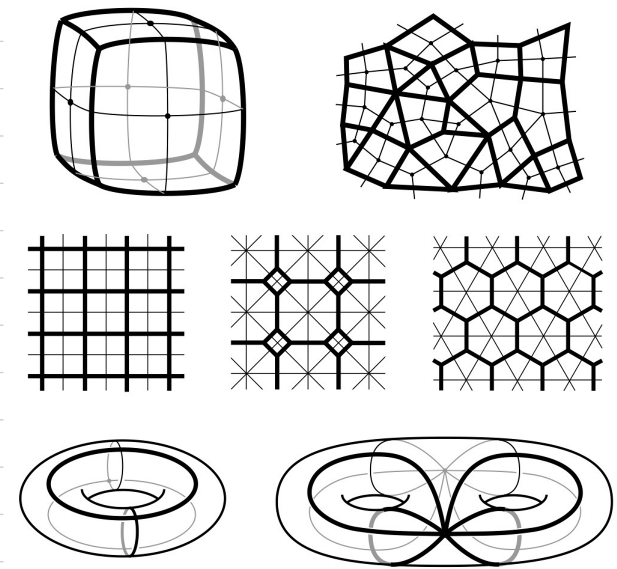
Section 3.3 Poincaré duality

We'll start with an easier version.
Formal version is Thm 3.30.
Recall a closed manifold is compact without boundary.

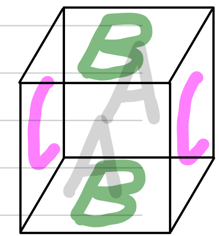


Poincaré Duality (easier version)
Let M be a closed n-dimensional manifold.
Let M be differentiable (or more generally, let M have a pair of "dual cell structures".)
Then $H^k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$.
Furthermore, if M is orientable, then $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$.

Rmk Thm 3.30 does not require differentiability or dual cell structures.
Rmk The isomorphism in Thm 3.30 takes cap products with the fundamental class.

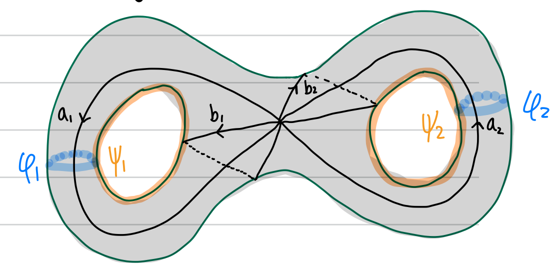
Orientable examples $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$

Sphere $H^k(S^n;\mathbb{Z}) \cong \begin{cases} 0 & k=0,n \\ \mathbb{Z} & \text{o.w.} \end{cases} \cong H_{n-k}(S^n;\mathbb{Z})$



Torus (n-dimensional) $H^k((S^1)^n;\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}} \cong \mathbb{Z}^{\binom{n}{n-k}} \cong H_{n-k}((S^1)^n;\mathbb{Z})$

Orientable surface of genus g $H^k(M_g;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^{2g} & k=1 \end{cases} \cong H_{n-k}(M_g;\mathbb{Z})$



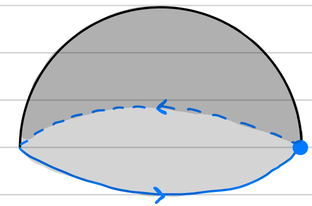
Projective space $\mathbb{R}P^n$ is orientable for n odd.

$H_i(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \text{ or } i=n \text{ odd} \\ \mathbb{Z}_2 & 0<i<n \text{ odd} \\ 0 & \text{o.w.} \end{cases}$ $H^i(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \text{ or } i=n \text{ odd} \\ \mathbb{Z}_2 & 0<i<n \text{ even} \\ 0 & \text{o.w.} \end{cases}$

So for n odd, $H^k(\mathbb{R}P^n;\mathbb{Z}) \cong H_{n-k}(\mathbb{R}P^n;\mathbb{Z})$.

Non-orientable examples $\quad H^k(M;\mathbb{Z}_2)\cong H_{n-k}(M;\mathbb{Z}_2)$

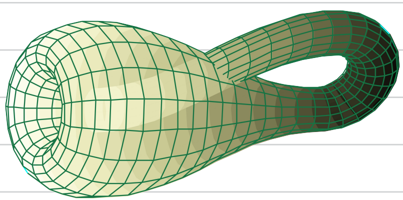
$\mathbb{RP}^n$ is not orientable for n even.

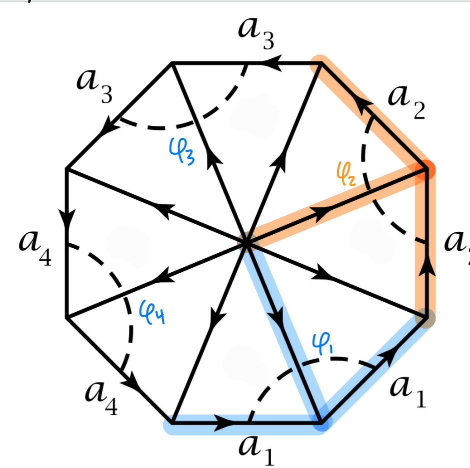


$$H^k(\mathbb{RP}^n;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2 & 0\le k\le n\\ 0 & \text{o.w.}\end{cases}\cong H_{n-k}(\mathbb{RP}^n;\mathbb{Z}_2)$$

Non-orientable surface of genus g

$$H^k(N_g;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2 & k=0,2\\ (\mathbb{Z}_2)^g & k=1\\ 0 & \text{o.w.}\end{cases}\cong H_{n-k}(N_g;\mathbb{Z}_2)$$





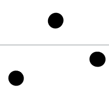
Rmk The local property of closed manifolds (locally homeomorphic to $\mathbb{R}^n$) imposes strong control on global properties (homology and cohomology).

Introduction to manifolds

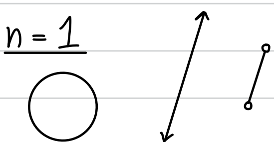
Def An <u>n-manifold</u> is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Means each $x \in M$ has a neighborhood $U \ni x$ homeomorphic to $\mathbb{R}^n$, or equivalently,
" " " " " " " " some open set in $\mathbb{R}^n$.

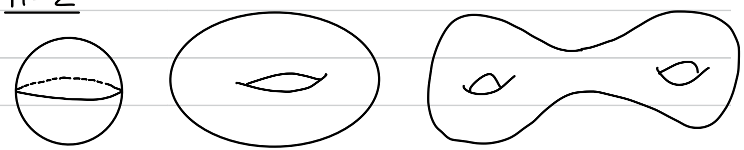
<u>$n=0$</u>



<u>$n=1$</u>



<u>$n=2$</u>



<u>Dim n</u>

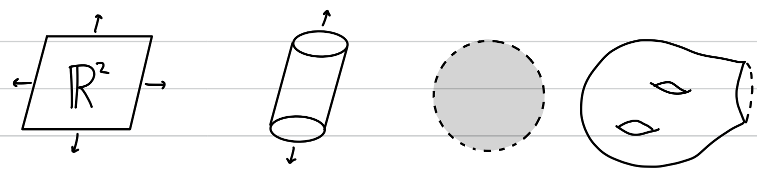
$\mathbb{R}^n$

S^n

$(S^1)^n$

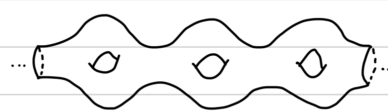
$\mathbb{RP}^n$

many others!

