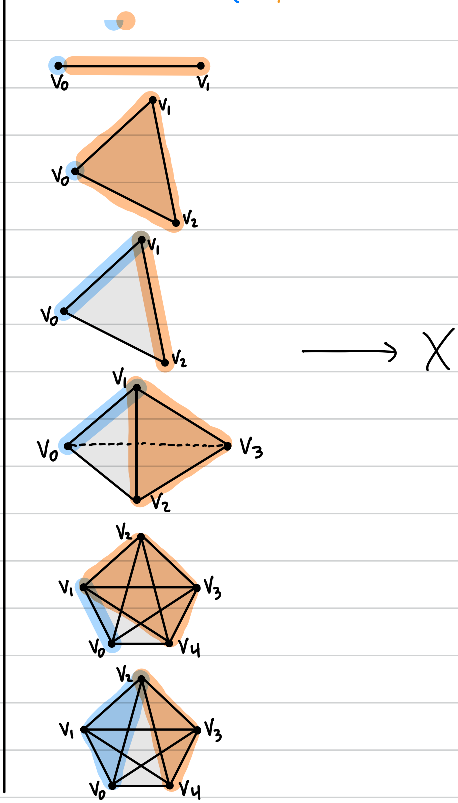
Section 3.2 Cup Product

Let X be a space.
Let R be a ring (often $\mathbb{R}, \mathbb{Q}$, or $\mathbb{Z}/n\mathbb{Z}$).
We'll define a cup product structure
$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$,
making $H^*(X;R)$ a (graded) ring.

First, define $C^k(X;R) \times C^\ell(X;R) \xrightarrow{\cup} C^{k+\ell}(X;R)$ with
$(\varphi, \psi) \longmapsto \varphi \cup \psi$
$\varphi \cup \psi$ acting on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ by
$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0, \dots, v_k]}) \, \psi(\sigma|_{[v_k, \dots, v_{k+\ell}]}).$$
product in R

- The product on cochains feels unnatural, but we'll it induces a map on cohomology.
- Certainly $\varphi \cup \psi \neq \pm \psi \cup \varphi \in C^{k+\ell}(X;R)$.
  But for R commutative and $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$, we'll see $\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$.

k	ℓ	Picture of $\varphi \cup \psi$
0	0	
0	1	
0	2	
1	1	
1	2	
1	3	
2	2	



Lemma 3.6 $\delta(\varphi\smile\psi) = (\delta\varphi\smile\psi) + (-1)^k(\varphi\smile\delta\psi)$
for $\varphi\in C^k(X;R)$ and $\psi\in C^\ell(X;R)$.

Pf Let $\sigma:\Delta^{k+\ell+1}\longrightarrow X$. Note

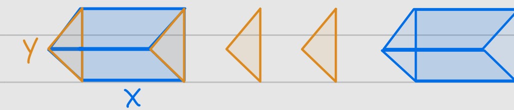
$$\begin{aligned}
\delta(\varphi\smile\psi)(\sigma) &= (\varphi\smile\psi)(\partial\sigma)\\
&= \sum_{i=0}^{k+\ell+1}(-1)^i(\varphi\smile\psi)(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]})\\
&= \sum_{i=0}^{k}(-1)^i\varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})\\
&\quad+\sum_{i=k+1}^{k+\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]}).
\end{aligned}$$

Note
$$\begin{aligned}
(\delta\varphi\smile\psi)(\sigma) &= \delta\varphi(\sigma|_{[v_0,\dots,v_{k+1}]})\,\psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})\\
&= \sum_{i=0}^{k+1}(-1)^i\varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}).
\end{aligned}$$

Note
$$\begin{aligned}
(-1)^k(\varphi\smile\delta\psi)(\sigma) &= (-1)^k\varphi(\sigma|_{[v_0,\dots,v_k]})\,\delta\psi(\sigma|_{[v_k,\dots,v_{k+\ell+1}]})\\
&= (-1)^k\varphi(\sigma|_{[v_0,\dots,v_k]})\sum_{i=k}^{k+\ell+1}(-1)^{i-k}\psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]})\\
&= \sum_{i=k}^{k+\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]}).
\end{aligned}$$

Hence we're done after observing the
$i=k+1$ term in $\delta\varphi\smile\psi$ cancels with the
$i=k$ term in $(-1)^k(\varphi\smile\delta\psi)$.

