Vietoris-Rips and Čech Complexes of Metric Gluings

(Finite Case)

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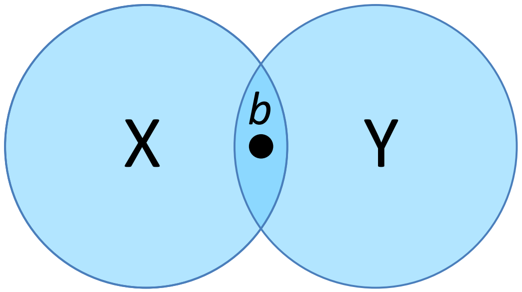
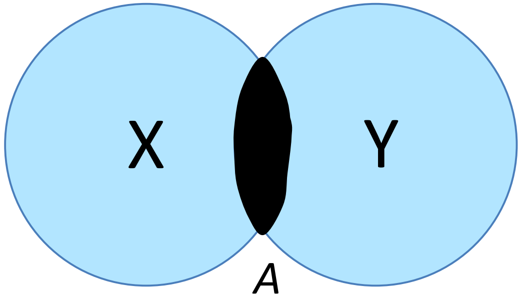
Homotopy results for…

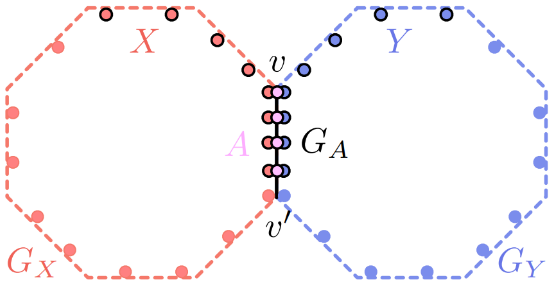
- Wedge sums



- General metric gluing

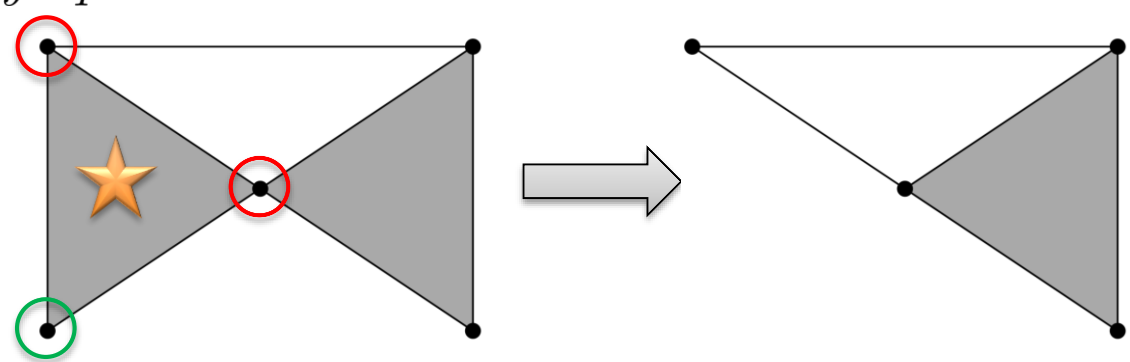


- Metric graphs



Definitions and notation – simplicial complex & collapse

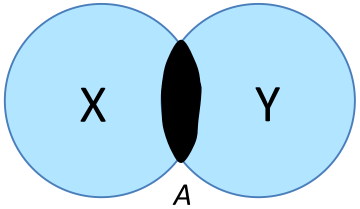
- Simplicial complex $K = \{\sigma\}$ is a set of simplices
  - $\sigma \cup \tau$ is a simplex on the union of the vertex sets (as opposed to a simplicial complex)
- Simplicial collapse:
  - K with maximal simplex σ, $\tau \subseteq \sigma$ is a *free face* of σ if σ is unique maximal coface of τ
  - A (τ, σ) *simplicial collapse* removes all ρ such that $\tau \subseteq \rho \subseteq \sigma$
  - *Elementary* simplicial collapse if $dim(\sigma) = dim(\tau) + 1$
  - Implies *homotopy equivalence*



