Persistent equivariant cohomology

arXiv 2408.17331

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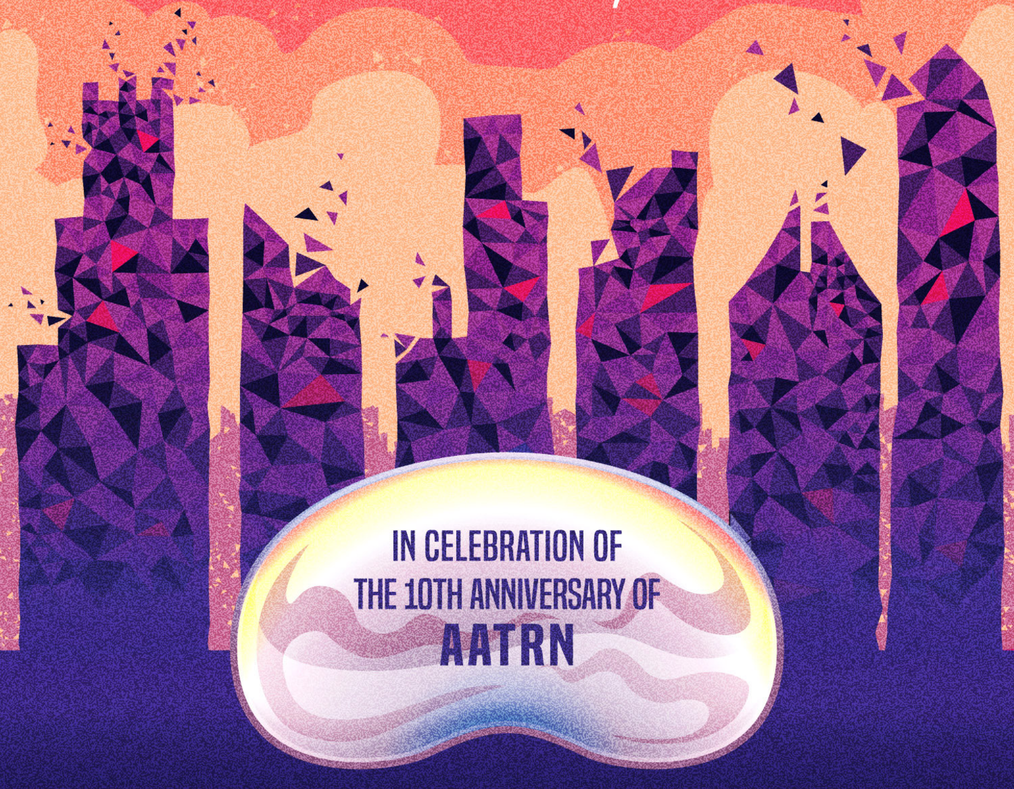
Joint with Evgeniya Lagoda, Michael Moy, Nikola Sadovek, Aditya de Saha

Reference: Maia Fraser, arXiv 2015
Reference: Loring Tu, What is equivariant cohomology?

THE GEOMETRIC REALIZATION OF AATRN

IMSI, CHICAGO, IL

AUG 18 - 22ND, 2025



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AATRN
Applied Algebraic Topology Research Network

iMSI
Institute for Mathematical and Statistical Innovation

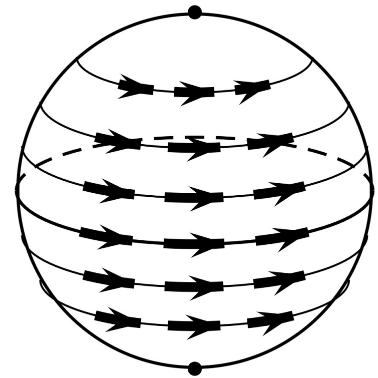
Setup

Filtration of spaces

$Y_1 \hookrightarrow Y_2 \hookrightarrow \dots \hookrightarrow Y_{k-1} \hookrightarrow Y_k$

