

EVASION PATHS IN MOBILE SENSOR NETWORKS

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Abstract

Suppose that ball-shaped sensors wander in a bounded domain. A sensor doesn't know its location but does know when it overlaps a nearby sensor. We say that an evasion path exists in this sensor network if a moving evader can avoid detection. In *Coordinate-free coverage in sensor networks with controlled boundaries via homology*, Vin de Silva and Robert Ghrist give a necessary condition, depending only on the time-varying connectivity graph of the sensors, for an evasion path to exist. Can we sharpen this result? We show that the existence of an evasion path depends not only on the fibrewise homotopy type of the region covered by sensors but also on the ambient isotopy class of its embedding in spacetime. For planar sensors that also measure weak rotation and distance information, we provide necessary and sufficient conditions for the existence of an evasion path. Furthermore, we study the entire space of evasion paths using a homotopy spectral sequence for diagrams of spaces due to Emmanuel Dror Farjoun and Alexander Zabrodsky.

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Chapter 1

Introduction

Topology is finding applications in diverse areas of science and engineering [EH10]. One example is topological data analysis [Car09], in which tools such as persistent homology [ELZ02, ZC05] and Mapper [SMC07] have been used to study data sets arising from image processing [CIdZ08], configuration spaces of molecules [MTCW10], and medical studies [NLC11]. Other applications include the study of nonlinear analysis and dynamical systems [KMM04], robotics [Far06], and sensor networks. In minimal sensor network problems one is given weak local data measured by many sensors and hopes to answer a global question. Tools from topology can be useful for this passage from local to global. For example, [BG09] combines redundant local counts of targets to obtain an accurate global count using integration with respect to Euler characteristic. Coverage problems are another class of problems in minimal sensing [dG06, dG07]: when we scatter sensors throughout a domain, can we determine if the entire domain is covered?

We are interested in the following mobile sensor network coverage problem. Suppose that ball-shaped sensors wander in a bounded domain. A sensor can't measure its location but does know when it overlaps a nearby sensor. We say that an evasion path exists if a moving evader can avoid being detected by the sensors. Can we

determine if an evasion path exists? This is the evasion problem and is the topic of Chapter 3. In [dG06] de Silva and Ghrist give a necessary homological condition for an evasion path to exist. Using zigzag persistence, we provide an equivalent homological condition that moreover can be computed in a streaming fashion. However, it turns out that homology alone is not sufficient. Indeed, neither the fibrewise homotopy type of the sensor network nor any invariants thereof determine if an evasion path exists; we show that the ambient isotopy class of the fibrewise embedding of the sensor network into spacetime also matters. Knowing this, we provide necessary and sufficient conditions for the existence of an evasion path for planar sensors that can also measure weak rotation and distance data.

The evasion problem motivates a natural extension: can we describe the entire space of evasion paths? Knowledge about the space of evasion paths may be helpful in determining how to best patch a sensor network that contains an evasion path. Alternatively, we may want to find the evasion path that maintains the largest separation between the evader and the sensors, that requires an evader to move the shortest distance, or that requires an evader to move at the lowest top speed. In Chapter 4 we develop tools from homotopy theory necessary for studying the space of evasion paths. In Chapter 5 we consider a generalization of the homotopy spectral sequence of a space to diagrams of spaces [DZ87]. For such a diagram modeling the uncovered region of a sensor network, this spectral sequence has input depending on unstable invariants of the uncovered region and under favorable circumstances converges to information about the space of evasion paths.

In Chapter 2 we provide background material on quivers and fibrewise spaces. We discuss the evasion problem in Chapter 3. We study diagrams of spaces in Chapter 4, in preparation for a homotopy spectral sequence for diagrams of spaces in Chapter 5. We conclude in Chapter 6 and describe directions for possible future work.

Chapter 2

Quivers and Fibrewise Spaces

The material in this chapter will be used to study the evasion problem in Chapter 3. In Section 2.1 we introduce quivers of type A_n and functor categories over quivers. The particular case when the functors map to vector spaces, called quiver theory or zigzag persistence, is described in Section 2.2. In Section 2.3 we show how to model a fibrewise space by a zigzag diagram of spaces, and we relate the existence of a section in a fibrewise space to the homology of its corresponding zigzag diagram.

2.1 Quivers of type A_n and functor categories

A quiver of type A_n is a directed graph with n vertices and $n - 1$ arrows

$$\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \bullet_3 \leftrightarrow \dots \leftrightarrow \bullet_{n-1} \leftrightarrow \bullet_n,$$

where each arrow points either to the left or to the right [DW05]. A path is a sequence of arrows $a_k a_{k-1} \dots a_2 a_1$ where the head of arrow a_i is the tail of arrow a_{i+1} . Fix a quiver of type A_n and let D be the free category it generates [Mac98]. The objects of D are the vertices of the quiver and the morphisms of D are the paths. Though

most of the results hold more generally, in this thesis we restrict attention to indexing categories generated by quivers of type A_n .

Let \mathcal{C} be a category. Since D is a small category we can form the functor category \mathcal{C}^D whose objects are functors $D \rightarrow \mathcal{C}$ and whose morphisms are natural transformations. We depict $C \in \mathcal{C}^D$ by the diagram

$$C_1 \leftrightarrow C_2 \leftrightarrow \dots \leftrightarrow C_n.$$

If there is a morphism in D from object j to i , then we let $p_{i,j}^C$ be the morphism from C_j to C_i . A morphism $f: C \rightarrow C'$ in \mathcal{C}^D is a diagram

$$\begin{array}{ccccccc} C_1 & \longleftarrow & C_2 & \longleftarrow & \dots & \longleftarrow & C_n \\ f_1 \downarrow & & f_2 \downarrow & & & & f_n \downarrow \\ C'_1 & \longleftarrow & C'_2 & \longleftarrow & \dots & \longleftarrow & C'_n \end{array}$$

where all of the squares commute.

2.2 Quiver representations and zigzag persistence

Let D be the free category generated by a quiver of type A_n , and let Vect_k be the category of vector spaces over a field k . To study the category Vect_k^D we use the path algebra kD , which is the k -algebra with basis the morphisms in D and with multiplication given by composition. More explicitly, let p and q be two morphisms in D . If the target of q is the source of p then path pq is the composition of the morphisms, and otherwise pq is defined to be zero. Let $kD\text{-Mod}$ be the category of left kD -modules.

Theorem 1. *The categories Vect_k^D and $kD\text{-Mod}$ are equivalent.*

This theorem can be found in [CB92, GR92] and holds more generally for finite quivers of arbitrary type. The categories Vect_k^D and $kD\text{-Mod}$ are abelian.

Let FVect_k be the category of finite-dimensional vector spaces. Then FVect_k^D is the category of quiver representations [Gab72, DW05]. A quiver representation V in FVect_k^D is a diagram

$$V_1 \leftrightarrow V_2 \leftrightarrow \dots \leftrightarrow V_n$$

of finite-dimensional vector spaces and linear maps. The direct sum of two quiver representations is given by $(V \oplus W)_i = V_i \oplus W_i$ and $p_{i,j}^{V \oplus W} = p_{i,j}^V \oplus p_{i,j}^W$. For birth and death indices $1 \leq b \leq d \leq n$, the interval representation $\mathbb{I}(b, d)$ is defined by

$$\mathbb{I}(b, d)_i = \begin{cases} k & \text{if } b \leq i \leq d \\ 0 & \text{otherwise,} \end{cases}$$

and for $j \rightarrow i$ in D ,

$$p_{i,j}^{\mathbb{I}(b,d)} = \begin{cases} \mathbb{1} & \text{if } b \leq j, i \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathbb{1}$ is the identity map on field k .

The study of FVect_k^D for D the free category generated by a quiver of type A_n is also called zigzag persistence [Cd10]. Zigzag persistence is a generalization of persistent homology [ELZ02, ZC05] in which the maps can go in either direction. As in persistent homology, a zigzag module is described up to isomorphism by its barcode decomposition [Gab72, Cd10].

Theorem 2 (Gabriel). *For D the free category generated by a quiver D of type A_n , a quiver representation $V \in \text{FVect}_k^D$ can be decomposed as*

$$V \cong \bigoplus_{i=1}^N \mathbb{I}(b_i, d_i)$$

where the factors in the decomposition are unique up to reordering.

A barcode is a multiset of intervals of the form $[b, d]$, and the barcode for a quiver representation $V \cong \bigoplus_{i=1}^N \mathbb{I}(b_i, d_i)$ is $\{[b_1, d_1], [b_2, d_2], \dots, [b_N, d_N]\}$.

2.3 Fibrewise spaces and diagrams of spaces

Let $I = [0, 1]$ be the closed unit interval. A fibrewise space is a topological space Y equipped with a map $Y \xrightarrow{p} I$. A map $f: Y \rightarrow W$ between two fibrewise spaces $Y \xrightarrow{p} I$ and $W \xrightarrow{p'} I$ is fibrewise if $p'f = p$. A section for a fibrewise space $Y \xrightarrow{p} I$ is a fibrewise map $s: I \rightarrow Y$. More explicitly, we need the composition ps to be the identity map on interval I . The space of all sections of Y is topologized as a subset of the space of all undirected paths in Y .