Definitions and notation – Metric stuff

- Metric space (X, d) and scale parameter $r \geq 0$, Vietoris-Rips complex is

$$VR(X; r) = \{\text{finite } \sigma \subseteq X | diam(\sigma) \leq r\}$$



- Gluing metric spaces (X, d_X) and (Y, d_Y)
  - $A_X \subset X$, $A_Y \subset Y$
  - A a metric space with isometries $\iota_X : A \to A_X$, $\iota_Y : A \to A_Y$
  - $X \cup_A Y$ quotient of $X \sqcup Y$ by equivalence between A_X and A_Y

$$X \cup_A Y = X \sqcup Y / \{\iota_X(a) \sim \iota_Y(a) \text{ for all } a \in A\}$$

  - Metric on $X \cup_A Y$

$$d_{X \cup_A Y}(s, t) = \begin{cases} d_X(s, t) & s, t \in X \\ d_Y(s, t) & s, t \in Y \\ \inf_{a \in A} d_X(s, \iota_X(a)) + d_Y(\iota_Y(a), t) & s \in X, t \in Y \end{cases}$$

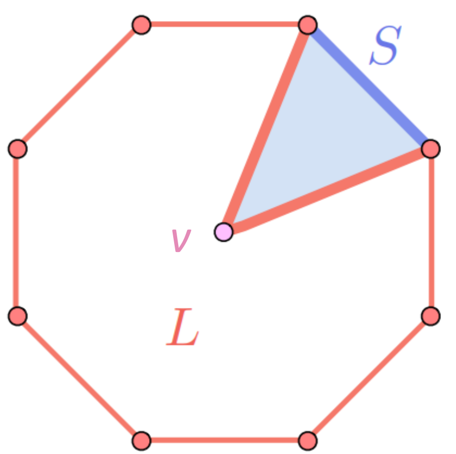
- Persistent homology...

A simple homotopy lemma

Lemma 1 (Barmak, Minian 2008 - Lemma 3.9) *Let:*

- L *be a subcomplex of a finite simplicial complex* K
- T *be a set of simplices* K *which are not in* L
- v *be a vertex of* L *which is contained in no simplex of* T*, but such that* $v \cup S$ *is a simplex of* K *for every* $S \in T$
- $K = L \cup \bigcup_{S \in T} \{S, v \cup S\}$

Then K *is homotopy equivalent to* L *via a sequence of elementary simplicial collapses.*



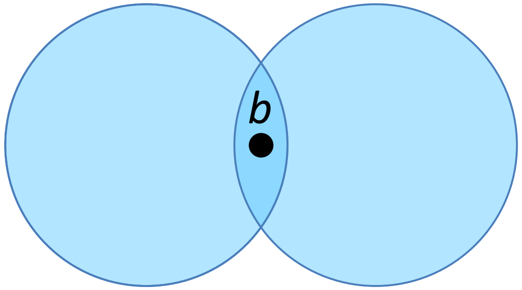
VR complexes of wedge sums

Proposition 1 (AAGGPSWWZ 2018) *For X and Y [finite] pointed metric spaces and $r > 0$ we have the homotopy equivalence*

$$VR(X;r) \vee VR(Y;r) \xrightarrow{\simeq} VR(X \vee Y;r)$$

Proof idea. Apply Lemma 1 with

$$L = VR(X;r) \vee VR(Y;r)$$
$$K = VR(X \vee Y;r)$$
$$T = \{\sigma \in K \setminus L | b \notin \sigma\}$$
$$a = b$$



Corollary 1 *Let X and Y be [finite] pointed metric spaces. For any homological dimension $i \geq 0$ and field k, the persistence modules $PH_i(VR(X;r) \vee VR(Y;r);k)$ and $PH_i(VR(X \vee Y;r);k)$ are isomorphic.*

Analogous statements are true for Čech complexes!

A slightly less simple homotopy lemma

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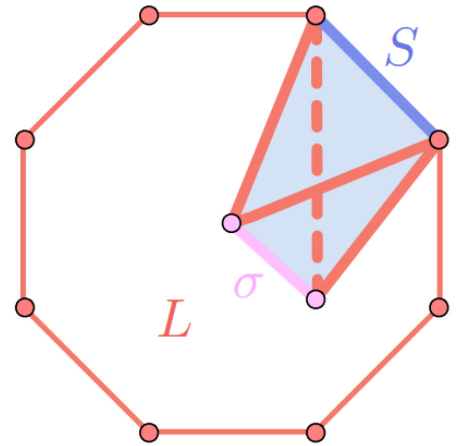
Lemma 2 (AAGGPSWWZ 2018) *Let:*

