Persistent equivariant cohomology

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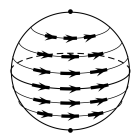
Early reference: Maia Fraser, arXiv 2015
Loring Tu: "What is equivariant cohomology?"

Setup

G a topological group
Filtration of G-spaces
$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_{k-1} \hookrightarrow Y_k$$
Apply G-equivariant cohomology
$$H^*_G(Y_1) \leftarrow H^*_G(Y_2) \leftarrow \cdots \leftarrow H^*_G(Y_{k-1}) \leftarrow H^*_G(Y_k)$$

Ex X a G-metric space
$$VR^m(X;r_1) \hookrightarrow VR^m(X;r_2) \hookrightarrow \cdots \hookrightarrow VR^m(X;r_k) \quad (\star)$$

Borel equivariant cohomology



$G \curvearrowright Y$.
When action is free, $H^*_G(Y) = H^*(Y/G)$.
More generally, let EG be a contractible space on which G acts freely.
The diagonal action of G on $EG \times Y$ is free
Define Borel equivariant cohomology as
$$H^*_G(Y) := H^*\left((EG \times Y)/G\right)$$

Ex

$$H^*_G(pt) = H^*\left((EG \times pt)/G\right) = H^*(EG/G) = H^*(BG).$$

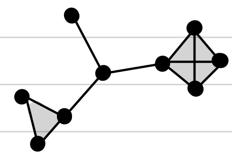
BG is called a classifying space since every principal G-bundle is a pullback of
$$\begin{array}{c} G \to EG \\ \downarrow \\ BG \end{array}$$

$Y \to pt$ gives $H^*_G(Y) \leftarrow H^*_G(pt) = H^*(BG)$
a module structure over the ring $H^*(BG)$.
This is realized by $(\star)$ when $r_k \geq \operatorname{diam}(X)$.

Vietoris-Rips metric thickenings

X a metric space, $r \geq 0$. Simplicial complex

$VR(X;r) = \{[x_0, \dots, x_k] \mid \mathrm{diam}(\{x_0, \dots, x_k\}) \leq r\}$.



Geometric realization

$|VR(X;r)| = \{\sum \lambda_i x_i \mid \lambda_i \geq 0,\ \sum \lambda_i = 1,\ \mathrm{diam}(\quad) \leq r\}$.

Metric thickening

$VR^m(X;r) = \{\sum \lambda_i \delta_{x_i} \mid \lambda_i \geq 0,\ \sum \lambda_i = 1,\ \mathrm{diam}(\quad) \leq r\}$,

equipped with an optimal transport metric.

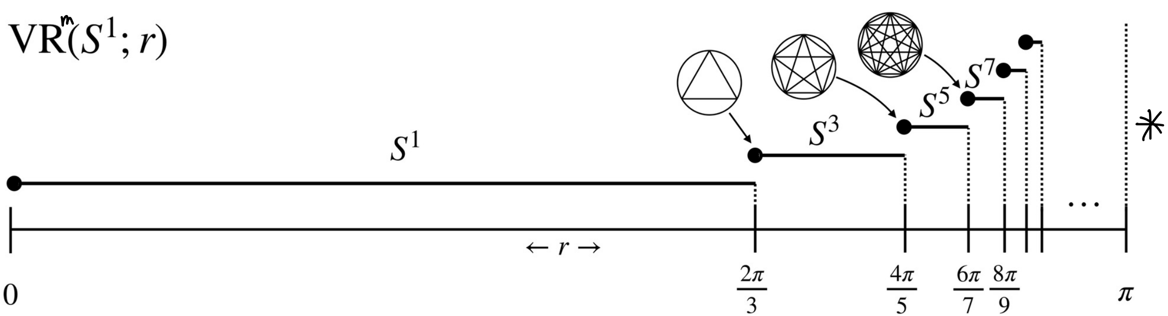
$X \hookrightarrow VR^m(X;r)$ is continuous but $X \hookrightarrow VR(X;r)$ may not be.