Two fibrewise maps $f_0, f_1: Y \rightarrow W$ are fibrewise homotopic if there is a homotopy $F: Y \times I \rightarrow W$ from f_0 to f_1 with each $F(\cdot, t)$ a fibrewise map. A fibrewise map $f: Y \rightarrow W$ is a fibrewise homotopy equivalence if there is a fibrewise map $f': W \rightarrow Y$ with compositions $f'f$ and ff' each fibrewise homotopic to the corresponding identity map; in this case we say that spaces Y and W are fibrewise homotopy equivalent. See [CJ98] for more information on fibrewise homotopy theory.

Given a discretization

$$0 = s_0 < \dots < s_n = 1$$

of interval I we build a zigzag diagram $ZY \in \text{Top}^D$ that models fibrewise space Y [CdM09]. Let $Y_i = p^{-1}(s_i)$ and let $Y_i^{i+1} = p^{-1}([s_i, s_{i+1}])$. Then ZY is the diagram

$$Y_0 \hookrightarrow Y_0^1 \hookleftarrow Y_1 \hookrightarrow \dots \hookleftarrow Y_{n-1} \hookrightarrow Y_{n-1}^n \hookleftarrow Y_n$$

of topological spaces and inclusion maps, and indexing category D is the free category

generated by the quiver below.

$$\bullet_1 \rightarrow \bullet_2 \leftarrow \bullet_3 \rightarrow \dots \leftarrow \bullet_{2n-1} \rightarrow \bullet_{2n} \leftarrow \bullet_{2n+1}$$

See Figure 2.1 for an example.

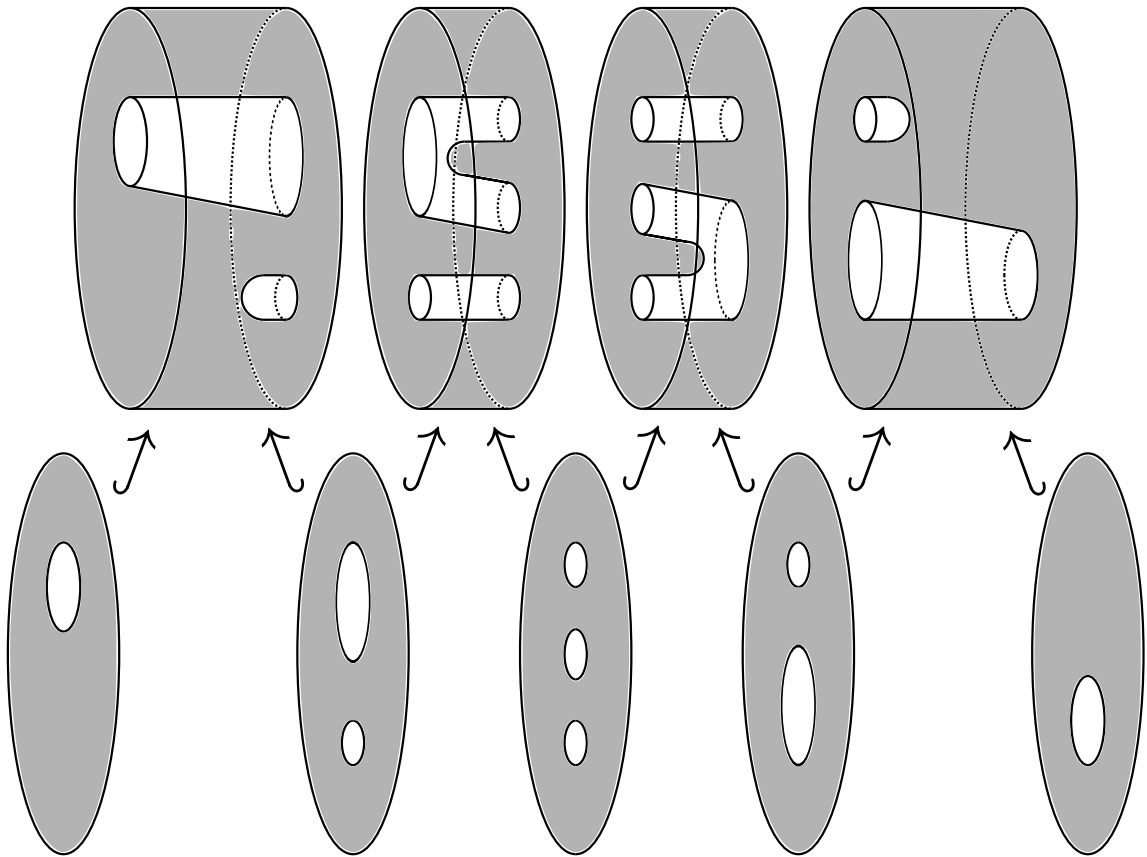


Figure 2.1: The zigzag diagram ZY for a fibrewise space Y with map $Y \rightarrow I$ given by projection onto the horizontal axis.

We now assume that each Y_i and Y_i^{i+1} have finite-dimensional homology and cohomology, each taken with coefficients in a field k . Applying the j -dimensional

homology functor H_j gives the diagram

$$H_j(Y_0) \rightarrow H_j(Y_0^1) \leftarrow H_j(Y_1) \rightarrow \dots \leftarrow H_j(Y_{n-1}) \rightarrow H_j(Y_{n-1}^n) \leftarrow H_j(Y_n),$$

denoted $H_j(ZY) \in \text{FVect}_k^D$. Applying the j -dimensional cohomology functor H^j gives the diagram

$$H^j(Y_0) \leftarrow H^j(Y_0^1) \rightarrow H^j(Y_1) \leftarrow \dots \rightarrow H^j(Y_{n-1}) \leftarrow H^j(Y_{n-1}^n) \rightarrow H^j(Y_n),$$

denoted $H^j(ZY) \in \text{FVect}_k^{D^{op}}$. Note D has been replaced with D^{op} because cohomology is contravariant.

The following lemmas will be useful in the proof of Theorem 3.

Lemma 1. *If Y and W are fibrewise homotopy equivalent then $H_j(ZY) \cong H_j(ZW)$ and $H^j(ZY) \cong H^j(ZW)$.*

Proof. Let $f: Y \rightarrow W$ be a fibrewise homotopy equivalence. This induces the commutative diagram

$$\begin{array}{ccccccccccc} Y_0 & \hookrightarrow & Y_0^1 & \longleftarrow & Y_1 & \hookrightarrow & \dots & \longleftarrow & Y_{n-1} & \hookrightarrow & Y_{n-1}^n & \longleftarrow & Y_n \\ f_0 \downarrow & & f_0^1 \downarrow & & f_1 \downarrow & & & & f_{n-1} \downarrow & & f_{n-1}^n \downarrow & & f_n \downarrow \\ W_0 & \hookrightarrow & W_0^1 & \longleftarrow & W_1 & \hookrightarrow & \dots & \longleftarrow & W_{n-1} & \hookrightarrow & W_{n-1}^n & \longleftarrow & W_n \end{array}$$

where each map f_i or f_i^{i+1} is defined via restriction and is a homotopy equivalence.

Since homology is a homotopy invariant, applying H_j gives the commutative diagram

$$\begin{array}{ccccccccccc} H_j(Y_0) & \longrightarrow & H_j(Y_0^1) & \longleftarrow & H_j(Y_1) & \longrightarrow & \dots & \longleftarrow & H_j(Y_{n-1}) & \longrightarrow & H_j(Y_{n-1}^n) & \longleftarrow & H_j(Y_n) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ H_j(W_0) & \longrightarrow & H_j(W_0^1) & \longleftarrow & H_j(W_1) & \longrightarrow & \dots & \longleftarrow & H_j(W_{n-1}) & \longrightarrow & H_j(W_{n-1}^n) & \longleftarrow & H_j(W_n) \end{array}$$

where each vertical map is an isomorphism. Hence $H_j(ZY) \cong H_j(ZW)$ in FVect_k^D . The proof for cohomology is analogous. \square

Lemma 2. *The barcodes for $H_j(ZY)$ and $H^j(ZY)$ are equal as multisets of intervals.*

Proof. The version of this lemma with persistent homology instead of zigzag persistence is given in Proposition 2.3 of [dMVJ11], and our proof is analogous. The quiver representations $H_j(ZY) \in \text{FVect}_k^D$ and $H^j(ZY) \in \text{FVect}_k^{D^{op}}$ live in different categories and hence cannot be isomorphic. However, consider the decomposition

$$H_j(ZY) \cong \bigoplus_{l=1}^N \mathbb{I}(b_l, d_l)$$

from Theorem 2. Applying the contravariant functor $\text{Hom}(\ ; k)$ produces the decomposition

$$\text{Hom}(H_j(ZY); k) \cong \bigoplus_{l=1}^N \mathbb{I}(b_l, d_l),$$

now in $\text{FVect}_k^{D^{op}}$. Naturality of the Universal Coefficient Theorem [Hat02, Theorem 3.2] with coefficients in a field gives $H^j(ZY) \cong \text{Hom}(H_j(ZY); k)$, and hence the barcodes for $H_j(ZY)$ and $H^j(ZY)$ are equal as multisets of intervals. \square