- L *be a subcomplex of a finite simplicial complex* K
- T *be a set of simplices* K *which are not in* L
- σ *be a simplex of* L *which is disjoint from all simplices of* T*, but such that* $\sigma \cup S$ *is a simplex of* K *for every* $S \in T$
- $K = L \cup \bigcup_{S\in T}\{\tau | S \subseteq \tau \subseteq \sigma \cup S\}$

Then K *is homotopy equivalent to* L *via a sequence of simplicial collapses.*

Proof idea. Order S_i so that $|S_i| \leq |S_{i+1}|$.
Let $K_i = L \cup \bigcup_{j=1}^{i}\{\tau | S_j \subseteq \tau \subseteq \sigma \cup S_j\}$.
Show that S_i is the free face of a simplicial collapse from K_i to K_{i-1}.



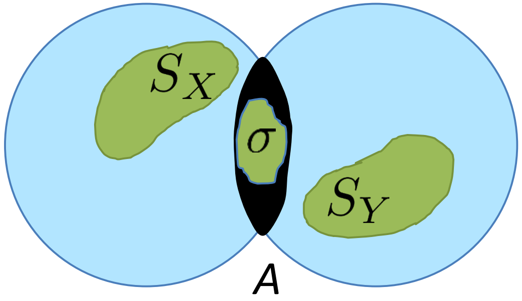
VR complexes of set-wise gluings

Theorem 1 (AAGGPSWWZ 2018) *X, Y metric spaces, A a closed subspace of both with $X \cap Y = A$, $r > 0$. Suppose that whenever $diam(S_X \cup S_Y) \leq r$ for some $\emptyset \neq S_X \subseteq X \setminus A$ and $\emptyset \neq S_Y \subseteq Y \setminus A$, there is a unique maximal nonempty $\sigma \subseteq A$ such that $diam(S_X \cup S_Y \cup \sigma) \leq r$. THEN,*

$$VR(X \cup_A Y; r) \simeq VR(X; r) \cup_{VR(A;r)} VR(Y; r).$$

If $VR(A; r)$ is contractible then

$$VR(X \cup_A Y; r) \simeq VR(X; r) \vee VR(Y; r).$$



VR complexes of set-wise gluings

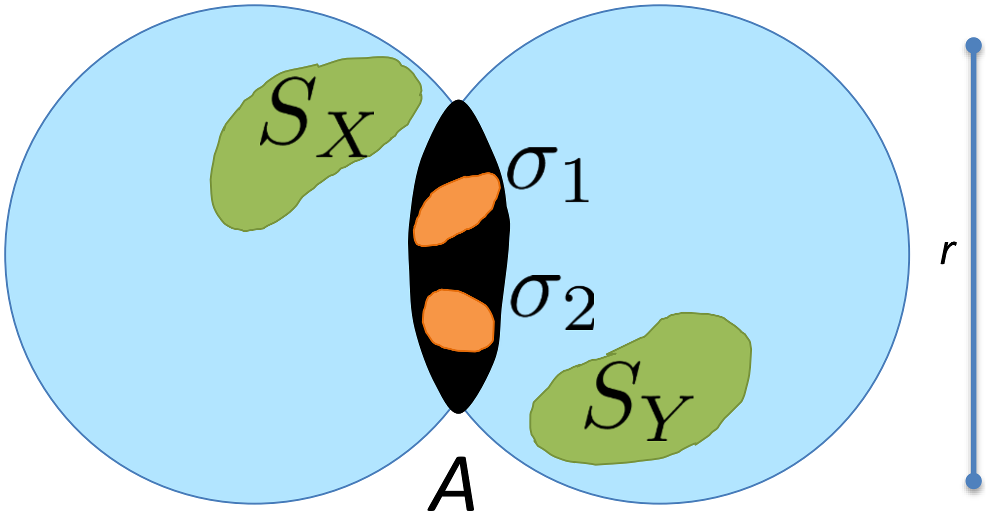
Corollary 2 (AAGGPSWWZ 2018) *X, Y metric spaces, $X \cap Y = A$ a closed subspace of both, $X \cup_A Y$ their metric gluing along A. Let $r > 0$ and suppose $diam(A) \leq r$. Then*

$$VR(X \cup_A Y; r) \simeq VR(X; r) \vee VR(Y; r).$$

Proof idea. If $diam(S_X \cup S_Y) \leq r$ then the set of all $\sigma \subseteq A$ satisfying $diam(S_X \cup S_Y \cup \sigma) \leq r$ is closed under unions (because $diam(A) \leq r$).