Aside: $\partial(X\times Y) = (\partial X\times Y)\cup(X\times\partial Y)$



Consequences

- If φ and ψ are cocycles, then so is $\varphi\smile\psi$ since
  $\delta(\varphi\smile\psi) = (\underbrace{\delta\varphi}_{=0}\smile\psi)+(-1)^k(\varphi\smile\underbrace{\delta\psi}_{=0}) = 0.$
- The cup product of a cocycle and a coboundary is a coboundary.
  Case 1 $\underset{\text{cocycle}}{\varphi}\smile\underset{\text{coboundary}}{\delta\psi} = \pm\delta(\varphi\smile\psi)$ if $\delta\varphi=0$.
  Case 2 $\underset{\text{coboundary}}{\delta\varphi}\smile\underset{\text{cocycle}}{\psi} = \delta(\varphi\smile\psi)$ if $\delta\psi=0$.
- Hence we get an induced map on cohomology:
  $H^k(X;R)\times H^\ell(X;R)\xrightarrow{\ \smile\ }H^{k+\ell}(X;R)$.
  Associative and distributive (also on cochains).

Example 3.7 M orientable surface genus g.
By UCT or Theorem 3.5 (on cellular cohomology),

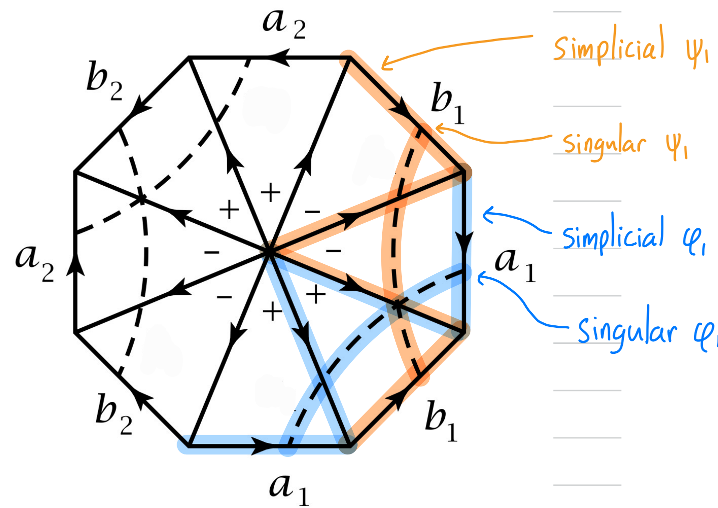
$$H^i(M;\mathbb{Z}) \cong \mathrm{Hom}(H_i(M),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{2g} & i=1 \\ \mathbb{Z} & i=2. \end{cases}$$

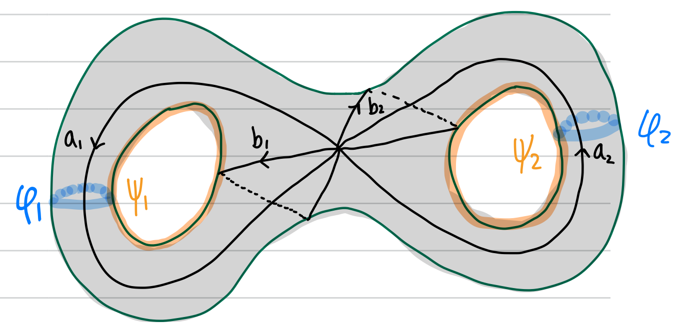
What is the cup product $H^1(M;\mathbb{Z}) \times H^1(M;\mathbb{Z}) \xrightarrow{\smile} H^2(M;\mathbb{Z})$?

$H_1(M)$ generated by cycles $a_1, b_1, \dots, a_g, b_g$.
$H^1(M;\mathbb{Z})$ generated by dual cocycles $\varphi_1, \psi_1, \dots, \varphi_g, \psi_g$.

Simplicial $H_2(M)$ generated by cycle c,
the sum of a 2-simplices with +/− signs indicated.
Simplicial $H^2(M;\mathbb{Z})$ generated by dual cocycle γ,
mapping one such 2-simplex to its sign.