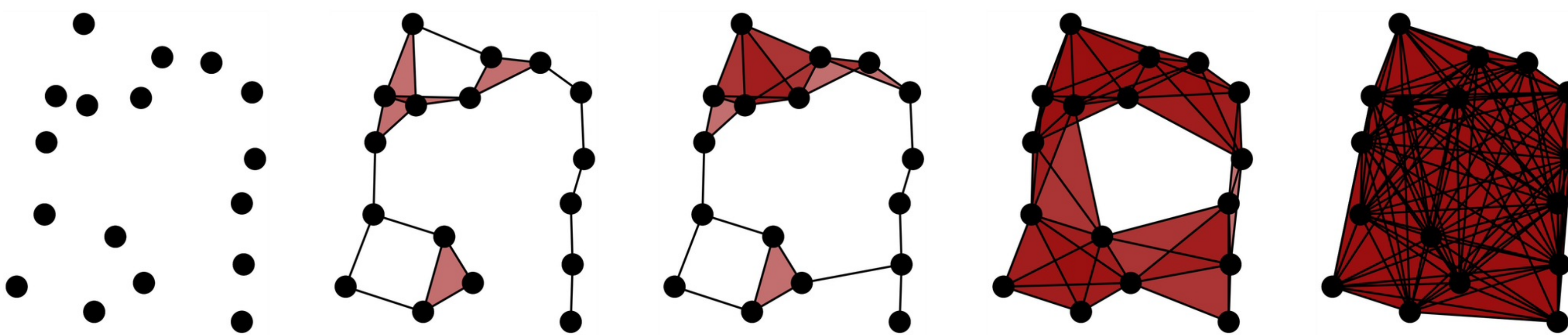
Apply cohomology

$H^*(Y_1) \leftarrow H^*(Y_2) \leftarrow \dots \leftarrow H^*(Y_{k-1}) \leftarrow H^*(Y_k)$



Ex X a metric space

$VR(X; r_1) \hookrightarrow VR(X; r_2) \hookrightarrow \dots \hookrightarrow VR(X; r_k)$



Setup

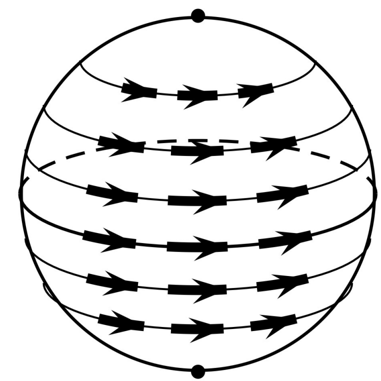
G a topological group

Filtration of G-spaces

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \dots \hookrightarrow Y_{k-1} \hookrightarrow Y_k$$

Apply G-equivariant cohomology

$$H^*_G(Y_1) \leftarrow H^*_G(Y_2) \leftarrow \dots \leftarrow H^*_G(Y_{k-1}) \leftarrow H^*_G(Y_k)$$



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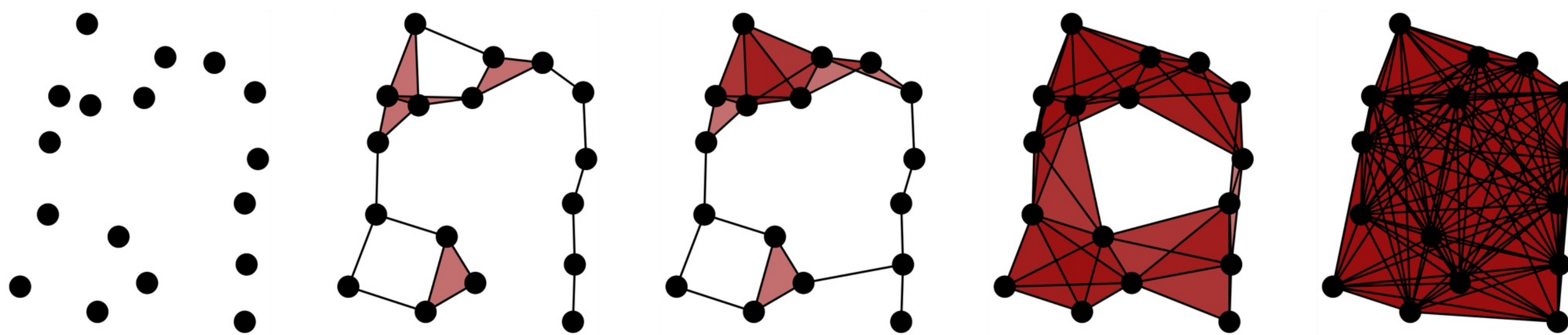