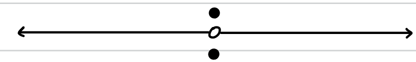


<u>Dim $2n$</u>

$\mathbb{CP}^n$

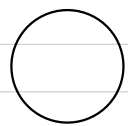


Second countable (topology has a countable basis) rules out the long line.

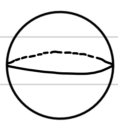
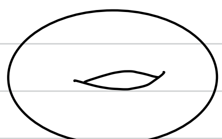
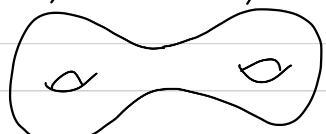
Hausdorff rules out the line with two origins 

The only closed connected n-dimensional manifolds (up to homeomorphism) are

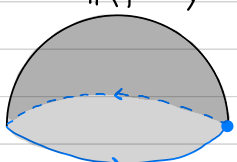
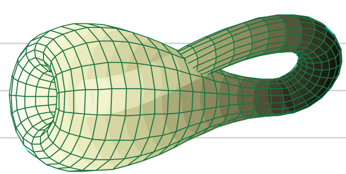
$\underline{n=0}$ The point 

$\underline{n=1}$ The circle 

$\underline{n=2}$ S^2, torus $\mathbb{T}^2 = S^1 \times S^1$, $\mathbb{T}^2 \# \mathbb{T}^2$, all genus g tori $M_g := \underbrace{\mathbb{T}^2 \# \ldots \# \mathbb{T}^2}_{g \text{ times}}$,

$\mathbb{RP}^2$, Klein bottle $\mathbb{RP}^2 \# \mathbb{RP}^2$, all genus g nonorientable surfaces $N_g := \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{g \text{ times}}$.

$\underline{n=3}$ Hard! | Every closed connected 2-manifold can be given a metric with constant curvature (positive, zero, or negative).

Proven by Perelman in 2006, declined Fields medal

Thurston's geometrization conjecture (now theorem) says each closed 3-manifold can be canonically decomposed into pieces with one of eight types of geometric structure. It implies the ...

Poincaré conjecture (now theorem) Every closed connected 3-manifold with trivial fundamental group is homeomorphic to S^3.

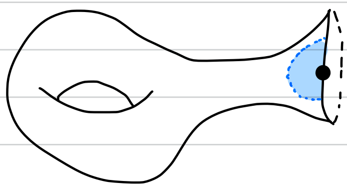
$\underline{n=4}$ Hard!

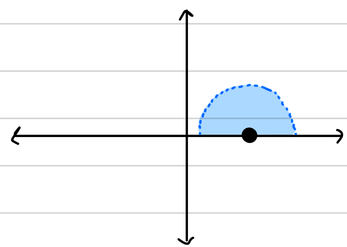
$\underline{n \geq 5}$ Hard, but some things get easier.

In 1961, Smale proved a generalized Poincaré conjecture (a homotopy n-sphere is homeomorphic to S^n) for $n \geq 5$.
In 1982, Freedman proved it for $n=4$.
The trivial π_1 assumption does not suffice for $n \geq 4$.

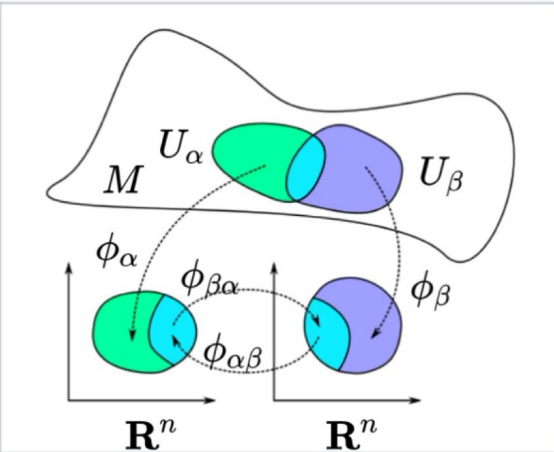
<u>Invariance of dimension</u> For $n \neq m$, a space cannot be both an n-manifold and an m-manifold.

<u>Def</u> An <u>n-manifold with boundary</u> (need not be a manifold!) is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$ or to $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.





From Wikipedia:



The transition map of two charts. $\phi_{\alpha\beta}$ denotes $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_{\beta\alpha}$ denotes $\phi_\beta \circ \phi_\alpha^{-1}$.

Given a topological space M...				
a C^k atlas	is a collection of charts	$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$	such that $\{U_\alpha\}_{\alpha\in A}$ covers M, and such that for all α and β in A, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is	a C^k map
a smooth or C^∞ atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$		a smooth map
an analytic or C^ω atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{R}^n\}_{\alpha\in A}$		a real-analytic map
a holomorphic atlas		$\{\varphi_\alpha : U_\alpha \to \mathbf{C}^n\}_{\alpha\in A}$		a holomorphic map

A <u>differentiable (C^k or C^∞)</u> manifold is a second-countable Hausdorff space M equipped with a maximal differentiable atlas.

<u>Whitney embedding theorem</u> A C^∞ n-manifold can be smoothly embedded in $\mathbb{R}^{2n}$.
(For n a power of 2, $\mathbb{RP}^n$ cannot be embedded in $\mathbb{R}^{2n-1}$.)

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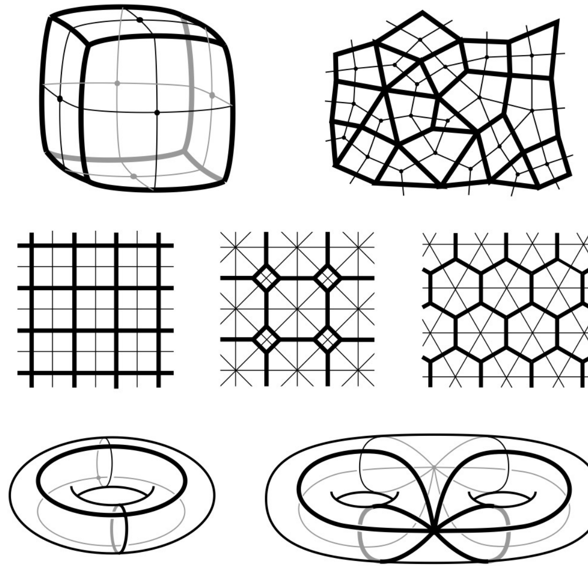

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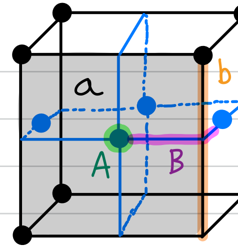
<u>Rmk</u> Thm 3.30 does not require differentiability or dual cell structures.
<u>Rmk</u> The isomorphism in Thm 3.30 takes cap products with the fundamental class.