$(\varphi_1 \smile \psi_1)(c) = 1$ generating $\mathbb{Z}$ $\Rightarrow [\varphi_1] \smile [\psi_1] = [\gamma]$
$(\psi_1 \smile \varphi_1)(c) = -1$ $\Rightarrow [\psi_1] \smile [\varphi_1] = -[\gamma]$.



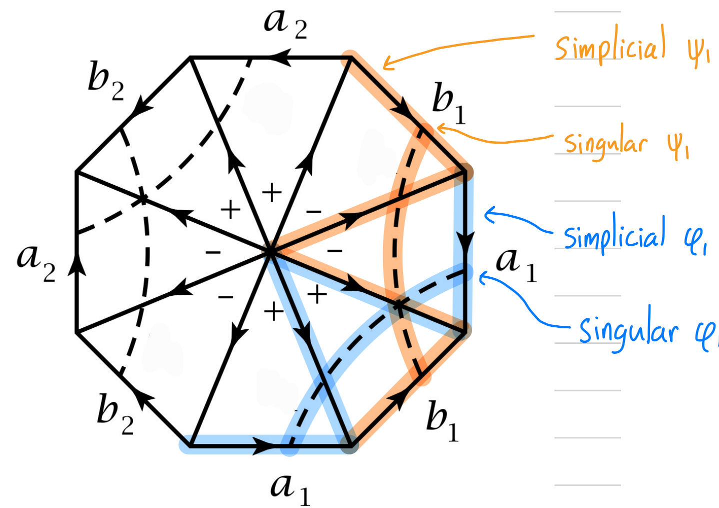


More generally,

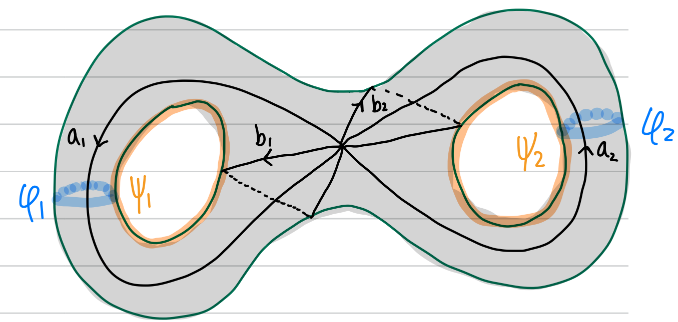
$$[\varphi_i]\smile[\psi_j] = \begin{cases} [\gamma] & i=j \\ 0 & i\neq j \end{cases} = -[\psi_j]\smile[\varphi_i].$$

$$[\varphi_i]\smile[\varphi_j] = 0 \quad \forall i,j.$$
$$[\psi_i]\smile[\psi_j] = 0 \quad \forall i,j.$$

By distributivity, this determines
$H^1(M;\mathbb{Z})\times H^1(M;\mathbb{Z}) \xrightarrow{\smile} H^2(M;\mathbb{Z})$ completely.



<u>Rmk</u> Nonzero cup products occur here precisely when the corresponding singular "loops" intersect. This works even for $[\varphi_i]\smile[\varphi_i]=0$ after deforming one copy to be disjoint from the other.

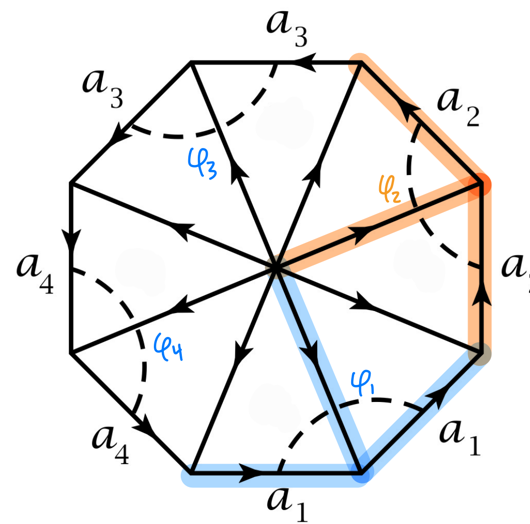


Example 3.8 N nonorientable surface genus g.