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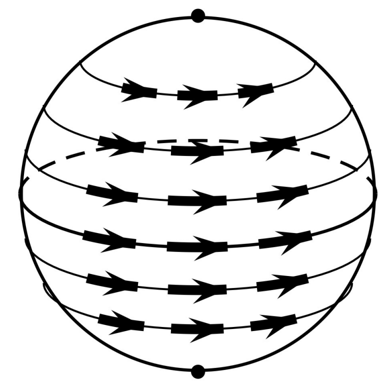

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Borel equivariant cohomology

$G \curvearrowright Y$.

When action is free, $H^*_G(Y) = H^*(Y/G)$.

More generally, let EG be a contractible space on which G acts freely.

The diagonal action of G on $EG \times Y$ is free

Define Borel equivariant cohomology as

$H^*_G(Y) := H^*\left((EG \times Y)/G\right)$

Ex $H^*_G(pt) = H^*\left((EG \times pt)/G\right) = H^*(EG/G) = H^*(BG)$.

BG is called a classifying space since every principal G-bundle is a pullback of:

$$\begin{array}{c} G \\ \downarrow \\ EG \\ \downarrow \\ BG \end{array}$$

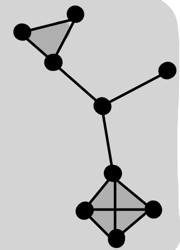
$Y \to pt$ gives $H^*_G(Y) \leftarrow H^*_G(pt) = H^*(BG)$ a module structure over the ring $H^*(BG)$.

Vietoris-Rips simplicial complexes

X a metric space, $r \geq 0$.

The simplicial complex $VR(X;r)$ has

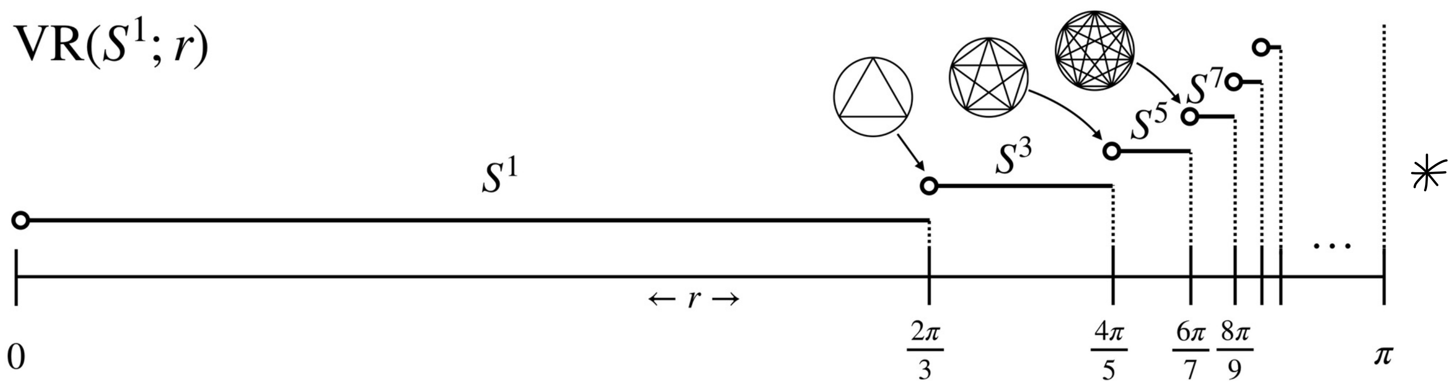
- vertex set X
- all simplices of diameter $< r$.



Theorem (Adamaszek, A. 2017, May 2023)

$$VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \frac{2\pi k}{2k+1} < r \leq \frac{2\pi(k+1)}{2k+3} \\ * & r \geq \pi \end{cases}$$

Our paper gives an equivariant proof.