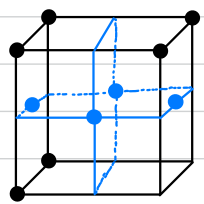
A dual cell decomposition of an n-manifold M is a pair of finite cell structures C, C^* such that

- each $(n-k)$-cell of C has a corresponding k-cell of C^*
- the boundary of an $(n-k)$-cell σ in C contains an $(n-k-1)$-cell $\Leftrightarrow$ the boundary of the dual $(k+1)$-cell in C^* contains the dual k-cell.



Picture $n=2$, $k=0$.

Ex $M = S^2$

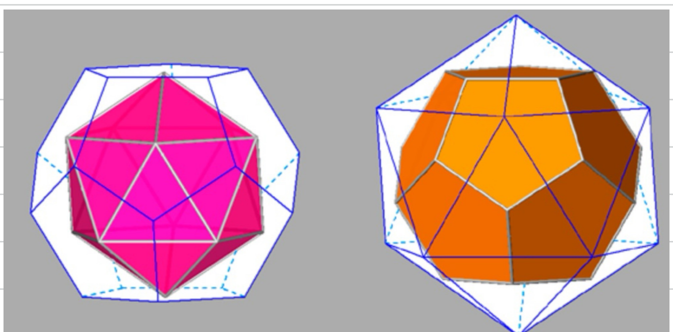


C: 8 vertices
12 edges
6 2-cells

Cube

C^*: 6 vertices
12 edges
8 2-cells

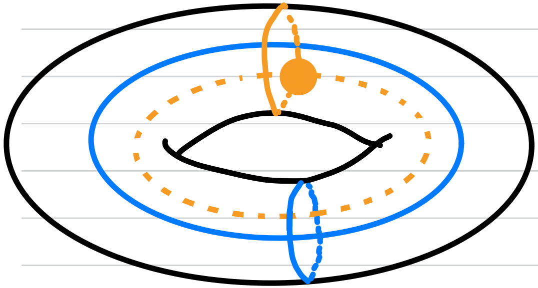
Octahedron



The icosahedron and dodecahedron are also dual.

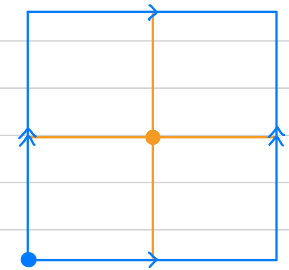
Image from cosmic-core.org.

Ex $M = S^1 \times S^1$

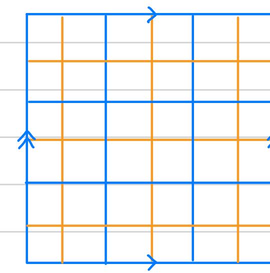


C: 1 vertex
2 edges
1 2-cell

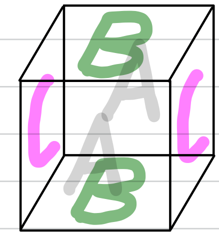
C^*: 1 vertex
2 edges
1 2-cell



We can get finer dual cell decompositions via



A similar construction works for $S^1 \times S^1 \times S^1$ with all squares replaced by cubes.



Proof of Poincaré Duality (easier version)

Use CW structure C to build a chain complex,
and CW structure C^* to build a cochain complex.

$$\begin{array}{ccccccccccccc}
0 \to & C_n(M;\mathbb{Z}_2) & \xrightarrow{\partial_n} & C_{n-1}(M;\mathbb{Z}_2) & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_2} & C_1(M;\mathbb{Z}_2) & \xrightarrow{\partial_1} & C_0(M;\mathbb{Z}_2) & \xrightarrow{\partial_0} 0 \\
& \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & \\
0 \to & C^{*0}(M;\mathbb{Z}_2) & \xrightarrow{\delta^0} & C^{*1}(M;\mathbb{Z}_2) & \xrightarrow{\delta^1} & \cdots & \xrightarrow{\delta^{n-2}} & C^{*(n-1)}(M;\mathbb{Z}_2) & \xrightarrow{\delta^{n-1}} & C^{*n}(M;\mathbb{Z}_2) & \xrightarrow{\delta^n} 0
\end{array}$$

For all k we have
$$\begin{aligned} C_{n-k}(M;\mathbb{Z}_2) &\cong \mathbb{Z}_2^{\oplus(\#\,(n-k)\text{-cells in } C)} \\ &\cong \mathbb{Z}_2^{\times(\#\,k\text{-cells in } C^*)} \\ &\cong C^{*k}(M;\mathbb{Z}_2) \end{aligned}$$

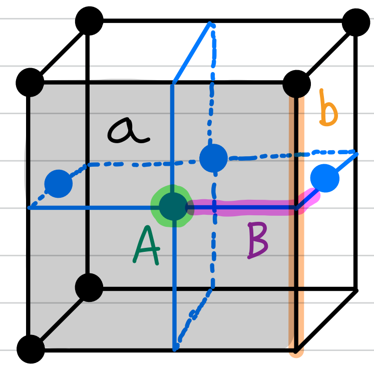
Furthermore, we'll see that under this correspondence, ∂_{n-k} corresponds to δ^k.
Hence $H^k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$.

When M is orientable, a similar proof works with $\mathbb{Z}$ coefficients.

To see ∂_{n-k} corresponds to δ^k:

Let a be an $(n-k)$-cell in C.
Let b be an $(n-k-1)$-cell in C.

Let A be the corresponding k-cell in C^*.
Let B be the corresponding $(k+1)$-cell in C^*.



Picture $n=2$, $k=0$.

By the second bullet defining dual cell decompositions,

b has coefficient 1 in ∂a

$\iff A$ has coefficient 1 in ∂B

$\iff B^*$ has coefficient 1 in δA^* (since $\delta A^*(B) := A^*(\partial B)$).

(Recall B^* is the cochain assigning 1 to the $(k+1)$-cell B, and A^* is the cochain assigning 1 to the k-cell A.)

Since this is true for all a, b, A, B, ∂_{n-k} corresponds to δ^k.

Let R be a commutative ring with identity (think $R = \mathbb{Z}$ or $\mathbb{Z}_2$).

<u>Thm 3.30 (Poincaré Duality)</u> If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism $\forall k$.

<u>Rmk</u> We need to define R-orientable, fundamental class, and the cap product $\frown$.