Recall $H_i(N) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & i=1 \\ 0 & i \geq 2. \end{cases}$

Cellular homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients gives

$H_i(N; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i=0 \\ (\mathbb{Z}/2\mathbb{Z})^g & i=1 \quad \text{gen. by } a_1, \dots, a_g. \\ \mathbb{Z}/2\mathbb{Z} & i=2 \quad \text{gen. by } c. \end{cases}$



By the UCT, we have a split SES

$0 \to \mathrm{Ext}(H_{n-1}(N), G) \to H^n(N; G) \to \mathrm{Hom}(H_n(N), G) \to 0.$

Recall $\mathrm{Ext}(\mathbb{Z}; G) = 0$ since $\mathbb{Z}$ is free.

Recall $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}; G) = G/2G.$

Choosing $G = \mathbb{Z}/2\mathbb{Z}$ gives

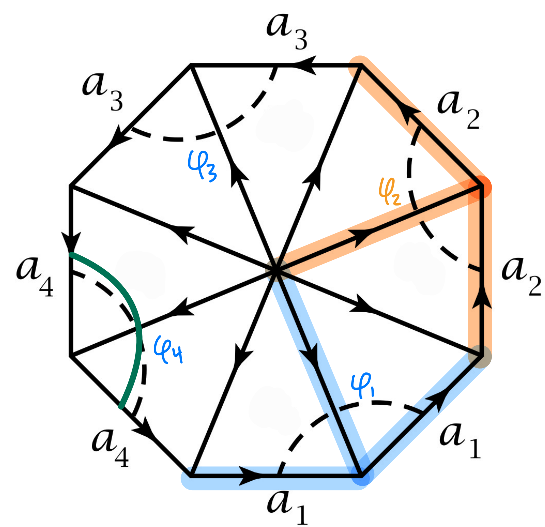
$H^i(N; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i=0 \\ (\mathbb{Z}/2\mathbb{Z})^g & i=1 \quad \text{gen. by } \varphi_1, \dots, \varphi_g \\ \mathbb{Z}/2\mathbb{Z} & i=2 \quad \text{gen. by } \gamma. \end{cases}$

Example 3.8 N nonorientable surface genus g.

What is the cup product
$H^1(N;\mathbb{Z}/2\mathbb{Z}) \times H^1(N;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\smile} H^2(N;\mathbb{Z}/2\mathbb{Z})$?

$(\varphi_i \smile \varphi_i)(c) = 1 \Rightarrow [\varphi_i] \smile [\varphi_i] = [\gamma]$.

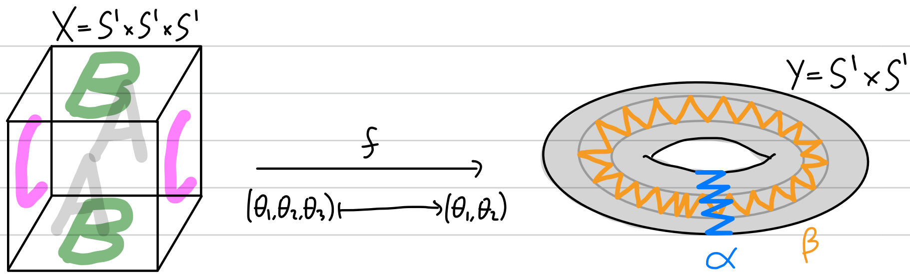
For $i \neq j$, $\varphi_i \smile \varphi_j = 0$ on all 2-chains
$\Rightarrow [\varphi_i] \smile [\varphi_j] = 0$.



Rmk Nonzero cup products occur here precisely when the corresponding singular "loops" intersect. This works even for $[\varphi_i] \smile [\varphi_i] = 1$ since any deformation has at least one intersection point.

Rmk When $g=1$, we get $H^*(\mathbb{RP}^2; \underset{\mathbb{Z}/2\mathbb{Z}}{\overset{!!}{\mathbb{Z}_2}}) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$ where $\underset{\text{dimension or degree}}{|\alpha|} = 1$ $(\alpha \in H^1(\mathbb{RP}^2;\mathbb{Z}_2))$.
More generally, $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ where $|\alpha| = 1$ (Thm 3.19).

Prop 3.10 For a map $f: X \to Y$, the induced maps $f^*: H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.



Pf This follows from the cochain-level formula

$$\begin{aligned}(f^\# \varphi \smile f^\# \psi)(\sigma) &= f^\#\varphi(\sigma|_{[v_0,\dots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= \varphi(f\sigma|_{[v_0,\dots,v_k]})\, \psi(f\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= (\varphi \smile \psi)(f\sigma) \\ &= f^\#(\varphi \smile \psi)(\sigma).\end{aligned}$$

Thm 3.11 For R a commutative ring, $\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$ for all $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$.