Lemma 3. *If fibrewise space Y has a section, then there is a full-length interval $[1, 2n + 1]$ in the barcodes for $H_0(ZY)$ and $H^0(ZY)$.*

Proof. A section $I \xrightarrow{s} Y$ gives the commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{s} & Y & \xrightarrow{p} & I \\ & \searrow & & \nearrow & \\ & & \mathbb{1} & & \end{array} \quad (2.1)$$

with $\mathbb{1}$ the identity map. Taking zigzag diagrams and applying H_0 gives the following.

$$\begin{array}{ccccc} H_0(ZI) & \xrightarrow{H_0(Zs)} & H_0(ZY) & \xrightarrow{H_0(Zp)} & H_0(ZI) \\ & \searrow & & \nearrow & \\ & & \mathbb{1} & & \end{array}$$

Since the identity map on $H_0(ZI) \cong \mathbb{I}(1, 2n+1)$ factors through $H_0(ZY)$, the splitting lemma implies that the barcode decomposition for $H_0(ZY)$ contains a summand isomorphic to $\mathbb{I}(1, 2n+1)$. The proof for cohomology is obtained by either applying H^0 to Diagram 2.1 or by using Lemma 2. □

Chapter 3

The Evasion Problem

Imagine that ball-shaped sensors wander in a bounded domain. A sensor can't measure its location but does know when it overlaps a nearby sensor. We say that an evasion path exists in this sensor network if a moving evader can avoid detection. Can we determine if an evasion path exists? We refer to this question as the evasion problem, and it is introduced in Section 11 of [dG06]. This problem is an example of a task in minimal sensing, where one is given only local data measured by limited sensors but tries to answer a global question.

We define the evasion problem in Section 3.1. In Section 3.2 we describe a theorem of de Silva and Ghrist that gives a necessary homological condition for the existence of an evasion path. We use zigzag persistence in Section 3.3 to obtain an equivalent homological condition that lends itself more easily to streaming computation. However, zigzag persistence does not give a complete solution to the evasion problem. Indeed, in Section 3.4 we show the existence of an evasion path depends not only on the fibrewise homotopy type of the sensor network but also on the ambient isotopy class of its embedding in spacetime. What extra capabilities might the sensors need to allow us to detect evasion paths? In Section 3.5 we suppose the sensors can also measure weak angular and distance information.

3.1 Evasion problem statement

The setting for our problem is Section 11 of [dG06] with a few minor changes. Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded domain homeomorphic to a d -dimensional ball, where $d \geq 2$. Suppose a finite set S of sensor nodes moves continuously inside this domain over the time interval $I = [0, 1]$, with each sensor $v \in S$ a continuous path $v: I \rightarrow \mathcal{D}$. We assume that two sensors are never at the same location. That is, for two sensors $v \neq \tilde{v}$ we have $v(t) \neq \tilde{v}(t)$ for all t . Let $B_{v(t)} = \{y \in \mathcal{D} \mid \|v(t) - y\| \leq 1\}$ be the unit ball covered by sensor v at time t . Then the union of the balls $X(t) = \cup_{v \in S} B_{v(t)}$ is the region covered by the sensors at time t , and $X(t)^c = \mathcal{D} \setminus X(t)$ is the uncovered region at time t . Let

$$X = \cup_{t \in I} X(t) \times \{t\} \subset \mathcal{D} \times I$$

be the subset of spacetime covered by sensors, and let $X^c = (\mathcal{D} \times I) \setminus X$ be the uncovered region in spacetime. Both X and X^c are fibrewise spaces, that is, spaces equipped with projection maps $X \rightarrow I$ and $X^c \rightarrow I$ to time.

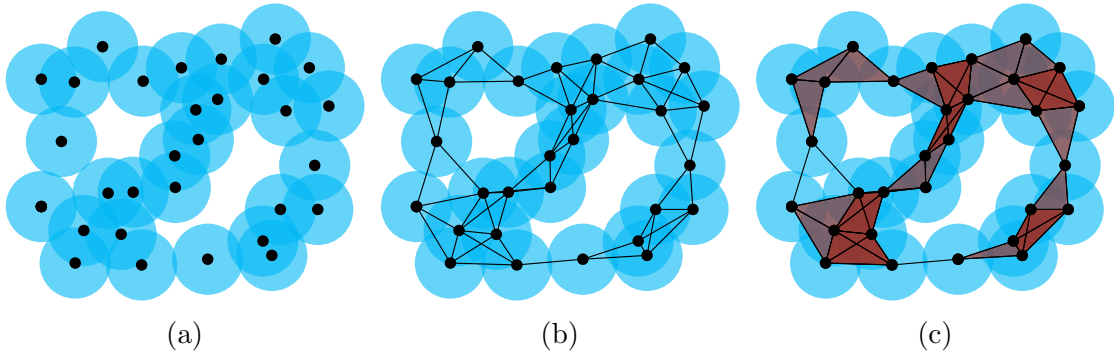


Figure 3.1: (a) A sensor network at a fixed point in time, (b) its connectivity graph, and (c) its Čech complex.

The sensors can't measure their locations but two sensors do know when they overlap. This allows us to measure the time-varying connectivity graph of the sensors.

The connectivity graph at time t has S as its vertex set and has an edge between sensors v and \tilde{v} if $B_{v(t)} \cap B_{\tilde{v}(t)} \neq \emptyset$; see Figure 3.1(b). We assume there is an immobile subset of fence sensors whose union of balls covers the boundary $\partial\mathcal{D}$ and is homotopy equivalent to $\partial\mathcal{D}$.

Potentially there are also evaders moving continuously in this domain. The evaders would like to avoid being detected by the sensors, but an evader is detected if it lies in the covered region $X(t)$ at time t . We say that an evasion path exists when it is possible for a moving evader to avoid being seen by the sensors.

Definition 1. An *evasion path* in a sensor network is a section $s: I \rightarrow X^c$ of fibrewise space $X^c \rightarrow I$. Equivalently, an evasion path is a continuous map $s: I \rightarrow \mathcal{D}$ such that $s(t) \notin X(t)$ for all t .

See Figure 3.2 for examples. Given a sensor network, we would like to determine whether or not an evasion path exists.

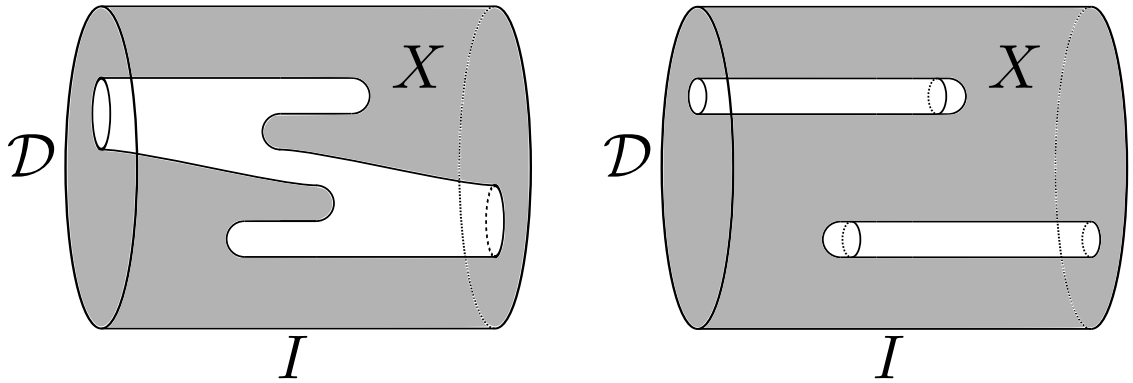


Figure 3.2: We have drawn two planar sensor networks with domain $\mathcal{D} \subset \mathbb{R}^2$ on the vertical axis and with time I on the horizontal axis. The region X in spacetime covered by the sensors is drawn in gray. The network on the left contains an evasion path. The network on the right does not contain an evasion path because an evader must move continuously and cannot teleport locations.

Connectivity graphs alone cannot determine the existence of evasion paths. Consider the two sensor networks in Figure 3.3, and suppose that in each case all three

sensors are immobile over the entire time interval. The two networks have the same connectivity graphs at each point in time, but the sensor network on the left contains an evasion path while the network on the right does not.

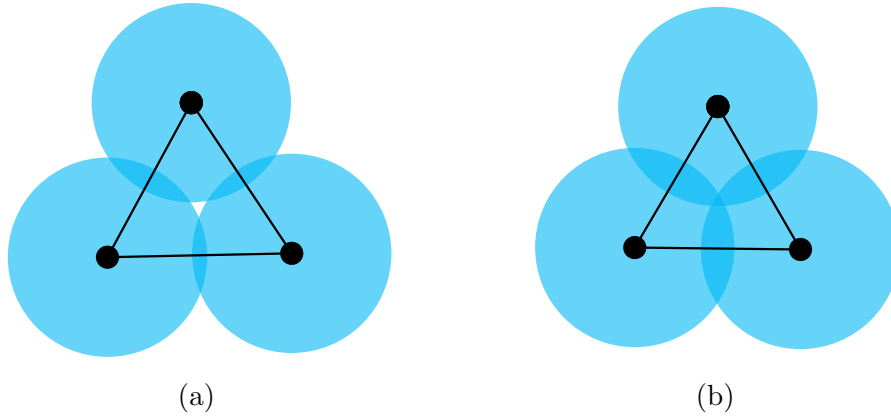


Figure 3.3: Let (a) and (b) be snapshots in time of two different sensor networks. Imagine that no sensor moves over the entire time interval. Then network (a) has an evasion path while network (b) does not, even though the two networks have the same connectivity graph at each point in time.