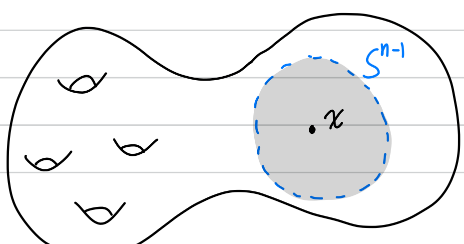
<u>Def (local)</u> An <u>R-orientation of M at $x \in M$</u> is a choice of generator/unit (element $\mu \in R$ with $R\mu = R$) of

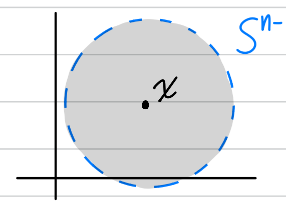
$H_n(M, M-\{x\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R)$ by excision and the def$^{\text{n}}$ of n-manifold

$\cong H_{n-1}(\mathbb{R}^n - \{x\}; R)$ by the LES of the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ with $\mathbb{R}^n \simeq *$

$\cong H_{n-1}(S^{n-1}; R)$ since $\mathbb{R}^n - \{x\} \simeq S^{n-1}$

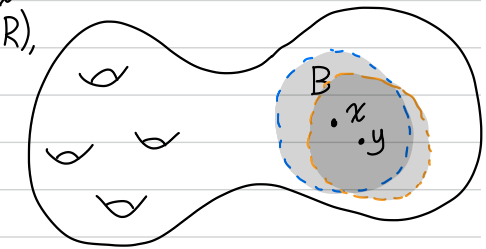
$\cong R.$





Def (global) An R-orientation of M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M-\{x\}; R)$, satisfying the consistency criterion that each $x \in M$ has an open neighborhood $x \in B \subseteq M$ with $\mu_B \in H_n(M, M-B; R)$, where $\forall y \in B$ we have

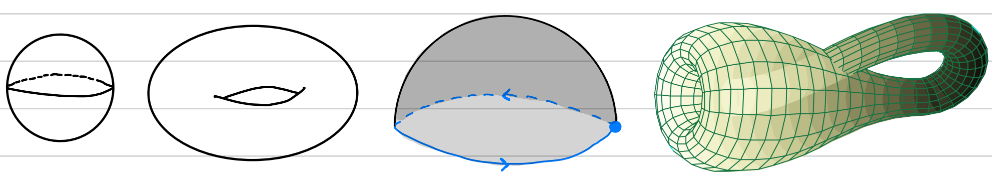
$$\begin{aligned} H_n(M, M-B; R) &\longrightarrow H_n(M, M-\{y\}; R) \\ \mu_B &\longmapsto \mu_y . \end{aligned}$$



If an R-orientation exists then M is R-orientable.

Ex Any manifold is $\mathbb{Z}_2$-orientable as there is only one choice of generator.

Ex Spheres and tori are $\mathbb{Z}$-orientable whereas the Klein bottle and $\mathbb{RP}^2$ are not.



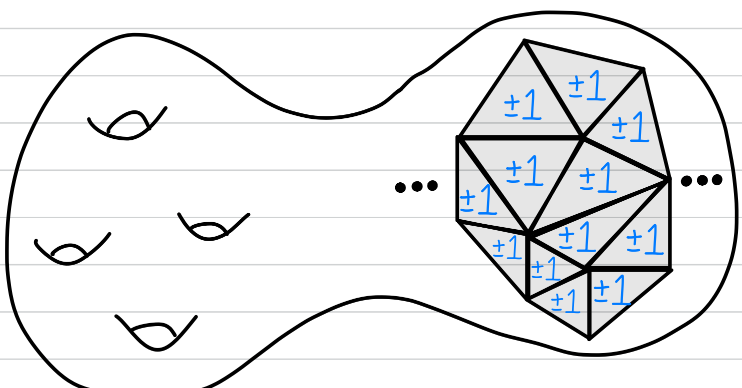
Def A fundamental class for M is an element $[M] \in H_n(M;R)$ satisfying

$$\begin{aligned} H_n(M;R) &\longrightarrow H_n(M, M-\{x\}; R) \\ [M] &\longmapsto \mu_x \quad \text{(local orientation)} \end{aligned}$$

for all $x \in M$.

Fact M is R-orientable $\iff$ a fundamental class exists.

Ex When M is a Δ-complex and $R = \mathbb{Z}$, a fundamental class $[M] \in H_n(M;\mathbb{Z})$ is an n-cycle with coefficient ± 1 on each n-simplex of M.

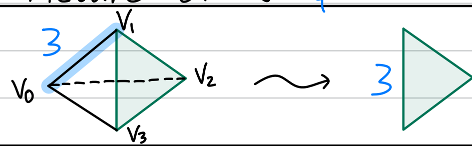


Cap product

For X a space, we'll have a cap product $H_k(X;R)\times H^\ell(X;R)\xrightarrow{\frown} H_{k-\ell}(X;R)$.

This has close connections to the cup product $H^k(X;R)\times H^\ell(X;R)\xrightarrow{\smile} H^{k+\ell}(X;R)$.

For $k\geq \ell$, define the R-bilinear map $C_k(X;R)\times C^\ell(X;R)\xrightarrow{\frown} C_{k-\ell}(X;R)$
by $\quad \sigma\frown\varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]} \quad$ for $\sigma:\Delta^k\to X$ a singular simplex.

<table>
<tr><th>k</th><th>ℓ</th><th>Picture of $\sigma\frown\varphi$</th></tr>
<tr><td>3</td><td>1</td><td>
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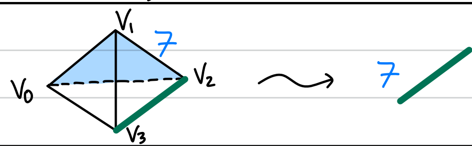
<tr><td>3</td><td>2</td><td>
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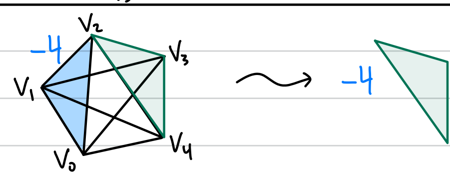
<tr><td>4</td><td>2</td><td>
</td></tr>
</table>

So φ "eats" the first ℓ dimensions to turn a k-simplex into a $(k-\ell)$-simplex.

Note $\partial(\sigma \frown \varphi) = (-1)^{\ell}(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$. The cap product of ...

- a cycle and a cocycle is a cycle since $\partial(\sigma\frown\varphi)=0$ if $\partial\sigma=0$ and $\delta\varphi=0$.
- a cycle and a coboundary is a boundary since $\sigma\frown\delta\varphi = \pm\,\partial(\sigma\frown\varphi)$ if $\partial\sigma=0$.
- a boundary and a cocycle is a boundary since $\partial\sigma\frown\varphi = \pm\,\partial(\sigma\frown\varphi)$ if $\delta\varphi=0$.

Hence we get an induced R-bilinear cap product $\underset{\frac{\operatorname{Ker}\partial_k}{\operatorname{Im}\partial_{k+1}}}{\underset{\shortparallel}{H_k(X;R)}} \times \underset{\frac{\operatorname{Ker}\delta^{\ell}}{\operatorname{Im}\delta^{\ell-1}}}{\underset{\shortparallel}{H^{\ell}(X;R)}} \xrightarrow{\frown} \underset{\frac{\operatorname{Ker}\partial_{k-\ell}}{\operatorname{Im}\partial_{k-\ell+1}}}{\underset{\shortparallel}{H_{k-\ell}(X;R)}}$.

