Im} \delta^{n-1}} = H^n(C; G)$.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} h \arrow[r] & H^n(C;G) \arrow[r, "h"] & \operatorname{Hom}(H_n(C), G) \arrow[r] \arrow[l, bend left, dashed, "s'"] & 0 \end{tikzcd}$$

Note $hs = \mathbb{1}$ (extend and then restrict).
Hence h is surjective <u>and</u> the above SES splits.

Ext$(-,G)$ is the (first) derived functor of Hom$(-,G)$

Let G be an abelian group.
Hom$(-,G)$ is left exact (Hatcher Ex pg 193):
If $A \to B \to C \to 0$ is exact, then so is
$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$.

Hom$(-,G)$ is not exact, however.
Maps SES's to SES's.

Ex Applying Hom$(-,\mathbb{Z})$ to $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$
yields $0 \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$, which is not exact.

However, Hom$(-,G)$ is exact on free abelian groups:
If $0 \to A \to B \to C \to 0$ is a SES of free abelian groups,
then so is $0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C) \to 0$.

A <u>free resolution</u> of an abelian group A is a chain complex

$$\cdots \to F_2 \to F_1 \to F_0 \to 0$$

of free groups with a map $F_0 \to A$ such that

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0 \quad \text{is exact.}$$

This replaces a (complicated) abelian group A with simpler groups on which $\mathrm{Hom}(-,G)$ is exact.

(Later: Can choose $F_2 = 0$.)

Let $H^n(F;G)$ be the cohomology of the cochain complex

$$0 \to \mathrm{Hom}(F_0,G) \to \mathrm{Hom}(F_1,G) \to \mathrm{Hom}(F_2,G) \to \cdots$$

<u>Language</u> $H^n(-;G)$ is the n-th derived functor of $\mathrm{Hom}(-,G)$.

<u>Exercise</u> $H^0(F;G) \cong \mathrm{Hom}(A,G)$.

<u>Lemma 3.1</u>

(a) Given free resolutions F and F' of A and A', every homomorphism $\alpha: A \to A'$ extends to a chain map from F to F'.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & F_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

Any two such chain maps are chain homotopic.

(b) For any two free resolutions F, F' of A, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ $\forall n$.

Def For abelian groups A and G,
$\mathrm{Ext}(A,G) := H^1(F;G)$
for F any free resolution of A.
So $\mathrm{Ext}(-,G)$ is the first derived functor of $\mathrm{Hom}(-,G)$.

Proposition 3F.11
If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups, then so is

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G) \to 0.$$

Pf The SES $0 \to A \to B \to C \to 0$ extends to a SES of chain complexes

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & F_2^A \arrow[r] \arrow[d] & F_2^B \arrow[r] \arrow[d] & F_2^C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_1^A \arrow[r] \arrow[d] & F_1^B \arrow[r] \arrow[d] & F_1^C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_0^A \arrow[r] \arrow[d] & F_0^B \arrow[r] \arrow[d] & F_0^C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Applying $\mathrm{Hom}(-,G)$ gives a SES of chain complexes
Snake lemma gives a LES in cohomology:

$$0 \to H^0(F^C;G) \to H^0(F^B;G) \to H^0(F^A;G) \to H^1(F^C;G) \to H^1(F^B;G) \to H^1(F^A;G) \to H^2(F^C;G) \to H^2(F^B;G) \to H^2(F^A;G) \to \cdots$$

Now, recall $H^0(F^A; G) \cong \mathrm{Hom}(A, G)$,

and $H^1(F^A; G) =: \mathrm{Ext}(A, G)$.

Also, $H^2(F^A; G) = 0$

Since each abelian group has a free resolution $0 \to F_1 \to F_0 \to A \to 0$ with $F_2 = 0$. $\square$

Indeed, let $F_0 \to A$ be surjective where free abelian group F_0 has basis in correspondence with a generating set of A.

The kernel F_1 of this map, as a subgroup of a free abelian group, is free abelian.

Hence $0 \to F_1 \to F_2 \to A$ is exact.

Thm 3.2 Universal coefficient theorem (for cohomology)
For a chain complex C of free abelian groups, the cohomology $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ is determined by the split SES

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G).$$

- $\mathrm{Ext}(A,G) = 0$ if A is free. Consider the free resolution $0 \to A \to A \to 0$.
- $\mathrm{Ext}(A \oplus A', G) \cong \mathrm{Ext}(A,G) \oplus \mathrm{Ext}(A', G)$ Direct sum of free resolutions $\begin{array}{c} \to F_1 \to F_0 \to A \to 0 \\ \oplus \quad \oplus \quad \oplus \\ \to F_1' \to F_0' \to A' \to 0 \end{array}$
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$ (Next page.)

- If group A is finitely generated, then $\mathrm{Hom}(A, \mathbb{Z}) \cong$ free part of A and $\mathrm{Ext}(A, \mathbb{Z}) \cong$ torsion part of A, giving:

$A \cong \mathbb{Z}^r \oplus \left(\oplus_{i=1}^{m} \mathbb{Z}/m_i\mathbb{Z}\right)$
$\mathbb{Z}^r$
$\oplus_{i=1}^{m} \mathbb{Z}/m_i\mathbb{Z}$

Cor 3.3 If a chain complex C of free abelian groups has finitely generated homology H_n and H_{n-1}, with torsion subgroups $T_n \subseteq H_n$ and $T_{n-1} \subseteq H_{n-1}$, then $H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$.

