Metric Reconstruction via Optimal Transport

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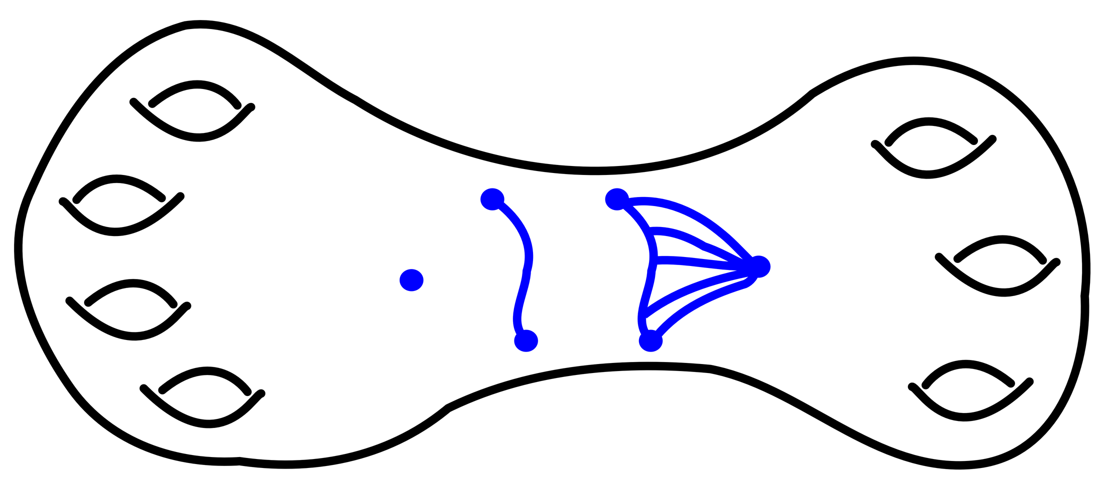
Vietoris–Rips simplicial complexes

Definition. For X a metric space and $r > 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X;r)$ has X as its vertex set, and a finite subset $\sigma \subset X$ as a simplex when $\mathrm{diam}(\sigma) \le r$.

Remark. If X is not discrete then the inclusion $X \hookrightarrow \mathrm{VR}(X;r)$ is not continuous, and if $\mathrm{VR}(X;r)$ is not locally finite then $\mathrm{VR}(X;r)$ is not metrizable.

In [3], Hausmann proves that for M a Riemannian manifold and r sufficiently small, there is a homotopy equivalence $\mathrm{VR}(M;r) \xrightarrow{\simeq} M$. This proof is not as straightforward as one might hope: map $\mathrm{VR}(M;r) \to M$ depends on the choice of a total ordering of all points in M, and the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not a homotopy inverse since it's not continuous.

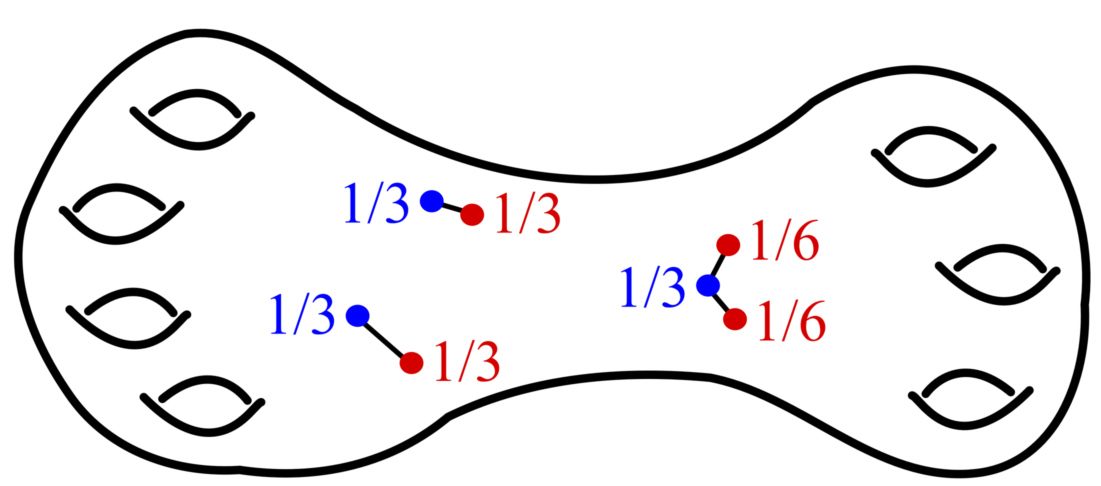


Vietoris–Rips metric thickenings

Definition. For X a metric space and $r > 0$, the *Vietoris–Rips metric thickening* is

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; \lambda_i \ge 0,\ \sum_i \lambda_i = 1,\ x_i \in X,\ \mathrm{diam}(\{x_0,\dots,x_k\}) \le r \right\},$$

equipped with the 1-Wasserstein metric.



It is not hard to check that $\mathrm{VR}^m(X;r)$ is a *metric r-thickening* of X.

Remark. As a set, $\mathrm{VR}^m(X;r)$ is naturally identified with the geometric realization of $\mathrm{VR}(X;r)$. However, these two topological spaces need not be homotopy equivalent.

Main Theorem ([2]). If M is a Riemannian manifold and r is sufficiently small, then $\mathrm{VR}^m(M;r) \simeq M$.