Since connectivity graphs alone are insufficient, we consider Čech simplicial complexes. The Čech complex $C(t)$ of the sensors is the nerve of the unit balls $\{B_{v(t)}\}_{v \in S}$ [EH10]. Hence the vertex set of $C(t)$ is the set of sensors S , and we have a k -simplex if the intersection of the corresponding $(k + 1)$ -balls is nonempty. That is, simplex $[v_0 v_1 \dots v_k]$ is in $C(t)$ when

$$B_{v_0(t)} \cap B_{v_1(t)} \cap \dots \cap B_{v_k(t)} \neq \emptyset.$$

See Figure 3.1(c) for an example. Note that the 1-skeleton of $C(t)$ is the connectivity graph at time t . The Čech complex $C(t)$ is homotopy equivalent to the union of the balls $X(t)$ by the nerve lemma [Hat02, Corollary 4G.3], and for this reason we would like to work with Čech complexes. We are now ready to state the evasion problem.

The Evasion Problem. *Given the Čech complexes $C(t)$ of a sensor network at all times $t \in I$, which determine the fibrewise homotopy type of covered region X , can one determine if an evasion path exists?*

In applications it is generally unreasonable to assume that our sensors can measure Čech complexes. This would require the task of detecting k -fold intersections, which is not possible under many models of minimal sensing. However, one can approximate the Čech complex from either above or below using the Vietoris–Rips complex [Vie27]; see Appendix A. The Vietoris–Rips complex is the maximal simplicial complex built on top of the connectivity graph, and hence is measured by our sensors. This approximation allows one to take results based on Čech complexes and produce analogous approximate results using only Vietoris–Rips complexes. For example, the results in [dG06] are stated in terms of Vietoris–Rips complexes. We avoid such approximations and instead use Čech complexes.

3.2 Stacked Čech complex and work of de Silva and Ghrist

The stacked Čech complex is a single cell complex encoding the Čech complexes $C(t)$ for all times $t \in I$. We assume there are only a finite number of times

$$0 < t_1 < \dots < t_n < 1$$

when the Čech complex changes. Hence for t and t' in either (t_i, t_{i+1}) , $[0, t_1)$, or $(t_n, 1]$, we have $C(t) = C(t')$. Moreover, we assume that at each time t_i simplices are either added or removed but not both. Since the sensors balls are closed, a simplex σ is

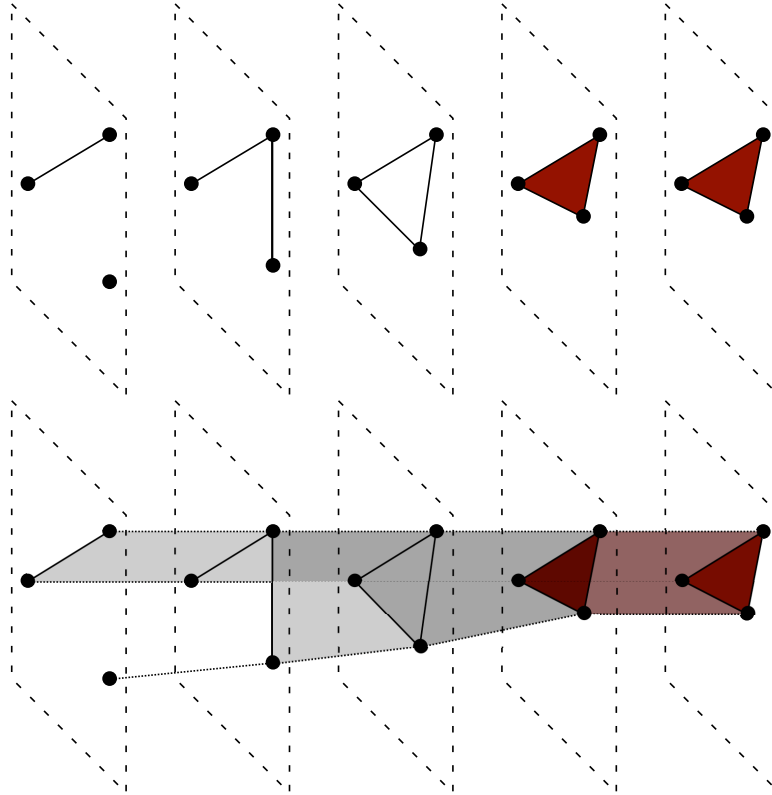


Figure 3.4: The stacked Čech complex for three sensor nodes. The top row shows how the Čech complex changes: initially the Čech complex consists of an edge and a vertex, and two edges and a 2-simplex are added as the sensors move closer together. The bottom row shows the stacked Čech complex with prism cells added.

- added at time t_i if $\sigma \in C(t_i)$ but $\sigma \notin C(t)$ for $t \in (t_{i-1}, t_i)$, and
- removed at time t_i if $\sigma \in C(t_i)$ but $\sigma \notin C(t)$ for $t \in (t_i, t_{i+1})$.

Choose interleaving times

$$0 = s_0 < t_1 < s_1 < \dots < t_n < s_n = 1.$$

Definition 2. The stacked Čech complex $SC \xrightarrow{p} I$ is the fibrewise space obtained from the disjoint union $\coprod_{i=0}^n C(s_i) \times [t_i, t_{i+1}]$, where $t_0 = 0$ and $t_{n+1} = 1$, by identifying

- $C(s_{i-1}) \times \{t_i\}$ as a subset of $C(s_i) \times \{t_i\}$ if simplices are added at t_i , and
- $C(s_i) \times \{t_i\}$ as a subset of $C(s_{i-1}) \times \{t_i\}$ if simplices are removed at t_i .

Map p is the projection onto the second coordinate, and note that $p^{-1}(t) = C(t)$.

This definition is similar to the stacked Vietoris–Rips complex of [dG06]. See Figure 3.4 for a small example.

De Silva and Ghrist give a partial answer to the evasion problem in Theorem 7 of [dG06]. We state their result in arbitrary dimensions and with the stacked Čech complex instead of the stacked Vietoris–Rips complex. Recall that a subset of immobile fence sensors covers the boundary $\partial\mathcal{D}$, and let $F \times [0, 1]$ be the subcomplex of the stacked Čech complex SC consisting of only these fence sensors.

Theorem 7 of [dG06], reformulated. *If there is an evasion path in the sensor network, then every $[\alpha] \in H_d(SC, F \times [0, 1])$ satisfies $0 = [\partial\alpha] \in H_{d-1}(F \times [0, 1])$.*

We explain the picture behind this theorem. Suppose there is some

$$[\alpha] \in H_d(SC, F \times [0, 1])$$

with $0 \neq [\partial\alpha]$. Let α be a relative d -cycle in SC with boundary in $F \times [0, 1]$ representing $[\alpha]$. The condition $0 \neq [\partial\alpha]$ means that the boundary of α wraps a nontrivial number of times around $F \times [0, 1]$. We think of α as a “sheet” in the region of spacetime covered by the sensors that separates time 0 from time 1; see Figure 3.5(a). If there is such a relative d -cycle α then no evasion path can exist. For example, Theorem 7 of [dG06] proves there is no evasion path in the sensor network in Figure 3.5(b).

The homological criterion in Theorem 7 of [dG06] is necessary but not sufficient for the existence of an evasion path. This can be seen in the sensor network in Figure 3.5(c). The homological criterion of Theorem 7 of [dG06] is satisfied but there