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "\rho"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "\rho"] \arrow[dl] & C_{n-1}(X) \arrow[d, "\rho"] \arrow[dl] \\
C_{n+1}(X) \arrow[r, "\partial"'] & C_n(X) \arrow[r, "\partial"'] & C_{n-1}(X)
\end{tikzcd}$$

Pf We'll build a chain map $\rho : C_n(X) \to C_n(X)$, chain homotopic to the identity (1 pg proof in Hatcher) (hence $\rho^* : C^n(X) \to C^n(X)$ is chain homotopic to the identity), satisfying $\rho^*\varphi \smile \rho^*\psi = (-1)^{k\ell} \rho^*(\psi \smile \varphi)$.

$$\begin{tikzcd}
C^{n+1}(X) & C^n(X) \arrow[l, "\delta"'] \arrow[d, dash, "\rho^*"] & C^{n-1}(X) \arrow[l, "\delta"'] \arrow[d, dash, "\rho^*"] \\
C^{n+1}(X) \arrow[u, dash, "\rho^*"] \arrow[ur] & C^n(X) \arrow[l, "\delta"] \arrow[ur] & C^{n-1}(X) \arrow[l, "\delta"]
\end{tikzcd}$$

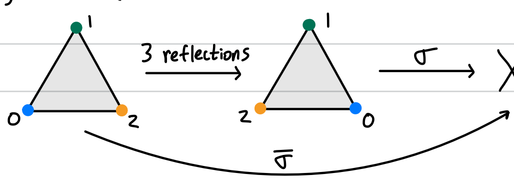
Passing to cohomology gives $[\varphi] \smile [\psi] = (-1)^{k\ell} [\psi] \smile [\varphi]$.

How to define $\rho : C_n(X) \to C_n(X)$?

The linear map $[v_0, \dots, v_n] \to [v_n, \dots, v_0]$ is a product of $\frac{n(n+1)}{2}$ transpositions/reflections.

For σ a singular n-simplex, let $\bar{\sigma}$ be σ precomposed with this linear map.

Let $\rho(\sigma) = (-1)^{\frac{n(n+1)}{2}} \bar{\sigma}$.



Note

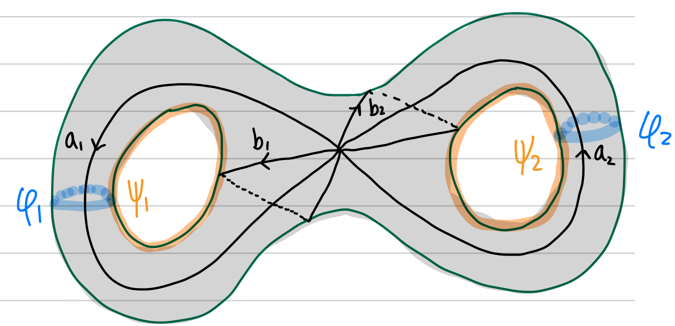
$$\begin{aligned}
&(\rho^*\varphi \smile \rho^*\psi)(\sigma) \\
&= \rho^*\varphi(\sigma|_{[v_0,\dots,v_k]})\, \rho^*\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\
&= (-1)^{\frac{k(k+1)}{2}} \varphi(\sigma|_{[v_k,\dots,v_0]})\, (-1)^{\frac{\ell(\ell+1)}{2}} \psi(\sigma|_{[v_{k+\ell},\dots,v_k]}) \\
&= (-1)^{\frac{k(k+1)}{2} + \frac{\ell(\ell+1)}{2}} \psi(\sigma|_{[v_{k+\ell},\dots,v_k]})\, \varphi(\sigma|_{[v_k,\dots,v_0]}) \quad [R \text{ commutative}] \\
&= (-1)^{k\ell} (-1)^{\frac{(k+\ell)(k+\ell+1)}{2}} \psi(\sigma|_{[v_{k+\ell},\dots,v_k]})\, \varphi(\sigma|_{[v_k,\dots,v_0]}) \quad [(-1)^{k\ell}(-1)^{k\ell} = 1] \\
&= (-1)^{k\ell} \rho^*(\psi \smile \varphi)(\sigma).
\end{aligned}$$

Thm 3.11 For R a commutative ring,
$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$ for all $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$.