<u>Verification</u> Let $\sigma:\Delta^k\to X$ and let $\varphi\in C^{\ell}(X;R)$ with $k\geq\ell$.

$$\begin{aligned}\partial\sigma\frown\varphi &= \Big(\textstyle\sum_{i=0}^{k}(-1)^i\,\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_k]}\Big)\frown\varphi\\ &= \textstyle\sum_{i=0}^{\ell}(-1)^i\,\varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]}\big)\,\sigma|_{[v_{\ell+1},\dots,v_k]}\\ &\quad+\textstyle\sum_{i=\ell+1}^{k}(-1)^i\,\varphi\big(\sigma|_{[v_0,\dots,v_\ell]}\big)\,\sigma|_{[v_\ell,\dots,\hat{v}_i,\dots,v_k]}\end{aligned}$$

$$\sigma\frown\delta\varphi = \textstyle\sum_{i=0}^{\ell+1}(-1)^i\,\varphi\big(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]}\big)\,\sigma|_{[v_{\ell+1},\dots,v_k]}$$

$$\partial(\sigma\frown\varphi) = \textstyle\sum_{i=\ell}^{k}(-1)^{i-\ell}\,\varphi\big(\sigma|_{[v_0,\dots,v_\ell]}\big)\,\sigma|_{[v_\ell,\dots,\hat{v}_i,\dots,v_k]}$$

Hence we're done upon observing the $i=\ell+1$ term in $\sigma\frown\delta\varphi$ aligns with the $i=\ell$ term in $\partial(\sigma\frown\varphi)$ after incorporating all signs.

The cap product has relative forms

$$H_k(X,A;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$
$$H_k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X;R).$$

For the first case, the cap product $C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$ restricts to $C_k(A;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(A;R) \subseteq C_{k-\ell}(X;R)$

So there is an induced cap product $\underset{\substack{\parallel \\ C_k(X;R)/C_k(A;R)}}{C_k(X,A;R)} \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X,A;R)$

with the formula for $\partial(\sigma \frown \varphi)$ maintained, and we can pass to (co)homology.

The cap product has relative forms

$$H_k(X,A;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$
$$H_k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X;R).$$

For the second case, recall $C^\ell(X,A;R) := \mathrm{Hom}(\underbrace{C_\ell(X,A)}_{= C_\ell(X)/C_\ell(A)}, R)$ can be viewed as the functions $C_\ell(X) \to R$ that vanish on singular ℓ-simplices in A.

As such, the cap product $C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$
restricts to $C_k(X;R) \times C^\ell(X,A;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$
which is zero on $C_k(A;R) \times C^\ell(X,A;R)$.

So there is an induced cap product $\underbrace{C_k(X,A;R)}_{= C_k(X;R)/C_k(A;R)} \times C^\ell(X,A;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$

with the formula for $\partial(\sigma \frown \varphi)$ maintained, and we can pass to (co)homology.

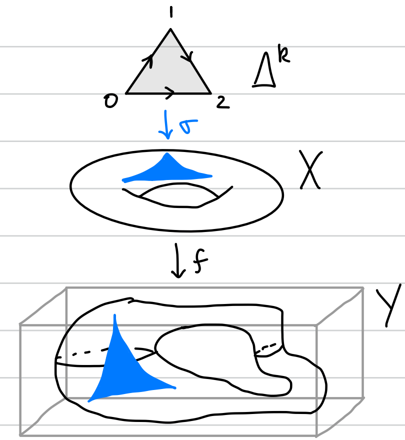
Naturality of cap product

For $f: X \to Y$, the diagram

$$\begin{tikzcd} \underset{\alpha}{\text{⋒}} \\ H_k(X;R) \arrow[d, "f_*"] & \times & H^\ell(X;R) \arrow[r, "\cap"] & H_{k-\ell}(X;R) \arrow[d, "f_*"] \\ H_k(Y;R) & \times & H^\ell(Y;R) \arrow[u, "f^*"'] \arrow[r, "\cap"] & H_{k-\ell}(Y;R) \\ & & \underset{\varphi}{\text{⋓}} \end{tikzcd}$$

satisfies $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$.

(This is the spirit of commutativity when both covariant and contrarariant functors are involved.)



$$\begin{tikzcd} C_\ell(X) \arrow[r, "f_\#"] \arrow[drr, dashed] & C_\ell(Y) \arrow[dr] \\ & & R \end{tikzcd}$$

Pf For $\sigma: \Delta^k \to X$ and $\varphi \in C^\ell(Y;R)$, we have

$$\begin{aligned} f_\#(\sigma) \cap \varphi &= f\sigma \cap \varphi \\ &= \varphi(f\sigma|_{[v_0,\dots,v_\ell]})\, f\sigma|_{[v_\ell,\dots,v_k]} \\ &= (f^\#\varphi)(\sigma|_{[v_0,\dots,v_\ell]})\, f_\#(\sigma|_{[v_\ell,\dots,v_k]}) \\ &= f_\#(\sigma \cap f^\#\varphi). \end{aligned}$$

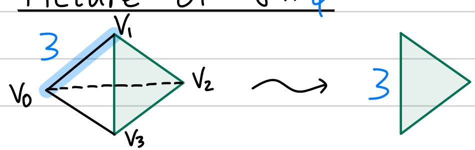
Extend linearly to $C_k(X;R)$, and then pass to (co)homology.

Recall:

Thm 3.30 (Poincaré Duality) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism $\forall k$.

For $k \geq \ell$, define the R-bilinear map $C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$ by $\sigma \frown \varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma|_{[v_\ell,\dots,v_k]}$ for $\sigma: \Delta^k \to X$ a singular simplex.

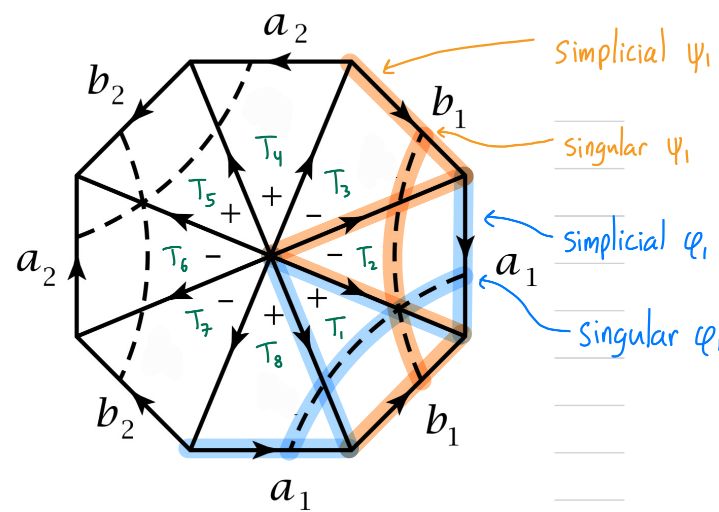
Picture of $\sigma \frown \varphi$



The formula $\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$ then gives a cap product $H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$.