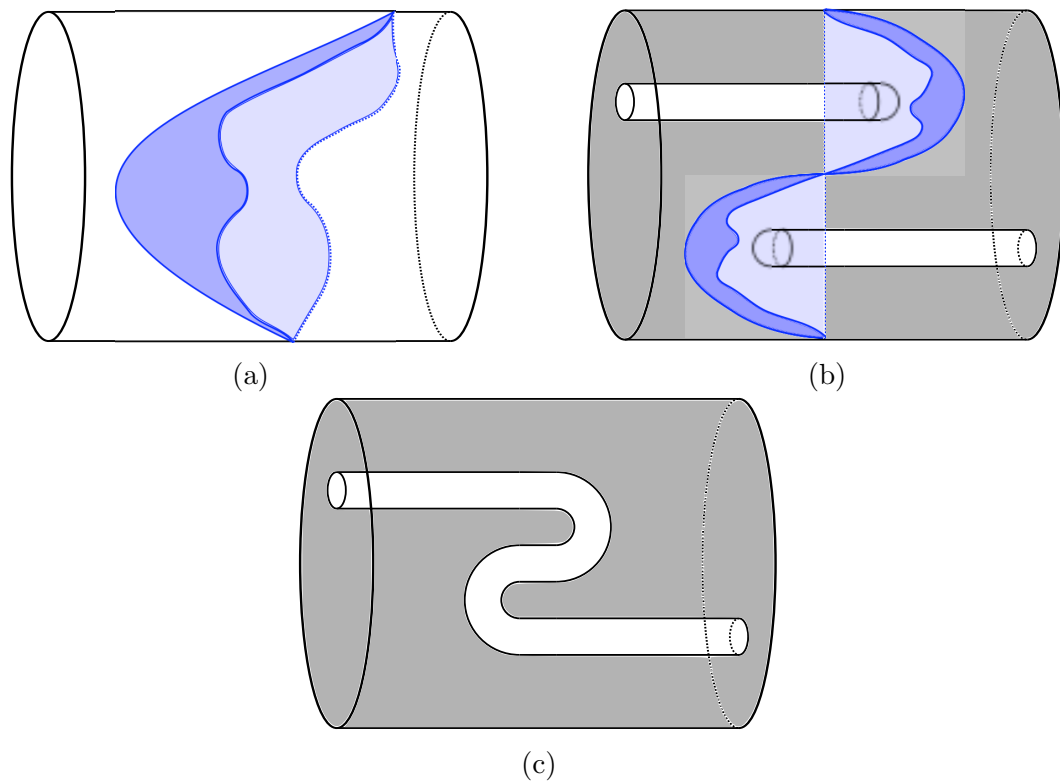


Figure 3.5: (a) A relative 2-cycle α from Theorem 7 of [dG06] is depicted in blue. (b) Theorem 7 of [dG06] proves that there is no evasion path in this sensor network. (c) Although there is no evasion path in this network, Theorem 7 of [dG06] does not apply.

is no evasion path since an evader cannot move backwards in time. Can we sharpen this theorem to get necessary and sufficient conditions?

3.3 Applying zigzag persistence

We began studying the evasion problem with the goal of finding an if-and-only-if criterion for the existence of an evasion path using zigzag persistence, which describes how the homology of the region covered by the sensors changes with time.

Let $0 = s_0 < \dots < s_n = 1$ be chosen as in Section 3.2. As in Section 2.3, we

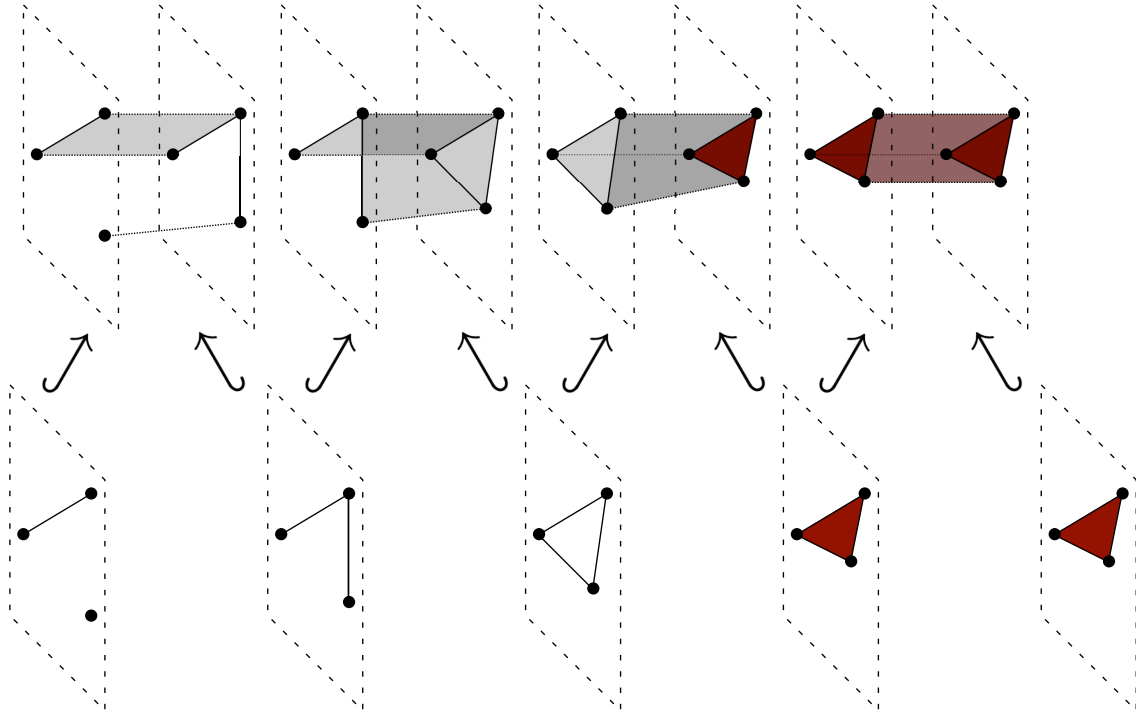


Figure 3.6: The zigzag diagram ZSC for the stacked Čech complex from the three sensor nodes in Figure 3.4.

use this discretization to form the diagrams of spaces ZX , ZX^c , and ZSC in Top^D from fibrewise spaces X , X^c , and SC , respectively. Figure 3.6 contains a depiction of ZSC . Though the stacked Čech complex is not fibrewise homotopy equivalent to the covered region X , by the nerve lemma we have the following commutative diagram with each vertical arrow a homotopy equivalence.

$$\begin{array}{ccccccccccc}
 SC_0 & \longleftarrow & SC_0^1 & \longleftarrow & SC_1 & \longleftarrow & \dots & \longleftarrow & SC_{n-1} & \longleftarrow & SC_{n-1}^n & \longleftarrow & SC_n \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 X_0 & \longleftarrow & X_0^1 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_{n-1} & \longleftarrow & X_{n-1}^n & \longleftarrow & X_n
 \end{array}$$

It follows from the proof of Lemma 1 that $H_j(ZX) \cong H_j(ZSC)$.

Our initial hypothesis was that an evasion path would exist in a sensor network if

and only if there were a full-length interval $[1, 2n + 1]$ in the barcode for $H_{d-1}(ZSC)$. For example, network (a) in Figure 3.7 has both an evasion path and a full-length interval, and network (b) has neither. One direction of this hypothesis is true.

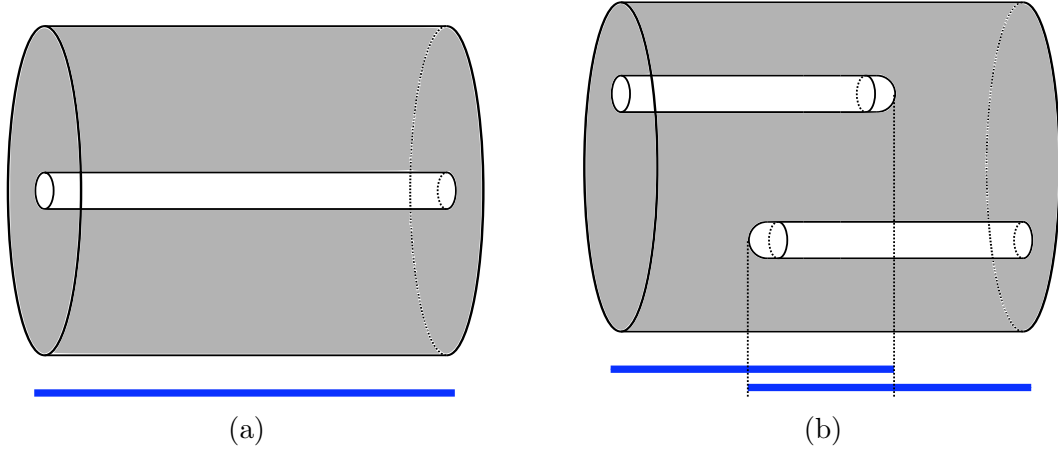


Figure 3.7: Two planar sensor networks and their barcode decompositions for $H_1(ZX)$.

Theorem 3. *If there is an evasion path in a sensor network, then there is a full-length interval $[1, 2n + 1]$ in the barcode for $H_{d-1}(ZSC)$.*

Proof. We follow Lemma 3. An evasion path is a section $s: I \rightarrow X^c$, that is, a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{s} & X^c \xrightarrow{p} I \\ & \searrow \mathbb{1} & \nearrow \end{array}$$

with $\mathbb{1}$ the identity map. Taking zigzag diagrams and applying H_0 gives the following.

$$\begin{array}{ccccc} H_0(ZI) & \xrightarrow{H_0(Zs)} & H_0(ZX^c) & \xrightarrow{H_0(Zp)} & H_0(ZI) \\ & \searrow \mathbb{1} & & \nearrow & \end{array}$$

Since the identity map on $H_0(ZI) \cong \mathbb{I}(1, 2n + 1)$ factors through $H_0(ZX^c)$, the splitting lemma implies that there is a full-length interval $[1, 2n + 1]$ in the barcode for

$H_0(ZX^c)$. To obtain a full-length interval in $H_{d-1}(ZX)$ we need a version of Alexander Duality [Hat02, Theorem 3.44]. We apply Theorem 3.11 of [Kal13], which uses the Diamond Principle of [Cd10] and our Lemma 2. Since $H_{d-1}(ZX) \cong H_{d-1}(ZSC)$, the proof is complete. \square

Remark. This theorem is as discerning as the reformulated version of Theorem 7 of [dG06]. That is, one theorem can be used to prove that no evasion path exists in a sensor network if and only if the other theorem can be used. However, suppose that the sensors move for a long period of time. In this case the amalgamated complex used in Corollary 3 of [dG06] to compute their homological criterion may become quite large. By contrast, the algorithm for computing zigzag persistence runs in a streaming fashion that does not require storing the sensor network across all times simultaneously [CdM09]. Hence computing our Theorem 3 may require less memory.