Thm

$$VR^m(S^1;r) \simeq \begin{cases} S^{2k+1} & \frac{2\pi k}{2k+1} \leq r < \frac{2\pi(k+1)}{2k+3} \\ * = S^\infty & r \geq \pi \end{cases}$$

Our paper gives an equivariant proof.

Note the fixed points of the S^1 action on $VR^m(S^1;r)$.



$\mathbb{Z} = \frac{\mathbb{Z}[u]}{1 \cdot u} \longleftarrow \frac{\mathbb{Z}[u]}{1 \cdot 3 \cdot u^2} \leftarrow \frac{\mathbb{Z}[u]}{1 \cdot 3 \cdot 5 u^3} \leftarrow \dots \leftarrow \mathbb{Z}[u]$

An incorrect guess

$S^1 \subseteq \mathbb{C}$ acts freely on $S^{2k+1} \subseteq \mathbb{C}^{k+1}$ by multiplication:

$\lambda(z_1, \ldots, z_{k+1}) = (\lambda z_1, \ldots, \lambda z_{k+1})$.

$$S^1 \hookrightarrow S^3 \hookrightarrow S^5 \hookrightarrow S^7 \hookrightarrow \cdots \hookrightarrow S^\infty \simeq *$$

$$ES^1 = S^\infty \qquad BS^1 = S^\infty/S^1 = \mathbb{C}P^\infty \qquad S^{2k+1}/S^1 = \mathbb{C}P^k$$

$$H^*_{S^1}(S^{2k+1}) = H^*(\mathbb{C}P^k) = \begin{cases} \mathbb{Z}[u]/u^{k+1} & k < \infty \\ \mathbb{Z}[u] & k = \infty \end{cases}$$

where $\deg(u) = 2$.

Is this what $H^*_{S^1}(VR^m(S^1; r))$ looks like?

No; the action is not free.

Correct theorem

$$H^*_{S^1}(VR^m(S^1; r)) \cong \begin{cases} \dfrac{\mathbb{Z}[u]}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k+1)\, u^{k+1}} & \text{if } \frac{2\pi k}{2k+1} \leq r < \frac{2\pi(k+1)}{2k+3} \\ \mathbb{Z}[u] & \text{if } r \geq \pi, \end{cases}$$

where $\deg(u) = 2$.

Elements of proof

- Equivariant homotopy equivalence

  $VR^m(S^1; r) \simeq S^1_1 * S^1_3 * S^1_5 * \ldots * S^1_{2k-1}$

  $= S(\mathbb{C}^k_{1,3,\ldots,2k-1})$

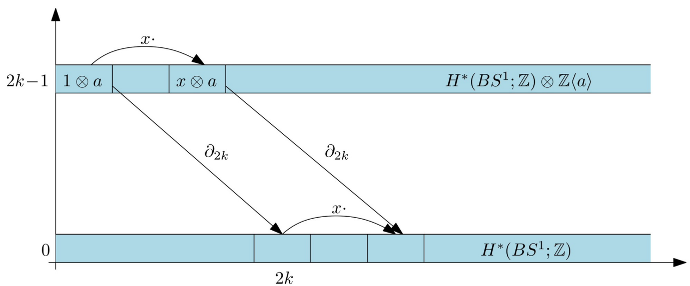
  where $\lambda(z_1, \ldots, z_k) = (\lambda z_1, \lambda^3 z_2, \ldots, \lambda^{2k-1} z_k)$.

- Serre spectral sequence
- Gysin homomorphism (Goodwillie post)

Serre spectral sequence

Complex vector bundle $\quad S(\mathbb{C}^k_{1,3,\dots,2k-1}) \longrightarrow (ES^1 \times S(\mathbb{C}^k_{1,3,\dots,2k-1}))/S^1$

$$\begin{array}{c} (ES^1 \times S(\mathbb{C}^k_{1,3,\dots,2k-1}))/S^1 \\ \downarrow \\ BS^1 \end{array}$$



The differential ∂_{2k} has image $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)\, u^k$ by the Gysin homomorphism, after which the spectral sequence stabilizes.

Hence $H^*_{S^1}(VR^m(S^1;r)) = H^*\left(ES^1 \times S(\mathbb{C}^k_{1,\dots,2k-1})/S^1\right)$ is as described.