Ex 3.31 Surfaces. Let $M = M_g$ be an orientable surface of genus $g \geq 1$.

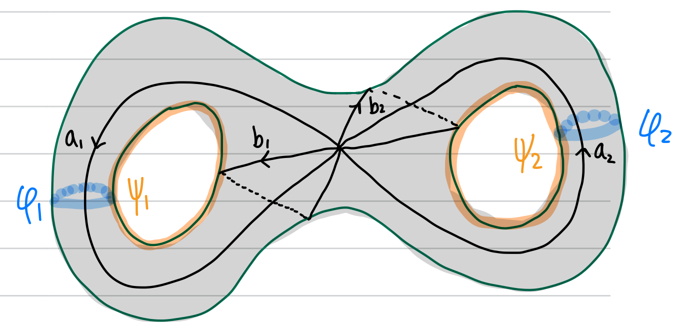
$H_1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ gen. $a_1, b_1, \ldots, a_g, b_g$ $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ gen. $\varphi_1, \psi_1, \ldots, \varphi_g, \psi_g$



Fundamental class $[M] \in H_2(M_g;\mathbb{Z})$ is generated by $\sum_{j=1}^{8} \pm T_j$ (signs shown).

$$
\begin{aligned}
[M] \frown \varphi_1 &= \textstyle\sum_{j=1}^{8} \pm (T_j \frown \varphi_1) \\
&= \textstyle\sum_{j=1}^{8} \pm \varphi_1(T_j|_{[v_0,v_1]})\, T_j|_{[v_1,v_2]} \\
&= \varphi_1(T_1|_{[v_0,v_1]})\, T_1|_{[v_1,v_2]} \quad \text{(other terms zero)} \\
&= b_1
\end{aligned}
$$

So φ_1 is Poincaré dual to b_1, and φ_i to b_i.



$$
\begin{aligned}
[M] \frown \psi_1 &= \textstyle\sum_{j=1}^{8} \pm (T_j \frown \psi_1) \\
&= \textstyle\sum_{j=1}^{8} \pm \psi_1(T_j|_{[v_0,v_1]})\, T_j|_{[v_1,v_2]} \\
&= -\psi_1(T_2|_{[v_0,v_1]})\, T_2|_{[v_1,v_2]} \\
&= -a_1
\end{aligned}
$$

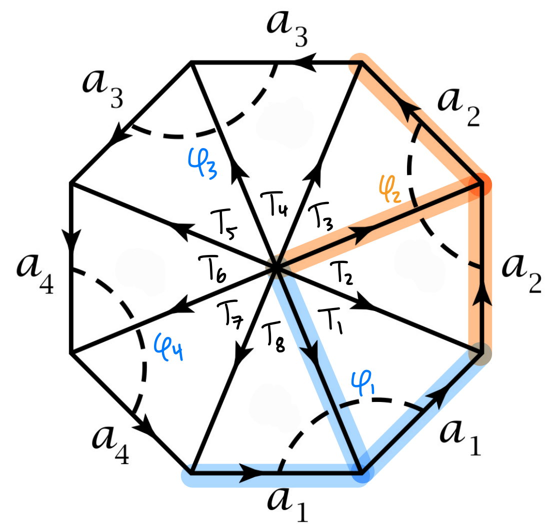
So ψ_1 is Poincaré dual to $-a_1$, and ψ_i to $-a_i$.

Geometrically, note singular "loop" φ_i is homotopic to b_i, and ψ_i to $-a_i$.

N nonorientable surface genus g.

$$H_i(N;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & i=0 \\ (\mathbb{Z}_2)^g & i=1 \quad \text{gen. by } a_1,\dots,a_g \\ \mathbb{Z}_2 & i=2 \quad \text{gen. by } [N] \end{cases}$$

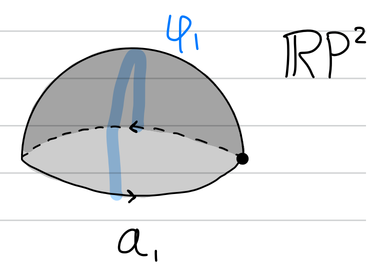
$$H^i(N;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & i=0 \\ (\mathbb{Z}_2)^g & i=1 \quad \text{gen. by } \varphi_1,\dots,\varphi_g \\ \mathbb{Z}_2 & i=2 \end{cases}$$



$$\begin{aligned} [N] \frown \varphi_1 &= \sum_{j=1}^{8} T_j \frown \varphi_1 \\ &= \sum_{j=1}^{8} \varphi_1\left(T_j|_{[v_0,v_1]}\right) T_j|_{[v_1,v_2]} \\ &= \varphi_1\left(T_1|_{[v_0,v_1]}\right) T_1|_{[v_1,v_2]} \\ &= a_1 \end{aligned}$$

So φ_1 is Poincaré dual to a_1, and φ_i to a_i.

Geometrically, note singular "loop" φ_i is homotopic to a_i.



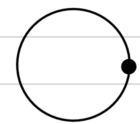
Thm 3.30 (Poincaré Duality) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism $\forall k$.

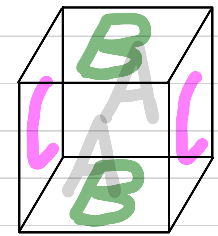
Corollary 3.37 The Euler characteristic of a closed odd-dimensional manifold is $\chi(M) = 0$.

Ex $\chi(S^1) = 1-1=0$ $\quad \chi(S^3) = 1-1=0$ $\quad \chi(S^{2k+1}) = 1-1=0$

$\chi((S^1)^3) = 1-3+3-1=0$ $\quad \chi(\mathbb{R}P^3) = 1-1+1-1=0$

$\chi(S^1 \times N) = 0$ for N a closed 2-manifold





This is a striking result! Orientability and Euler characteristic suffice to classify closed connected 2-manifolds. Not so with 3-manifolds.

Pf sketch Let M be a closed n-manifold. M has finitely generated homology by Corollaries A.8, A.9. For M orientable,

$H_k(M) \cong \mathbb{Z}^r \oplus \left(\oplus_{j=1}^{\ell} \mathbb{Z}_{m_j}\right)$

$$\begin{aligned} \operatorname{rank} H_i(M) &= \operatorname{rank} H^{n-i}(M) && \text{by Poincaré duality} \\ &= \operatorname{rank} H_{n-i}(M) && \text{by Universal Coefficient Theorem,} \end{aligned}$$

Corollary 3.3: $H^k \cong (H_k/T_k) \oplus T_{k-1}$

So for n odd, the terms of $\chi(M) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(M)$ cancel in pairs.