Interestingly, the converse to Theorem 3 is incorrect. This is demonstrated by the sensor network in Figure 3.8(a). It is tempting to guess that the barcode for this network consists of the intervals drawn on top in red, but they are crossed out because they are incorrect. The correct barcode beneath contains a full-length interval $[1, 2n + 1]$ even though there is no evasion path. We explain this counterintuitive barcode in Figure 3.8(b).

Caution 2.9 from [Cd10] explains that although every submodule isomorphic to an interval in a persistent homology module corresponds to a direct summand, the same is not true for zigzag modules. The sensor networks in Figures 3.7(b) and 3.8(a) are good examples of this caution. The blue cycle in Figure 3.8(b) is a summand of the zigzag module in Figure 3.8(a), but the same cycle is a submodule that is not a summand of the zigzag module in Figure 3.7(b).

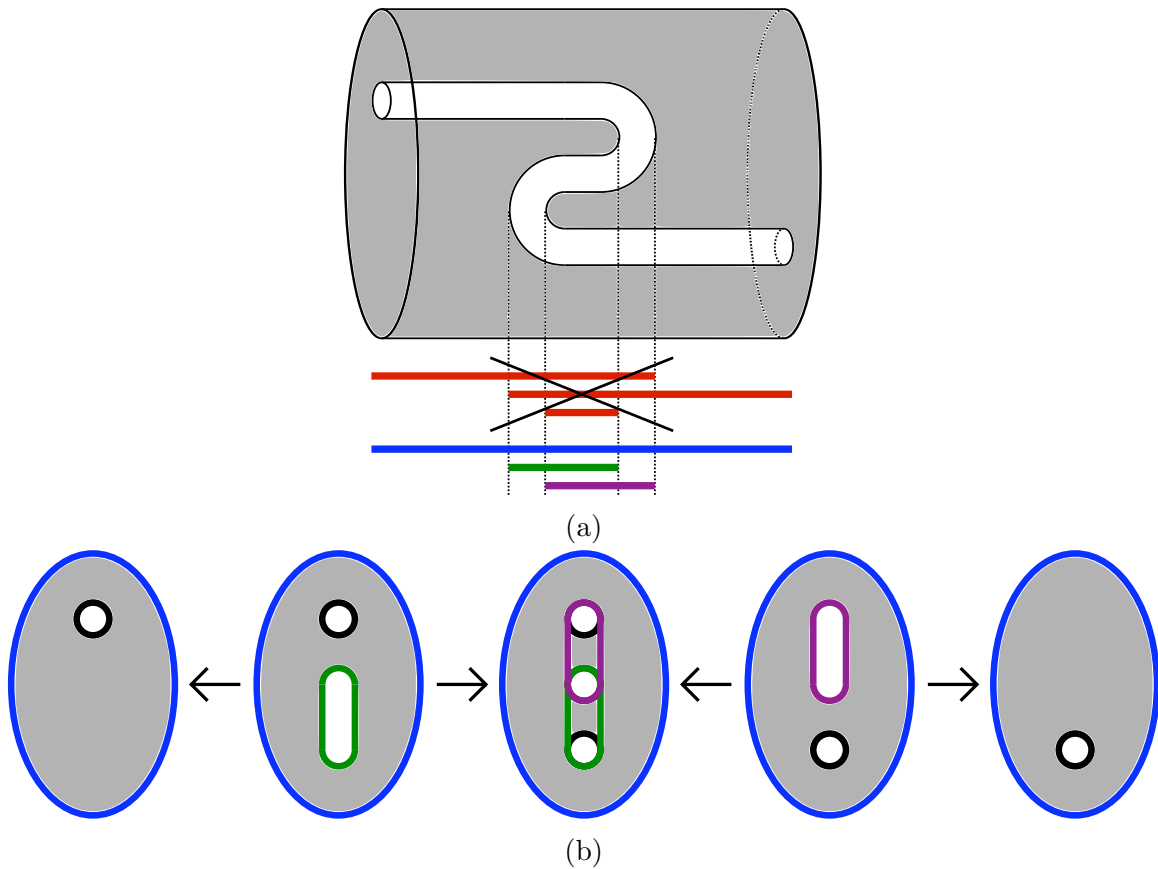


Figure 3.8: (a) It is tempting to guess that the barcode for $H_1(ZX)$ consists of the crossed-out intervals on top in red, but instead the correct barcode is drawn beneath in blue, green, and purple. Note there is a full-length interval $[1, 2n + 1]$ even though there is no evasion path in this network. (b) A coarsened version of diagram ZX . The cycles drawn in blue, green, and purple are generators for the three intervals in $H_1(ZX)$.

3.4 Dependence on the embedding

It turns out that the answer to the evasion problem is no: given the Čech complexes $C(t)$ of a sensor network at all times $t \in I$, which determine the fibrewise homotopy type of covered region X , one cannot always determine if an evasion path exists. The ambient isotopy class of the fibrewise embedding of X into spacetime $\mathcal{D} \times I$ also matters.

We demonstrate this impossibility result using the planar sensor networks (a) and (b) in Figure 3.9. Let us describe network (a). Initially, the bottom half of domain \mathcal{D} is covered by sensors. These sensors retreat to the boundary, leaving a horizontal line of sensors. Two sensors on this line jut out towards the top of \mathcal{D} , forming three sides of a square. These two sensors move closer together, completing the square. The bottom two sensors in this square move apart, breaking the bottom edge of the square. The curvy line of sensors straightens out. Finally, sensors flood from the boundary to cover the top half of \mathcal{D} . Network (b) is identical to (a) except that the square opens towards the bottom of \mathcal{D} .

At each time $t \in I$ the Čech complexes $C(t)$ in the two networks are identical. Nevertheless, network (a) contains an evasion path while network (b) does not. We emphasize that these examples are realizable by continuously moving sensors. Moreover, in domain $\mathcal{D} \subset \mathbb{R}^d$ for any $d \geq 2$ there exists a pair of analogous examples¹.

The covered regions for networks (a) and (b) are fibrewise homotopy equivalent, and the stacked Čech complexes are fibrewise homeomorphic. However, the uncovered regions for networks (a) and (b) are not fibrewise homotopy equivalent, and in particular the first has a section while the second does not. Thus the existence of an evasion path depends not only on the fibrewise homotopy type of the sensor network but also on how the sensor network is fibrewise embedded in spacetime $\mathcal{D} \times I$.

¹Analogous examples in $\mathcal{D} \subset \mathbb{R}^1$ require the sensors to turn off and then back on.