Ex For k odd and $\alpha = \beta \in H^k(X;R)$,
$2(\alpha \smile \alpha) = 2\alpha^2 = 0$ in $H^{2k}(X;R)$.
So if $H^{2k}(X;R)$ has no elements of order two, then $\alpha^2 = 0$.

Ex $\alpha^2 = 0$ for any $\alpha \in H^1(M;\mathbb{Z})$ for M an orientable surface of genus g.

$[\varphi_i] \smile [\varphi_j] = 0 \quad \forall i,j.$
$[\psi_i] \smile [\psi_j] = 0 \quad \forall i,j.$
$[\varphi_i] \smile [\psi_j] = \begin{cases} [\gamma] & i=j \\ 0 & i \neq j \end{cases} = -[\psi_j] \smile [\varphi_i].$



For example,
$([\varphi_i]+[\psi_j])^2 = [\varphi_i]^2 + [\varphi_i] \smile [\psi_j] + [\psi_j] \smile [\varphi_i] + [\psi_j]^2 = 0.$

The cohomology ring

Define $H^*(X;R) = \bigoplus_{i\geq 0} H^i(X;R)$

Elements $\sum_i \alpha_i$ are finite sums, $\alpha_i \in H^i(X;R)$.

Addition: $\sum_i \alpha_i + \sum_i \beta_i = \sum_i \alpha_i + \beta_i$

Multiplication: $\left(\sum_i \alpha_i\right)\left(\sum_i \beta_i\right) = \sum_{i,j} \alpha_i \cup \beta_j = \sum_{k\geq 0} \sum_{i+j=k} \alpha_i \cup \beta_j$.

This makes $H^*(X;R)$ a ring, with an identity if R has an identity.

It is a graded ring: $A \cong \bigoplus_{k\geq 0} A_k$ with $A_k \times A_\ell \xrightarrow{\cdot} A_{k+\ell}$.

An element $a \in A_k$ has <u>dimension</u> or <u>degree</u> $|a| = k$.

Many elements of A do not live in some A_k.

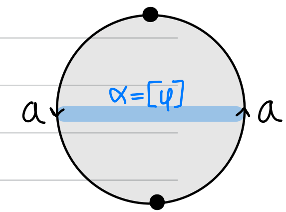
$\boxed{\alpha \cup \beta = (-1)^{k\ell}\, \beta \cup \alpha \text{ for } \alpha \in H^k(X;R),\ \beta \in H^\ell(X;R),\ R \text{ commutative.}}$

If $ab = (-1)^{|a||b|} ba$ for $a \in A_{|a|}$, $b \in A_{|b|}$, as in Thm 3.11,
then the graded ring A is referred to as
<u>commutative</u>, <u>graded commutative</u>, <u>anticommutative</u>, or <u>skew commutative</u>.

What are some example graded commutative rings?

Ex 3.12 Polynomial rings $R[\alpha]$ and $R[\alpha]/(\alpha^{n+1})$ with $|\alpha|$ even or $2=0$ in R.

Recall from Ex 3.8 (with genus $g=1$) that $H^i(\mathbb{R}P^2;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & i=0,1,2 \\ 0 & \text{o.w.} \end{cases}$



where if α generates $H^1(\mathbb{R}P^2;\mathbb{Z}_2)$, then α^2 generates $H^2(\mathbb{R}P^2;\mathbb{Z}_2)$.

$$H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \{c_2\alpha^2 + c_1\alpha + c_0 \mid c_i \in \mathbb{Z}_2\}$$
$$\cong \mathbb{Z}_2[\alpha]/(\alpha^3) \text{ where } |\alpha| = 1.$$

$$(c_2\alpha^2+c_1\alpha+c_0)(d_2\alpha^2+d_1\alpha+d_0)$$
$$= (c_2d_0+c_1d_1+c_0d_2)\alpha^2+(c_1d_0+c_0d_1)\alpha + c_0d_0$$

Economic representation!

$H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ and $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ where $|\alpha|=1$ (Thm 3.19).