Also $\{\sigma\} \neq \emptyset$.

By Theorem 1 on previous slide and $VR(A; r)$ is contractible we're done!

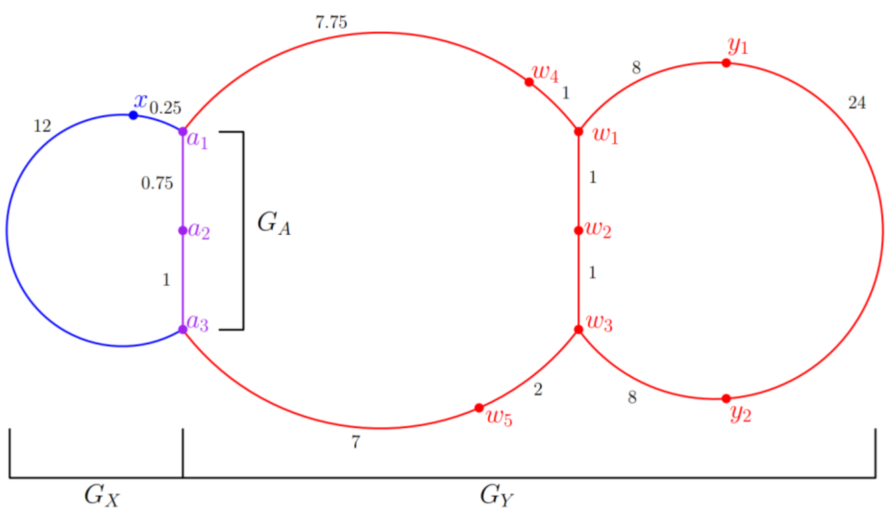


Remark on Čech case for set-wise gluings

- An analogous Theorem 1 can hold true for

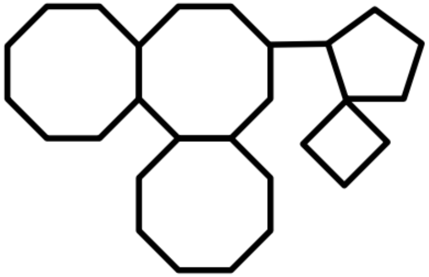
$$\check{\mathrm{C}}\mathrm{ech}(X;r) \cup_{\check{\mathrm{C}}\mathrm{ech}(A;r)} \check{\mathrm{C}}\mathrm{ech}(Y;r) \simeq \check{\mathrm{C}}\mathrm{ech}(X \cup_A Y;r)$$

  by replacing $diam(S_X \cup S_Y \cup \sigma) \leq r$ with $\bigcap_{z \in S_X \cup S_Y \cup \sigma} B(z;r) \neq \emptyset$
- But the argument for Corollary 2 does not hold
  - For a counterexample see the full version of our paper
- Does not mean that Corollary 2 is not true, just that the argument must be different



Metric graphs – Definitions

- *Graph* has vertices $V = \{v_i\}$ and edges $E = \{e_j\} \subseteq V \times V$
- *Metric graph* assigns positive finite length l_j to each edge e_j. AND each point along an edge has a proportional distance to each endpoint
- Natural metric on metric graph G: distance between any two points (not necessarily vertices) is infimum of length of all paths between them



- *Loop* of a metric graph is a continuous map $c : \mathbb{S}^1 \to G$. Also use *loop* to refer to the image of the map.
- *Length* of a loop is the length of its image in G

VR complexes of gluings of metric graphs

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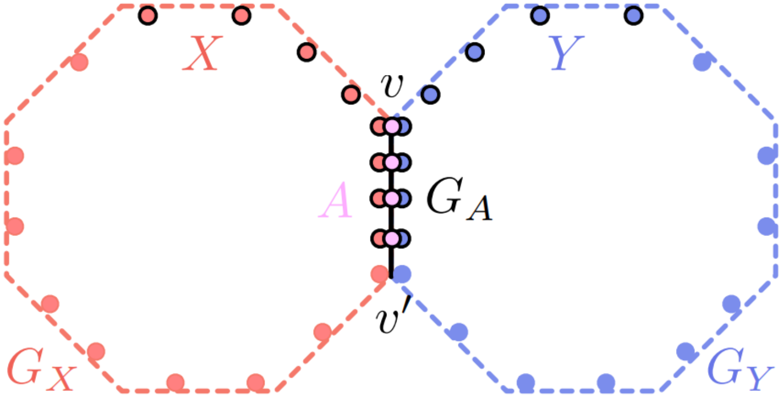
Theorem 2 (AAGGPSWWZ 2018) *Suppose*