An incorrect guess

$S^1 \subseteq \mathbb{C}$ acts freely on $S^{2k+1} \subseteq \mathbb{C}^{k+1}$ by multiplication:

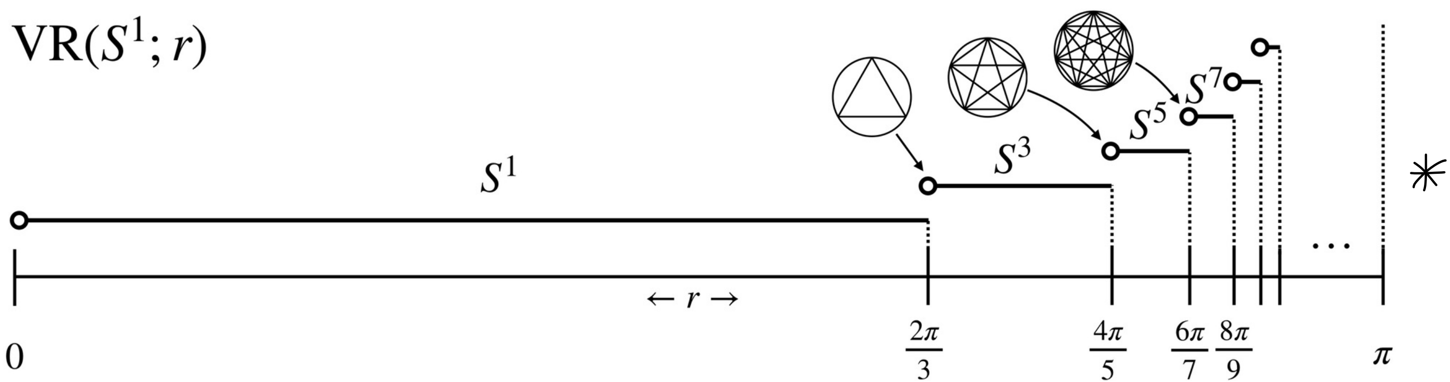
$\lambda(z_1, \ldots, z_{k+1}) = (\lambda z_1, \ldots, \lambda z_{k+1})$.

$$S^1 \hookrightarrow S^3 \hookrightarrow S^5 \hookrightarrow S^7 \hookrightarrow \ldots \hookrightarrow S^\infty \simeq *$$

$$ES^1 = S^\infty \qquad BS^1 = S^\infty/S^1 = \mathbb{C}P^\infty \qquad S^{2k+1}/S^1 = \mathbb{C}P^k$$

$$H^*_{S^1}(S^{2k+1}) = H^*(\mathbb{C}P^k) = \begin{cases} \mathbb{Z}[u]/u^{k+1} & k < \infty \\ \mathbb{Z}[u] & k = \infty \end{cases}$$

where $\deg(u) = 2$.



$$\mathbb{Z} = \frac{\mathbb{Z}[u]}{1 \cdot u} \longleftarrow \frac{\mathbb{Z}[u]}{u^2} \longleftarrow \frac{\mathbb{Z}[u]}{u^3} \longleftarrow \cdots \longleftarrow \mathbb{Z}[u]$$

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where $\deg(u) = 2$.

Correct theorem (A.L.M.S.&S. 2024)

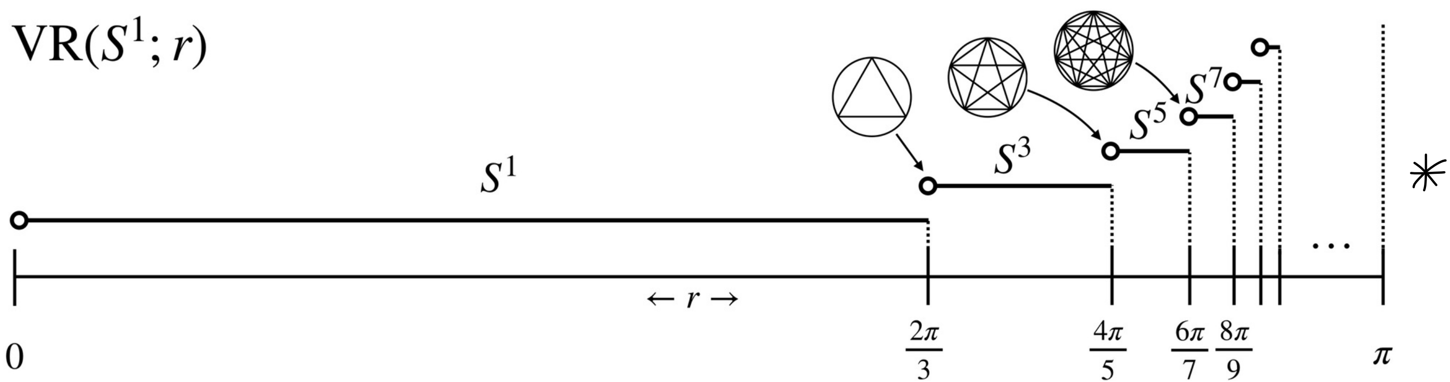
$$H^*_{S^1}(VR(S^1;r)) \cong \begin{cases} \dfrac{\mathbb{Z}[u]}{1\cdot 3\cdot 5\cdot \ldots \cdot(2k+1)u^{k+1}} & \text{if } \frac{2\pi k}{2k+1} \le r < \frac{2\pi(k+1)}{2k+3} \\ \mathbb{Z}[u] & \text{if } r \ge \pi, \end{cases}$$