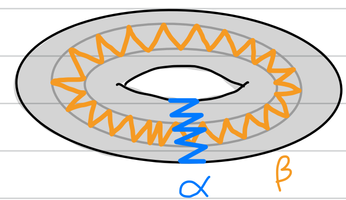
$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$ where $|\alpha|=2$ (Thm 3.19).

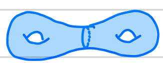
$H^*(\prod_{i=1}^k \mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha_1,\ldots,\alpha_k]$ where $|\alpha_i| = 1$ (Example 3.20).

Ex 3.13 Exterior algebras $\Lambda_R[\alpha_1,\dots,\alpha_n]$ with $|\alpha_i|$ odd.

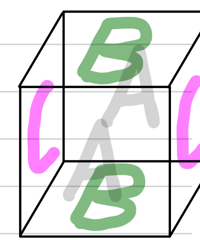
Let R be a commutative ring with identity 1.
The *exterior algebra* $\Lambda_R[\alpha_1,\dots,\alpha_n]$ is the free R-module with basis the finite products $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}$, $i_1<\dots<i_k$ with associative, distributive multiplication defined by $\alpha_i\alpha_j=-\alpha_j\alpha_i$ for $i\neq j$ and $\alpha_i^2=0$.
The empty product of α_i's is the identity, denoted 1.
$\Lambda_R[\alpha_1,\dots,\alpha_n]$ is graded commutative if $|\alpha_i|$ is odd $\forall i$.

$H^*(S^1\times S^1;\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha,\beta]$ with $|\alpha|=|\beta|=1$ (Ex 3.7).



The cohomology of  is not of this form — why?

$H^*((S^1)^n;\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha_1,\dots,\alpha_n]$ with $|\alpha_i|=1$ $\forall i$ (Ex 3.16).



Product of odd spheres
$H^*(\prod_{i=1}^n S^{2k_i+1};\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha_1,\dots,\alpha_n]$ with $|\alpha_i|=2k_i+1$ $\forall i$.

Recall induced homomorphisms are ring homomorphisms (Prop 3.10). For example:

Ex 3.14 The inclusions $i_\alpha : X_\alpha \to \coprod_\alpha X_\alpha$ induce a ring isomorphism $H^*(\coprod_\alpha X_\alpha; R) \longrightarrow \prod_\alpha H^*(X_\alpha; R)$

coordinate-wise multiplication

Similarly $\tilde{H}^*(\vee_\alpha X_\alpha; R) \xrightarrow{\cong} \prod_\alpha \tilde{H}^*(X_\alpha; R)$ as rings.
Reduced cohomology is cohomology relative a basepoint.
To get $\cong$ we assume basepoints x_α are deformation retracts of neighborhoods, i.e., the (X_α, x_α) are good pairs.

Ring structures can distinguish spaces from wedge sums.
Consider $\mathbb{C}P^2 := S^2 \cup_f D^4$ with $f: S^3 \to S^2$ the Hopf map and $S^2 \vee S^4$.

$$H^i(S^2 \vee S^4) \cong \begin{cases} \mathbb{Z} & i = 0,2,4 \\ 0 & \text{o.w.} \end{cases} \cong H^i(\mathbb{C}P^2; \mathbb{Z})$$

Note $H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3)$ with $|\alpha| = 2$ has nontrivial cup products $(\alpha^2 \neq 0)$, whereas $S^2 \vee S^4$ does not $(\tilde{H}^*(S^2 \vee S^4; \mathbb{Z}) \cong \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^4; \mathbb{Z}))$.

Realization problem

Which graded commutative R-algebras occur as cup product rings $H^*(X; R)$ for some space X?

- $R = \mathbb{Q}$, essentially all (Quillen 1969)
- $R = \mathbb{Z}_p$, p prime ?
- $R = \mathbb{Z}$??

More generally (see §4.B on the Hopf invariant), let $n \geq 2$. For $f: S^{2n-1} \to S^n$, let $C_f : S^n \cup_f D^{2n}$.

For $f, g: S^{2n-1} \to S^n$, if $f \simeq g$, then $C_f \simeq C_g$. Conversely, $C_f \not\simeq C_g$ implies $f \not\simeq g$.