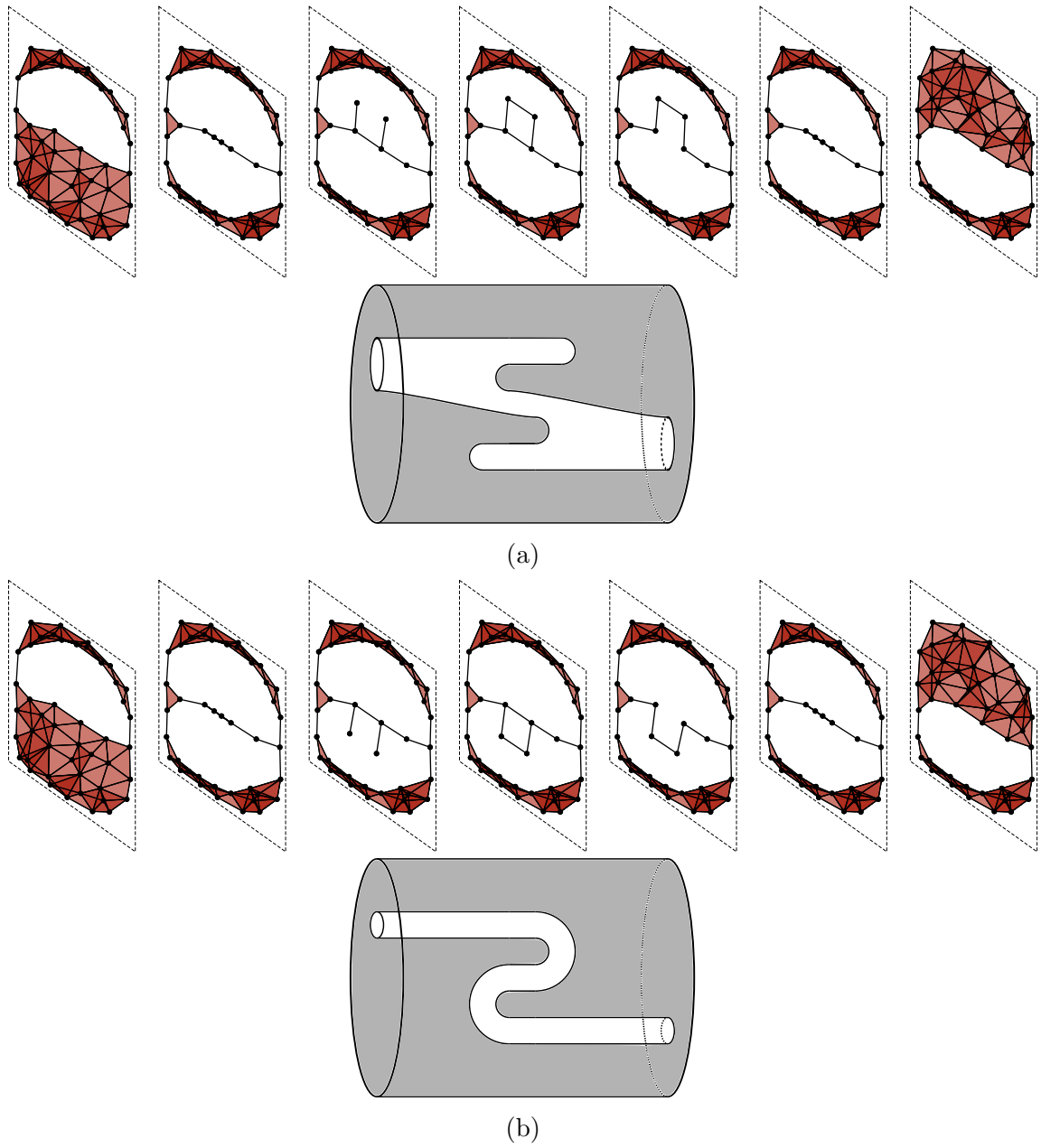


Figure 3.9: Each subfigure is a sensor network represented both as seven sequential Čech complexes and as a covered region X in spacetime $\mathcal{D} \times I$. At each time the Čech complexes in (a) and (b) are identical. Moreover, the two covered regions are fibrewise homotopy equivalent. Nevertheless, network (a) contains an evasion path, but network (b) does not because the evader cannot travel backwards in time.

3.5 Sensors measuring cyclic orders

Since neither the Čech complex at each time nor the fibrewise homotopy type of covered region X are sufficient to determine if an evasion path exists, what minimal sensing capabilities might we add? In this section we assume the sensors live in a planar domain $\mathcal{D} \subset \mathbb{R}^2$ and that each sensor measures the cyclic order of its neighbors, as in [GLPS08]. It is not uncommon for sensors to measure this weak angular data, for example by performing circular radar sweeps. In Theorem 4 we give necessary and sufficient conditions for the existence of an evasion path based on this rotation information.

Theorem 4 relies on the alpha complex of the sensors. Let $V_{v(t)}$ be the Voronoi cell

$$V_{v(t)} = \{y \in \mathcal{D} \mid \|v(t) - y\| \leq \|\tilde{v}(t) - y\| \text{ for all } \tilde{v} \in S\}$$

of all points in \mathcal{D} closest to sensor v at time t . The alpha complex $A(t)$ is the nerve of the convex sets $\{B_{v(t)} \cap V_{v(t)}\}_{v \in S}$ [EM94, EH10]. It is a subcomplex of both the Čech complex and of the Delaunay triangulation, and is homotopy equivalent to the Čech complex. Recovering the alpha complex requires significantly stronger sensors. However, if each sensor measures the local distances to its overlapping neighbors, which may be approximated by time-of-flight, then this data determines the alpha complex [FM09].

We assume there are only a finite number of times $0 < t_1 < \dots < t_n < 1$ when the alpha complex changes. Hence for t and t' in (t_i, t_{i+1}) , $[0, t_1)$, or $(t_n, 1]$ we have identical alpha complexes $A(t) = A(t')$. Moreover, we assume that at each t_i one of the following changes to the alpha complex occurs.

1. A single edge is added or removed.
2. A single 2-simplex is added or removed.

3. A free pair consisting of a 2-simplex and a face edge with no other cofaces is added or removed.
4. A Delaunay edge flip occurs.

We assume that each sensor measures the clockwise cyclic order of its neighbors in the Alpha complex. This cyclic order data is necessarily fixed in each interval (t_i, t_{i+1}) , $[0, t_1)$, or $(t_n, 1]$.

Theorem 4. *Suppose $X(t)$ is connected at each time t . Then from the time-varying alpha complex and the time-varying cyclic orderings of the neighbors about each sensor, we can determine whether or not an evasion path exists.*

Proof. We give a sketch of the proof. Let $A^1(t)$ be the 1-skeleton of the alpha complex at time t . We have cyclic permutations π_v of the edges incident with each vertex v , where $\pi_v(e)$ is the successor of edge e in the clockwise ordering around v . This gives $A^1(t)$ the structure of a rotation system [MT01], also called a fat graph or ribbon graph [Igu02]. A rotation system partitions the directed edges of $A^1(t)$ into sets of boundary cycles. Each boundary cycle is a loop of directed edges $(e_1 e_2 \dots e_k)$ constructed so that if v_i the target vertex of directed edge e_i , then $\pi_{v_i}(e_i) = e_{i+1}$ and $\pi_{v_k}(e_k) = e_1$. See Figure 3.10 for an example, and note that this cyclic ordering data allows one to distinguish the two sensor networks in Figure 3.9.

The boundary cycles of $A^1(t)$ are in bijective correspondence with the connected components of $\mathbb{R}^2 \setminus A^1(t)$. Removing the boundary cycles of length three that are filled by 2-simplices (and also the boundary cycle corresponding to the outside of $\partial\mathcal{D}$) produces a bijection with the connected components of the uncovered region $X(t)^c$. Hence by tracking the boundary cycles of $A^1(t)$ we can measure how the connected components of the uncovered region merge, split, appear, and disappear. In other words, we can reconstruct the Reeb graph of $X^c \rightarrow I$.

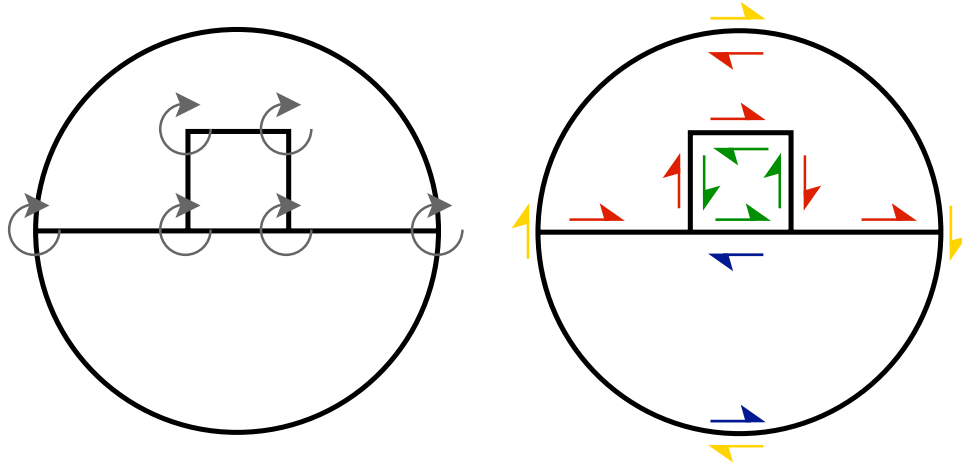


Figure 3.10: An example rotation system. The cyclic orderings are drawn on the left in gray, and the four boundary cycles are drawn on the right in red, green, blue, and yellow.

We will maintain labels on the boundary cycles of $A^1(t)$ so that a boundary cycle is labeled *true* if the corresponding connected component of $\mathbb{R}^2 \setminus A^1(t)$ may contain an evader and *false* if not. At time $t = 0$ we label the boundary cycles of length three filled by 2-simplices in $A(0)$ (and also the boundary cycle corresponding to the outside of $\partial\mathcal{D}$) as *false*. All other boundary cycles are labeled *true*. When we pass a time t_i when the alpha complex changes, we update the labels as follows.

1. If a single edge is added, then since $X(t)$ is connected a single boundary cycle splits in two. Each new boundary cycle maintains the original label. If a single edge is removed then two boundary cycles merge, and the new cycle is labeled *true* if either of the original two cycles were labeled *true*.
2. If a single 2-simplex is added, then the label on the corresponding boundary cycle of length three is set to *false*. If a single 2-simplex is removed then the label on the corresponding boundary cycle of length three remains *false*.
3. If a free pair consisting of a 2-simplex and a face edge is added, then a boundary

cycle splits into two with one label unchanged. The other label corresponding to the added 2-simplex is set to *false*. If a free pair is removed then the boundary cycle of length three corresponding to the 2-simplex is removed and the label on the other altered boundary cycle is unchanged.

4. If a Delaunay edge flip occurs, then two boundary cycles labeled *false* are replaced by two different boundary cycles also labeled *false*.