For M nonorientable, Poincaré duality gives

$$\dim H_i(M;\mathbb{Z}_2) = \dim H^{n-i}(M;\mathbb{Z}_2)$$
$$= \dim H_{n-i}(M;\mathbb{Z}_2)$$

by the R-module UCT on page 196 of Hatcher, which for $R=\mathbb{Z}_2$ gives

$$0 \longrightarrow \underbrace{\cancel{\mathrm{Ext}_{\mathbb{Z}_2}(H_{k-1}(M;\mathbb{Z}_2),\mathbb{Z}_2)}}_{\text{0 since } \mathbb{Z}_2 \text{ is free over } \mathbb{Z}_2} \longrightarrow H^k(M;\mathbb{Z}_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_2}(H_k(M;\mathbb{Z}_2),\mathbb{Z}_2) \longrightarrow 0$$

hence $H^k(M;\mathbb{Z}_2) \cong H_k(M;\mathbb{Z}_2)$

So, the terms of $\sum_{i=0}^n (-1)^i \dim H_i(M;\mathbb{Z}_2) = 0$ cancel in pairs.

We show $\chi(M) = \sum_{i=0}^n (-1)^i \operatorname{rank} H_i(M) = \sum_{i=0}^n (-1)^i \dim H_i(M;\mathbb{Z}_2)$.

Use the standard UCT in Thm 3.2:

$$0 \longrightarrow \mathrm{Ext}(H_{i-1}(M),\mathbb{Z}_2) \longrightarrow H^i(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_i(M),\mathbb{Z}_2) \longrightarrow 0$$

- Each $\mathbb{Z}$ summand of $H_i(M)$ gives a $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2) \cong H_i(M;\mathbb{Z}_2)$
- Each $\mathbb{Z}_m$ summand of $H_i(M)$ with m even gives $\mathbb{Z}_2$ summands of $H^i(M;\mathbb{Z}_2)$ and $H^{i+1}(M;\mathbb{Z}_2)$ (recall $\mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_2) \cong \frac{\mathbb{Z}_2}{m\mathbb{Z}_2}$) whose contributions to $\sum_{i=0}^n (-1)^i \dim H_i(M;\mathbb{Z}_2)$ cancel.
- Each $\mathbb{Z}_m$ summand of $H_i(M)$ with m odd contributes nothing to $H^*(M;\mathbb{Z}_2)$. $\square$

Connection with cup product

The cap and cup products are related:

$$(\varphi \cup \psi)(\alpha) = \psi(\alpha \cap \varphi) \qquad (*)$$

for $\alpha \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$.

Indeed, for $\sigma : \Delta^{k+\ell} \to R$ we have

$$\begin{aligned}\psi(\sigma \cap \varphi) &= \psi\big(\varphi(\sigma|_{[v_0,\dots,v_k]})\, \sigma|_{[v_k,\dots,v_{k+\ell}]}\big)\\ &= \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})\\ &= (\varphi \cup \psi)(\sigma).\end{aligned}$$

We have maps

$$C^\ell(X;R) \xrightarrow{\varphi\cup} C^{k+\ell}(X;R)$$
$$C_\ell(X;R) \xleftarrow{\cap\varphi} C_{k+\ell}(X;R).$$

Formula $(*)$ gives that map $\varphi\cup$ is dual to $\cap\varphi$:

$$\begin{tikzcd}
\psi \arrow[r, mapsto] & \varphi\cup\psi \\
C^\ell(X;R) \arrow[r, "\varphi\cup"] \arrow[d, equal] & C^{k+\ell}(X;R) \arrow[d, equal] \\
\mathrm{Hom}_R(C_\ell(X;R),R) \arrow[r, "(\cap\varphi)^*"'] & \mathrm{Hom}_R(C_{k+\ell}(X;R),R) \\
\psi \arrow[r, mapsto] & (\alpha \mapsto \psi(\alpha\cap\varphi))
\end{tikzcd}$$

Pass to (co)homology. If the h maps are isomorphisms (say R is a field, or $R=\mathbb{Z}$ and $H_*(X,\mathbb{Z})$ is free $\forall i$), then $\varphi\cup$ is dual to $\cap\varphi$.

If furthermore (co)homolgy is finitely generated, so that homology and cohomology determine each other, then the cup and cap product determine each other.

$$\begin{tikzcd}
\mathrm{Ext}_R(H_{\ell-1}(X;R),R) \arrow[d, dashed] & \mathrm{Ext}_R(H_{k+\ell-1}(X;R),R) \arrow[d, dashed] \\
H^\ell(X;R) \arrow[r, "\varphi\cup"] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowright"] & H^{k+\ell}(X;R) \arrow[d, "h"] \\
\mathrm{Hom}_R(H_\ell(X;R),R) \arrow[r, "(\cap\varphi)^*"'] & \mathrm{Hom}_R(H_{k+\ell}(X;R),R)
\end{tikzcd}$$

Let M be a closed R-orientable n-manifold.

We have a cup product pairing $H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R$

$$(\varphi, \psi) \longmapsto (\varphi \cup \psi)[M]$$

Such a bilinear pairing $A \times B \xrightarrow{p} R$ is <u>nonsingular</u> if the associated linear maps

$$A \longrightarrow \mathrm{Hom}_R(B, R) \qquad B \longrightarrow \mathrm{Hom}_R(A, R)$$
$$a \longmapsto (b \longmapsto p(a,b)) \qquad b \longmapsto (a \longmapsto p(a,b))$$

are both isomorphisms.

<u>Prop 3.38</u> The cup product pairing is nonsingular for closed R-orientable manifolds when R is

- a field, or
- $\mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.

$$H^k(M;R) \longrightarrow \mathrm{Hom}_R(H^{n-k}(M;R), R)$$
$$\varphi \longmapsto (\psi \longmapsto (\varphi \cup \psi)[M])$$
$$H^{n-k}(M;R) \longrightarrow \mathrm{Hom}_R(H^k(M;R), R)$$
$$\psi \xmapsto{D^*h} (\varphi \longmapsto (\varphi \cup \psi)[M])$$

<u>Pf</u> For map h from the UCT and $D: H^k \to H_{n-k}$ from Poincaré duality, consider
$\varphi \longmapsto [M] \cap \varphi$

$$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R), R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R), R)$$
$$\psi \qquad (\alpha \longmapsto \psi(\alpha)) \qquad (\varphi \longmapsto \psi([M] \cap \varphi) = (\varphi \cup \psi)[M])$$

For R as specified, h and D^* are isomorphisms, hence so is $\psi \xmapsto{D^*h} (\varphi \mapsto (\varphi \cup \psi)[M])$.
The anticommutativity of the cup product $\varphi \cup \psi = \pm \psi \cup \varphi$ (and swapping k and $n-k$) gives that $\varphi \longmapsto (\psi \mapsto (\varphi \cup \psi)[M])$ is also an isomorphism.