Let $\alpha \in H^n(C_f; \mathbb{Z}) \cong \mathbb{Z}$ and $\beta \in H^n(C_f; \mathbb{Z}) \cong \mathbb{Z}$ be generators (choose the sign of β carefully). The <u>Hopf invariant of f</u> is the integer $H(f)$ satisfying $\alpha^2 = H(f)\beta$.

- f nullhomotopic $\Rightarrow C_f \simeq S^n \vee S^{2n}$ and $H(f) = 0$.
- $f: S^3 \to S^2$ the Hopf map $\Rightarrow C_f = \mathbb{C}P^2$ and $H(f) = 1$.
- For n odd, $\alpha^2 = -\alpha^2$, hence $\alpha^2 = 0$ and $H(f) = 0$.
- For n even, the Hopf invariant distinguishes infinitely many homotopy classes of maps $S^{2n-1} \to S^n$. So $\pi_{2n-1}(S^n)$ is infinite for n even.

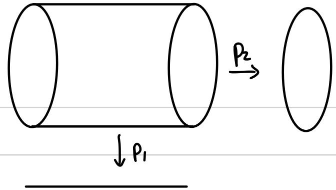
Furthermore, $H: \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism (Prop. 4B.1) and $\pi_{2n-1}(S^n)$ contains $\mathbb{Z}$ as a direct summand for n even (Cor 4B.2).

<u>Thm</u> (J.F. Adams, 1960) A map $f: S^{2n-1} \to S^n$ with Hopf invariant $H(f) = 1$ exists only when $n = 2, 4, 8$.

<u>Consequences</u>

an algebra over a field with division (except by 0)

- $\mathbb{R}^n$ is a <u>division algebra</u> only for $n = 1, 2, 4, 8$.
- S^n is an H-space only for $n = 0, 1, 3, 7$.
- S^n has n linearly independent vector fields only for $n = 0, 1, 3, 7$.
- The only fiber bundles $S^p \to S^q \to S^r$ occur when $(p, q, r) = (0, 1, 1)$, $(1, 3, 2)$, $(3, 7, 4)$, and $(7, 15, 8)$.



A Künneth formula

R a commutative ring and X, Y spaces.

Projections $p_1 : X \times Y \to X$ $\quad p_1(x,y) = x$

$p_2 : X \times Y \to Y$ $\quad p_2(x,y) = y$

The cross product is the map

$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R)$ given by

$$(a,b) \longmapsto a \times b := p_1^*(a) \cup p_2^*(b).$$

Since the cup product $\cup$ is distributive, the cross product $\times$ is bilinear:

$(ra + r'a') \times b = r(a \times b) + r'(a' \times b)$

$a \times (rb + r'b') = r(a \times b) + r'(a \times b')$

$\forall r \in R,\ a \in H^*(X;R),\ b \in H^*(Y;R).$

Pf $(ra + r'a') \times b = p_1^*(ra + r'a') \cup p_2^*(b) = (r p_1^*(a) + r' p_1^*(a')) \cup p_2^*(b)$

$= r(p_1^*(a) \cup p_2^*(b)) + r'(p_1^*(a') + p_2^*(b)) = r(a \times b) + r'(a' \times b).$

Is the cross product an isomorphism? Not even a ring homomorphism, since

$$r(a,b) = (ra, rb) \overset{\times}{\longmapsto} r p_1^*(a) \cup r p_2^*(b) = r^2(a \times b).$$

Transform this R-bilinear map into an R-linear one by replacing $H^*(X;R) \times H^*(Y;R)$ with the tensor product $H^*(X;R) \otimes_R H^*(Y;R)$.

not yet defined

Thm 3.15 The cross product

$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R)$

gen. $(a,b) \longmapsto a \times b := p_1^*(a) \cup p_2^*(b)$

is a ring isomorphism if X, Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module $\forall k$.

Universal property

For M and N R-modules,

the <u>tensor product</u> $M \otimes_R N$ is an R-module

equipped with a bilinear map $M \times N \xrightarrow{\otimes} M \otimes_R N$

such that

$$\begin{tikzcd}
 & M \otimes_R N \arrow[d, "\exists! L \text{ linear}"] \\
M \times N \arrow[ur, "\otimes"] \arrow[r, phantom, "\circlearrowright"] \arrow[dr, "B"'] & {} \\
 & P
\end{tikzcd}$$

for each bilinear $M \times M \xrightarrow{B} P$.