- Why is $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$?

  Free resolution F: $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \cancel{\mathbb{Z}/n\mathbb{Z}} \to 0$

  Remove $A = \mathbb{Z}/n\mathbb{Z}$ and dualize:

$$\begin{array}{ccccccc} 0 & \xleftarrow{\delta^1} & \mathrm{Hom}(\mathbb{Z},G) & \xleftarrow{\delta^0} & \mathrm{Hom}(\mathbb{Z},G) & \longleftarrow & 0 \\ \| & & \wr\| & & \wr\| & & \| \\ 0 & \longleftarrow & G & \xleftarrow{n} & G & \longleftarrow & 0 \end{array}$$

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) := H^1(F;G) = {}^{\mathrm{Ker}\,\delta^1}/_{\mathrm{Im}\,\delta^0} \cong G/nG .$$

- Free resolution $\longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A$.

  Recall $H^n(F;G)$ is the cohomology of the cochain complex

  $0 \longrightarrow \mathrm{Hom}(F_0,G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1,G) \xrightarrow{f_2^*} \mathrm{Hom}(F_2,G) \longrightarrow \ldots$ Language n-th derived functor

  Why is $H^0(F;G) \cong \mathrm{Hom}(A,G)$?

  Since $\mathrm{Hom}(-,G)$ is left exact, the augmented sequence

  $0 \to \mathrm{Hom}(A,G) \longrightarrow \mathrm{Hom}(F_0,G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1,G)$ is exact, yielding

$$H^0(F;G) \cong \mathrm{Ker}(f_1^*) \cong \mathrm{Hom}(A,G).$$

Lemma 3.1

(a) Given free resolutions F and F' of A and A', every homomorphism $\alpha : A \to A'$ extends to a chain map from F to F'.

Any two such chain maps are chain homotopic.

(b) For any two free resolutions F, F' of A, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ $\forall n$.

$$
\begin{tikzcd}
 & x \arrow[d, phantom, sloped, "\ni"] & & x \arrow[d, phantom, sloped, "\ni"] & & \\
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F'_2 \arrow[r, "f'_2"'] & F'_1 \arrow[r, "f'_1"'] & F'_0 \arrow[r, "f'_0"'] & A' \arrow[r] & 0
\end{tikzcd}
$$

Proof (a) For each basis element $x \in F_0$, define $\alpha_0(x) = x'$ for some $x' \in F'_0$ with $f'_0(x') = \alpha f_0(x)$ [f'_0 surjective].

Inductively, for each basis element $x \in F_i$, define $\alpha_i(x) = x'$ for some $x' \in F_i'$ with $f_i'(x') = \alpha_{i-1} f_i(x)$, which exists since $\operatorname{Im} f_i' = \operatorname{Ker} f'_{i-1}$ and $f'_{i-1} \alpha_{i-1} f_i = \alpha_{i-2} f_{i-1} f_i = 0$.

Lemma 3.1

(a) Given free resolutions F and F' of A and A', every homomorphism $\alpha : A \to A'$ extends to a chain map from F to F'.
Any two such chain maps are chain homotopic.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\beta_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1"] \arrow[dl, "\lambda_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0"] \arrow[dl, "\lambda_0"'] & A \arrow[r] \arrow[d, "\beta=0"] \arrow[dl, "\lambda_{-1}"'] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & A' \arrow[r] & 0
\end{tikzcd}$$

<u>Proof</u> (a) Suppose we have two chain maps α_i, α_i' extending α.
Their difference $\beta_i = \alpha_i - \alpha_i'$ is a chain map extending $\beta = \alpha - \alpha = 0$.
<u>Goal</u> Define $\lambda_i : F_i \to F_{i+1}'$ with $\beta_i = \alpha_i - \alpha_i' = f_{i+1}'\lambda_i + \lambda_{i-1} f_i$.

For $i=0$, let $\lambda_{-1}=0$. For each basis element $x \in F_0$,
define $\lambda_0(x) = x' \in F_1'$ with $f_1'(x') = \beta_0(x)$,
which exists since $\mathrm{Im}\, f_1' = \mathrm{Ker}\, f_0'$ and $f_0'\beta_0 = \beta f = 0$.

Inductively, for each basis element $x \in F_i$,
define $\lambda_i(x) = x' \in F_{i+1}'$ with $f_{i+1}'(x') = \beta_i(x) - \lambda_{i-1} f_i(x)$,
which exists since $\mathrm{Im}\, f_{i+1}' = \mathrm{Ker}\, f_i'$ and
$f_i'(\beta_i - \lambda_{i-1} f_i) = \beta_{i-1} f_i - f_i'\lambda_{i-1} f_i = (\beta_{i-1} - f_i'\lambda_{i-1}) f_i = \lambda_{i-2} f_{i-1} f_i = 0.$

$\beta_{i-1} = f_i'\lambda_{i-1} + \lambda_{i-2} f_{i-1}$ by induction