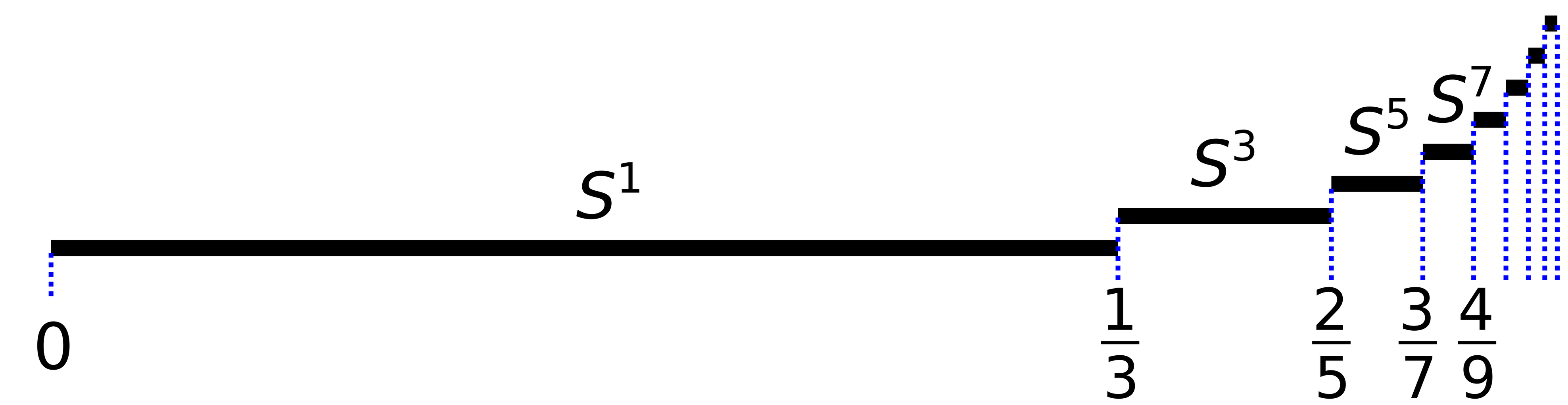
Proof. Consider the (now canonical) continuous map $f\colon \mathrm{VR}^m(M;r) \to M$ defined by sending a point $\sum_i \lambda_i \delta_{x_i}$ to its Fréchet mean in M. This map has the (now continuous) inclusion $\iota\colon M \hookrightarrow \mathrm{VR}^m(M;r)$ as a homotopy inverse. We have $f \circ \iota = \mathrm{id}_M$, and via a linear homotopy we get $\iota \circ f \simeq \mathrm{id}_{\mathrm{VR}^m(M;r)}$. □

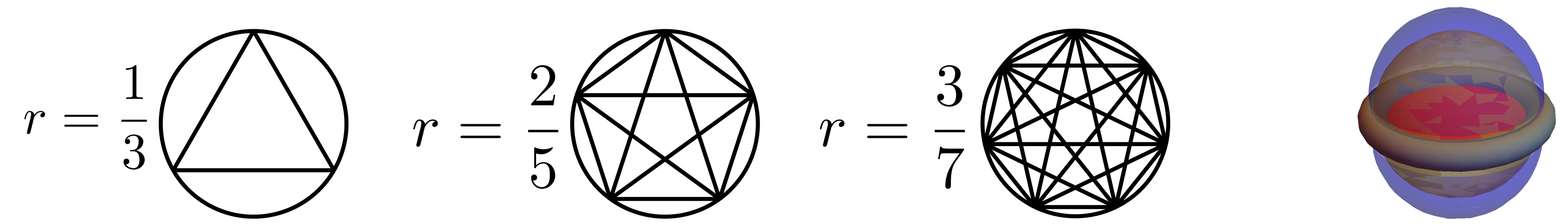
Example differences between $\mathrm{VR}(X;r)$ and $\mathrm{VR}^m(X;r)$

The circle. Let S^1 be the circle of unit circumference with the geodesic metric. We have

$$\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N} \text{ (see [1])}.$$

Though $\mathrm{VR}(S^1;\frac{1}{3}) = \bigvee^{\infty} S^2$, we have $\mathrm{VR}^m(S^1;\frac{1}{3}) \simeq S^3$.





The n-sphere. Let S^n be the n-sphere, and let r_n be the scale parameter when the first inscribed regular $(n+1)$-simplex Δ^{n+1} appears. We have $\mathrm{VR}(S^n;r_n) \simeq \bigvee^{\infty} S^{n+1}$, whereas $\mathrm{VR}^m(S^n;r_n) \simeq \frac{\mathrm{SO}(n+1)}{A_{n+2}}$.

Questions

- Are the homotopy types of $\mathrm{VR}^m(S^n;r)$ related to strongly self-dual polytopes [4]?
- For M a Riemannian manifold, are $\mathrm{conn}(\mathrm{VR}(M;r))$ and $\mathrm{conn}(\mathrm{VR}^m(M;r))$ non-decreasing functions of r [3]?
- Are $\mathrm{VR}_<(X;r)$ and $\mathrm{VR}^m_<(X;r)$ homotopy equivalent?

References

[1] Michał Adamaszek and Henry Adams, *The Vietoris–Rips complexes of a circle*, Pacific Journal of Mathematics, 290:1–40, 2017.

[2] Michał Adamaszek, Henry Adams, and Florian Frick, *Metric reconstruction via optimal transport*, `arXiv:1706.04876`, 2017.

[3] Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.

[4] Lásló Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Scientiarum Mathematicarum 45 (1983), 317–323.