Corollary 3.39

- For M a closed connected orientable n-manifold, $\varphi \in H^k(M;\mathbb{Z})$ generates an infinite cyclic summand $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z})$ with $\varphi \smile \psi$ generating $H^n(M;\mathbb{Z}) \cong H_0(M;\mathbb{Z}) \cong \mathbb{Z}$.
- For M a closed connected n-manifold, $\varphi \in H^k(M;\mathbb{Z}_2)$ is nonzero $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z}_2)$ with $\varphi \smile \psi$ generating $H^n(M;\mathbb{Z}_2) \cong H_0(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Pf

- φ generates a $\mathbb{Z}$ summand of $H^k(M;\mathbb{Z})$

  $H^k(M;\mathbb{Z}) \cong \mathbb{Z}^r \oplus \left(\oplus_{j=1}^{\ell} \mathbb{Z}_{m_j}\right)$

  $\varphi \longleftrightarrow (1,0,\dots,0)$

  $\tau(a_1,\dots,a_{r+\ell}) = a_1$

  $\Leftrightarrow$ $\exists$ homomorphism $\tau : H^k(M;\mathbb{Z}) \to \mathbb{Z}$ with $\tau(\varphi) = \pm 1$

  $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z})$ with $(\varphi \smile \psi)[M] = \pm 1$

  Since ignoring torsion, $H^{n-k}(M;\mathbb{Z}) \cong \mathrm{Hom}(H^k(M;\mathbb{Z}),\mathbb{Z})$ by Prop. 3.38

  $\psi \xrightarrow{\text{surj}} \big(\varphi \mapsto (\varphi \smile \psi)[M]\big) = \tau$

  $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z})$ with $\varphi \smile \psi$ generating $H^n(M;\mathbb{Z}) \cong H_0(M;\mathbb{Z}) \cong \mathbb{Z}$.

  $\varphi \smile \psi \longmapsto [M] \frown (\varphi \smile \psi) \longmapsto (\varphi \smile \psi)[M] = \pm 1$

- $\varphi \neq 0$ in $H^k(M;\mathbb{Z}_2)$

  $\Leftrightarrow$ $\exists$ homomorphism $\tau : H^k(M;\mathbb{Z}_2) \to \mathbb{Z}_2$ with $\tau(\varphi) = \pm 1$

  $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z}_2)$ with $(\varphi \smile \psi)[M] = 1$

  Since $H^{n-k}(M;\mathbb{Z}_2) \cong \mathrm{Hom}(H^k(M;\mathbb{Z}_2),\mathbb{Z}_2)$

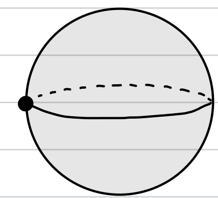
  $\psi \longleftrightarrow \big(\varphi \mapsto (\varphi \smile \psi)[M]\big) = \tau$

  $\Leftrightarrow$ $\exists \psi \in H^{n-k}(M;\mathbb{Z}_2)$ with $\varphi \smile \psi$ generating $H^n(M;\mathbb{Z}_2) \cong H_0(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

  $\varphi \smile \psi \longmapsto [M] \frown (\varphi \smile \psi) \longmapsto (\varphi \smile \psi)[M] = 1$

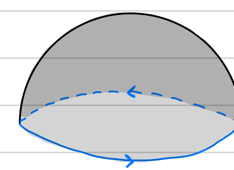
Example 3.40 Projective spaces

Let's deduce $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha|=2$.



$\mathbb{C}P^n$ is orientable (all complex manifolds are).
Inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces an iso on H^i for $i \le 2n-2$,
so by induction on n, $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ is generated by α^i for $i<n$.
By Corollary 3.39, $\exists m \in \mathbb{Z}$ such that $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$.
This means $m=\pm 1$ (else α^n is not generated), so $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$.

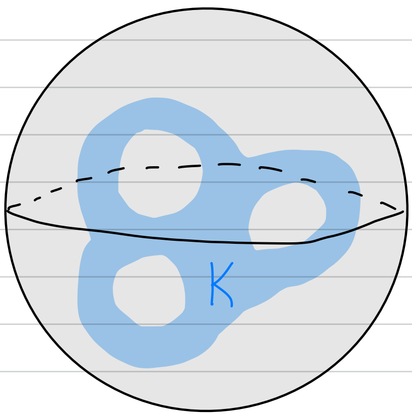
Let's deduce $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha|=1$.

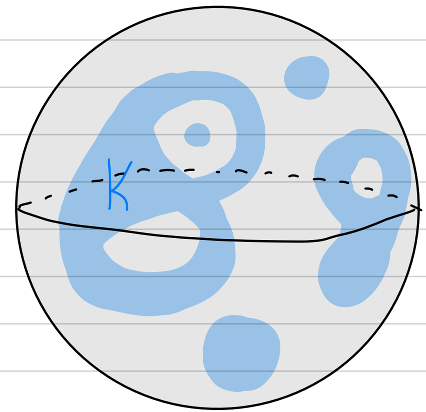


Inclusion $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$ induces an iso on $H^i(-;\mathbb{Z}_2)$ for $i \le n-1$
since the n-cell is attached wrapping around twice.
So by induction on n, $H^i(\mathbb{R}P^n;\mathbb{Z}_2)$ is generated by α^i for $i<n$.
By Corollary 3.39, $\alpha \neq 0$ implies $\alpha \smile \alpha^{n-1} = \alpha^n$ generates $H^n(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$.
So $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha|=1$.

Alexander duality

Corollary 3.45 Let K be a compact, locally contractible, nonempty, proper subset of S^n.
Then $\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$.





$\tilde{H}_0(S^2 - K; \mathbb{Z}) \cong \mathbb{Z}^3 \cong \tilde{H}^1(K; \mathbb{Z})$

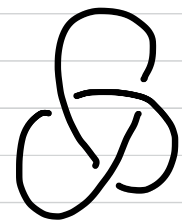
$\tilde{H}_1(S^2 - K; \mathbb{Z}) \cong 0 \cong \tilde{H}^0(K; \mathbb{Z})$

$\tilde{H}_0(S^2 - K; \mathbb{Z}) \cong \mathbb{Z}^3 \cong \tilde{H}^1(K; \mathbb{Z})$

$\tilde{H}_1(S^2 - K; \mathbb{Z}) \cong \mathbb{Z}^4 \cong \tilde{H}^0(K; \mathbb{Z})$

Alexander duality

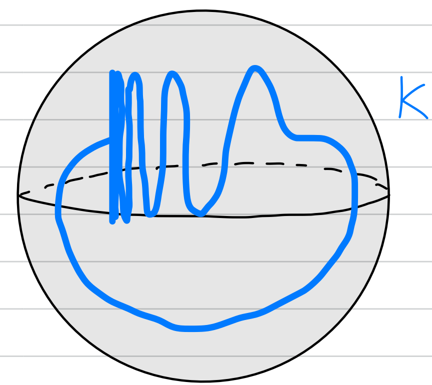
Corollary 3.45 Let K be a compact, locally contractible, nonempty, proper subset of S^n. Then
$$\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z}).$$



Rmk The homology of $S^n - K$ does not depend on how K is embedded in S^n!

Interesting already for $n=3$, $K = S^1$ (knot theory).

Rmk Local contractibility is needed.
Consider $n=2$ and K the quasi-circle.



$$\tilde{H}_0(S^n - K; \mathbb{Z}) \cong \mathbb{Z} \ncong 0 \cong \tilde{H}^1(K; \mathbb{Z}).$$