- $G = G_X \cup_{G_A} G_Y$ *a metric graph,* $G_A = G_X \cap G_Y$
- G_A *is a path without branching*
- *Any simple loop going through* G_A *has length at least* ℓ
- *Length of* G_A *is at most* $\ell/3$
- $X \subseteq G_X$ *and* $Y \subseteq G_Y$ *with* $X \cap G_Y = Y \cap G_X = X \cap Y = A$

Then $VR(X \cup_A Y; r) \simeq VR(X; r) \cup_{VR(A;r)} VR(Y; r)$ *for all* $r > 0$.

Proof idea. Let the length of G_A be $\alpha \leq \frac{\ell}{3}$.
If $r \geq \alpha$ then we use Corollary 2.

diameter of the gluing region

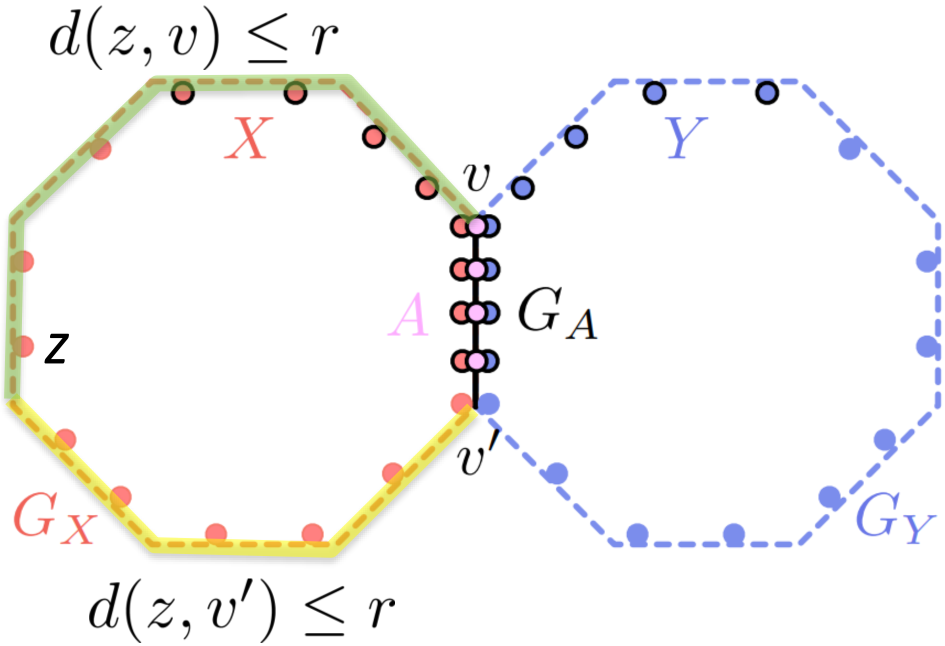


VR complexes of gluings of metric graphs

Proof idea. Let the length of G_A be $\alpha \leq \frac{\ell}{3}$. If $r < \alpha$ we will use Theorem 1. Assume z with distances as in the figure. Then there is a loop of length

$$d(z,v) + d(z,v') + \alpha < 3\alpha \leq \ell.$$

Such a z cannot exist. Use this to imply the condition of Theorem 1 is true.



How does this help?

- We know the homotopy type of the VR complex of a circle (cycle graph) for all *r* values!

Theorem 3 (Adamaszek, Adams 2017) *For* $0 \leq r < \frac{1}{2}$ *we have a homotopy equivalence*

$$VR_{\leq}(S^1; r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, l = 0, 1, \ldots \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{l}{2l+1}, \end{cases}$$