where $\deg(u) = 2$.

Proof Equivariant homotopy equivalence

$$VR(S^1;r) \simeq S^1_1 * S^1_3 * S^1_5 * \dots * S^1_{2k-1}$$
$$= S(\mathbb{C}^k_{1,3,\dots,2k-1})$$

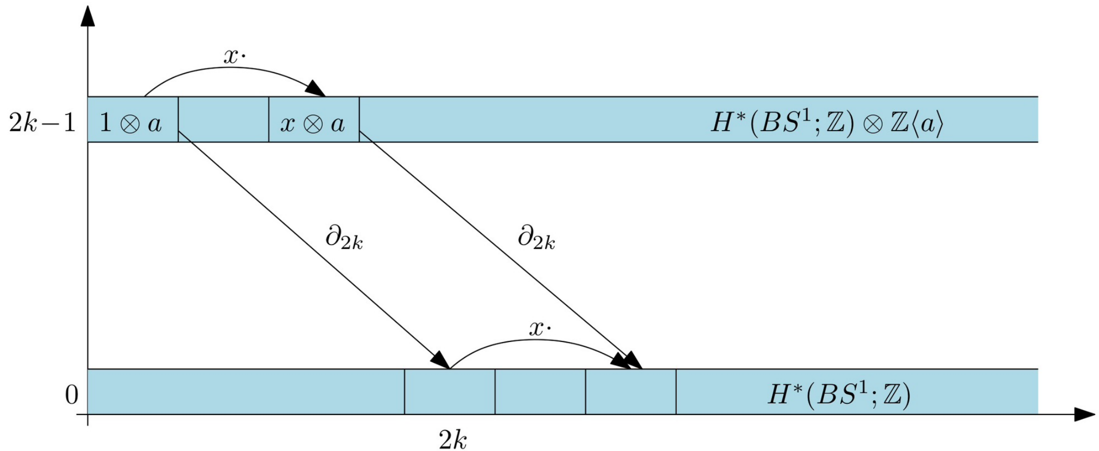
where $\lambda(z_1, \dots, z_k) = (\lambda z_1, \lambda^3 z_2, \dots, \lambda^{2k-1} z_k)$.



$$\mathbb{Z} = \frac{\mathbb{Z}[u]}{1\cdot u} \longleftarrow \frac{\mathbb{Z}[u]}{1\cdot 3\cdot u^2} \longleftarrow \frac{\mathbb{Z}[u]}{1\cdot 3\cdot 5 u^3} \longleftarrow \cdots \longleftarrow \mathbb{Z}[u]$$

Serre spectral sequence

Complex vector bundle $\quad S(\mathbb{C}^k_{1,3,\dots,2k-1}) \longrightarrow \left(ES^1 \times S(\mathbb{C}^k_{1,3,\dots,2k-1})\right)/S^1 \longrightarrow BS^1$



The differential ∂_{2k} has image $1\cdot 3\cdot 5\cdot \ldots \cdot(2k-1)\,u^k$
by the Gysin homomorphism (Goodwillie post),
after which the spectral sequence stabilizes.

Hence $H^*_{S^1}(VR(S^1;r)) = H^*\left(ES^1 \times S(\mathbb{C}^k_{1,\dots,2k-1})/S^1\right)$ is as described. $\square$