where $\mathfrak{c}$ *is the cardinality of the continuum.*

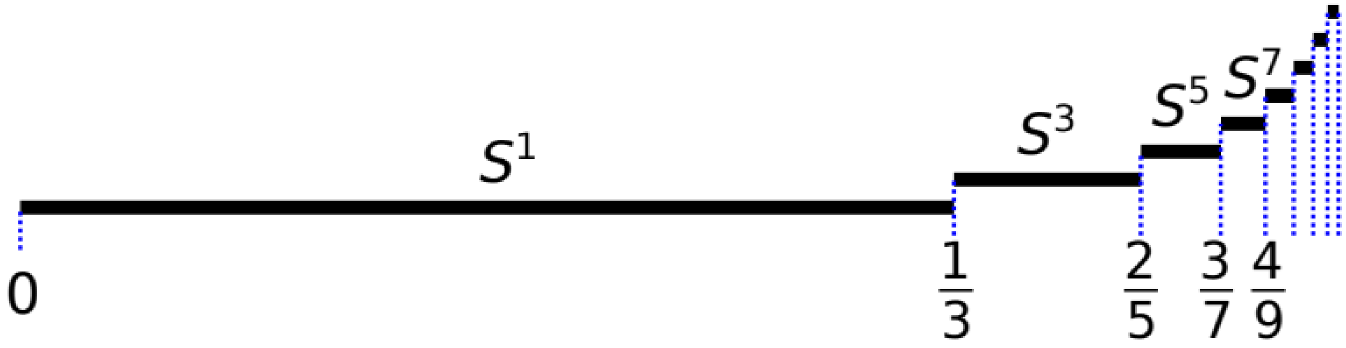
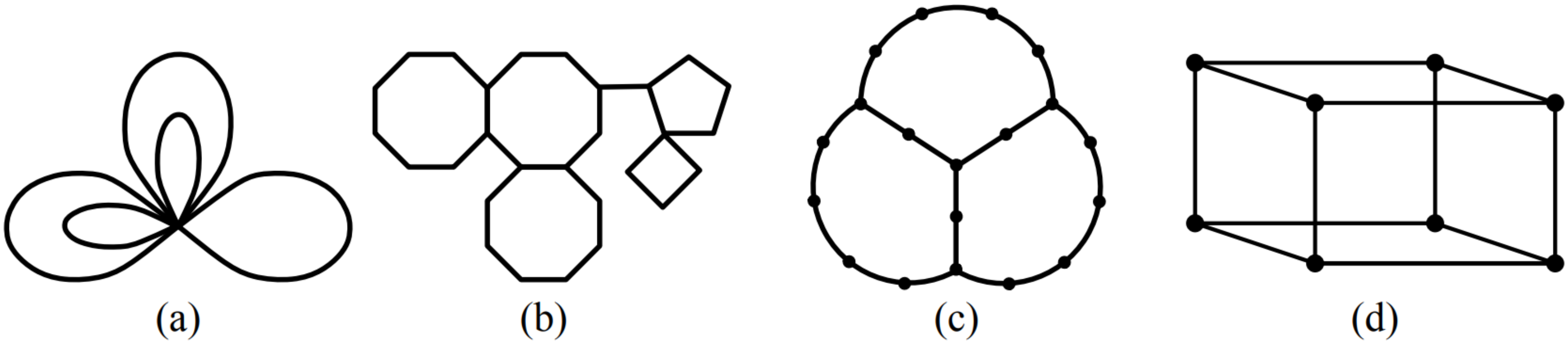


Image from Henry Adams and Michał Adamaszek

Persistence for VR complexes of gluings of metric graphs

Corollary 3 (AAGGPSWWZ 2018) *Let* G, G_X, G_Y, G_A, X, Y, A *be as before. Suppose* $VR(A; r)$ *is contractible for all* $r > 0$. *Then, for any homological dimension* $i \geq 0$ *and field* k, *the persistence modules* $PH_i(VR(X; r) \vee VR(Y; r); k)$ *and* $PH_i(VR(X \vee Y; r); k)$ *are isomorphic.*

Families of graphs



- Graphs we can characterize with our results
  - (a) is a simple wedge sum
  - (b) involves gluing cycles along vertices, single edges (or short paths)
  - More general case of gluing k-cycles and trees
- Graphs we cannot characterize with our results
  - (c) requires gluing along a non-simple path
  - (d) involves “self-gluing”

Future work

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- Using our theoretical results to build computational algorithms that simplify homotopy and homology calculations when metric graphs and metric spaces can be decomposed
- Gluing beyond a simple path
  - Along a tree or self-gluing
- Use to produce a generative model for metric graphs with easily computable homotopy type
  - Specify a gluing rule and randomly glue component graphs together
- Approximations of persistence profiles of graphs using stability results

References

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