An evasion path exists in the sensor network if and only if there is a boundary cycle in $A^1(1)$ labeled *true*. □

Figure 3.11 shows that the connectedness assumption in Theorem 4 is necessary.

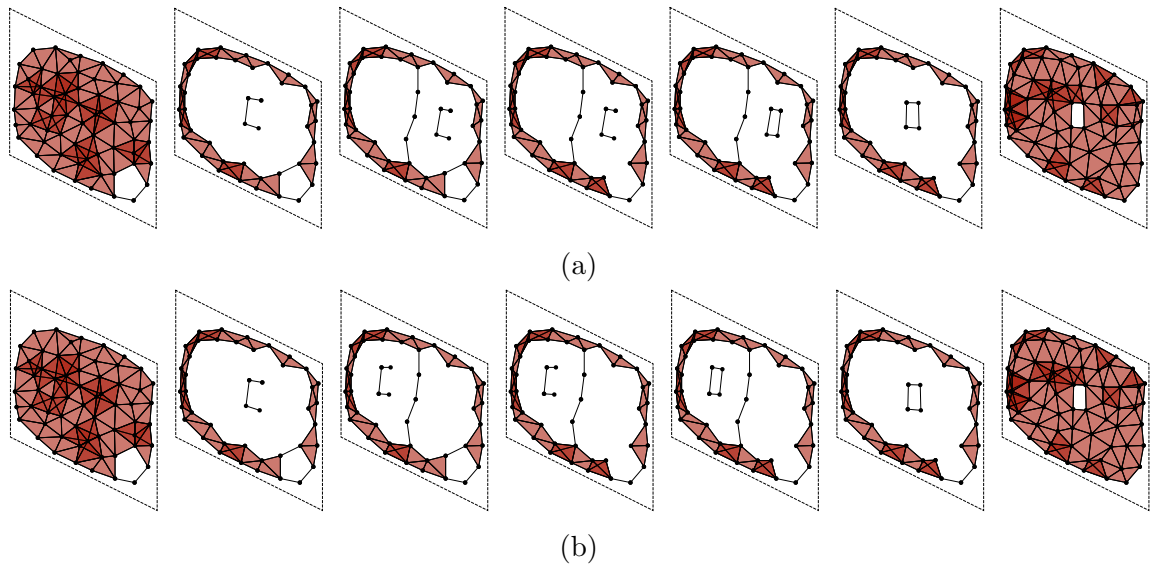


Figure 3.11: Each subfigure is a sensor network represented by seven sequential Čech complexes. At each time the Čech complexes, alpha complexes, and cyclic order information are identical. Nevertheless, network (a) contains an evasion path but network (b) does not. These networks show that it is necessary in Theorem 4 to assume that each $X(t)$ is connected.

It is an open question if Čech complexes and rotation information suffice.

Open Question. Suppose we have a sensor network with $X(t)$ connected at each time t . Using only the time-varying Čech complex and the time-varying cyclic orderings of the neighbors about each sensor, is it possible to determine if an evasion path exists?

An answer to the open question would fill the gap between Theorem 7 of [dG06], which uses only minimal sensor capabilities but is not sharp, and Theorem 4, which is sharp but requires more advanced sensors measuring cyclic orders and alpha complexes or local distances.

Chapter 4

Diagrams of Spaces

The evasion problem motivates a natural extension: can we describe the space of evasion paths? That is, what information must we measure about covered region X and the ambient isotopy class of its fibrewise embedding in spacetime $\mathcal{D} \times I$ to describe the space of sections $I \rightarrow X^c$?

We are primarily interested in this question from a theoretical point of view, but it also relates to the following applied problems. Suppose an evasion path exists in a mobile sensor network. We may be interested in determining the best way to patch the network so that evasion paths no longer exist. Alternatively, we may want to find the evasion path that maintains the largest separation between the evader and the sensors, that requires an evader to move the shortest distance, or that requires an evader to move at the lowest top speed. Knowledge about the space of sections may be helpful for such problems.

In Section 4.1 we relate the space of sections of a fibrewise space $Y \rightarrow I$ to the homotopy limit of a diagram of spaces built from Y . This is our motivation for studying homotopy limits in this chapter and next. In Sections 4.2–4.7 we describe simplicial and cosimplicial objects, a model category structure for diagrams of spaces, adjoints for the forgetful functor on diagrams, projective and injective diagrams, monads, and

the homotopy spectral sequence of a cosimplicial space. With this framework we define homotopy limits in Section 4.8, and moreover, a spectral sequence converging to the homotopy groups of a homotopy limit.

4.1 The space of evasion paths and homotopy limits

In this section we relate the space of evasion paths to homotopy limits. This is our motivation for studying homotopy limits, which we define in Section 4.8. We work with topological spaces in this section but for the remainder of this chapter and the next we work with simplicial sets.

Example 1. Consider the sensor network drawn in Figure 4.1, in which the uncovered region consists of a cylinder and two thin tubes. The blue evasion path wraps once

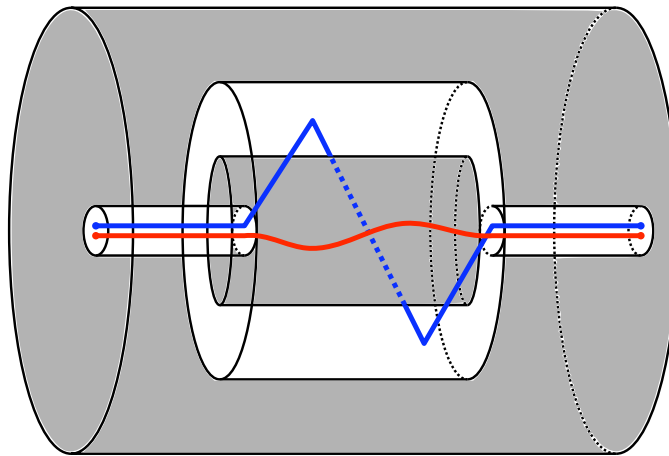


Figure 4.1: The uncovered region consists of a cylinder in spacetime along with two thin tubes connecting the cylinder to the starting and ending times. The blue evasion path wraps once around the cylinder while the red evasion path travels straight across.

around the cylinder while the red evasion path travels straight across; these two

evasion paths are not fibrewise isotopic. We can model the uncovered region by the diagram

$$* \rightarrow S^1 \leftarrow *$$

whose homotopy limit is ΩS^1 , the loop space of the circle. This is a discrete space with an infinite number of components, one for each integer winding number around the circle. The homotopy limit of the diagram is homotopy equivalent to the space of evasion paths in the sensor network.

Example 2. The space of evasion paths need not be discrete. For example, if Y is a space and $Y \times I$ is the uncovered region of a sensor network, then the space of evasion paths is homotopy equivalent to Y . The case when $Y = S^1$ is drawn in Figure 4.2.

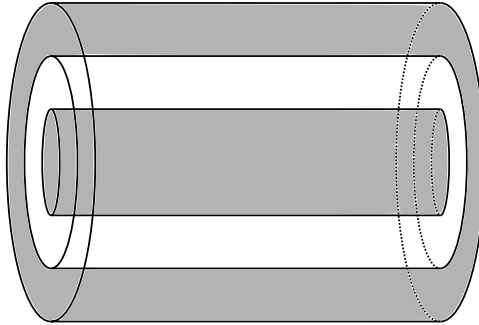


Figure 4.2: The uncovered region for this sensor network consists of a cylinder in spacetime.

We now formalize the relationship between spaces of sections and homotopy limits. Let $Y \xrightarrow{p} I$ be a fibrewise space. Recall $Y_i = p^{-1}(s_i)$ and $Y_i^{i+1} = p^{-1}([s_i, s_{i+1}])$. Let PY_i^{i+1} be the space of undirected paths

$$PY_i^{i+1} = \{ \alpha: [s_i, s_{i+1}] \rightarrow Y_i^{i+1} \mid p(\alpha(s_i)) = s_i \text{ and } p(\alpha(s_{i+1})) = s_{i+1} \},$$

and let $\vec{P}Y_i^{i+1}$ be the corresponding space of fibrewise paths

$$\vec{P}Y_i^{i+1} = \{\alpha: [s_i, s_{i+1}] \rightarrow Y_i^{i+1} \mid p(\alpha(t)) = t \text{ for all } t \in [s_i, s_{i+1}]\}.$$

We say that a discretization $0 = s_0 < \dots < s_n = 1$ is section-preserving for Y if each PY_i^{i+1} deformation retracts onto $\vec{P}Y_i^{i+1}$. For example, a discretization chosen as in Section 3.2 is section-preserving for the stacked Čech complex SC , for the region X covered by the sensors, and for the region X^c not covered by the sensors.

Lemma 4. *If diagram $ZY \in \text{Top}^D$ is formed from a section-preserving discretization, then the space of sections $\Gamma(Y)$ is homotopy equivalent to the homotopy limit $\text{holim}_D ZY$.*

Proof. We have

$$\begin{aligned} \text{holim}_D ZY &\cong \text{hom}_D(|B(D \downarrow -)|, ZY) \\ &\cong \{(\alpha_0 \dots \alpha_{n-1}) \in PY_0^1 \times \dots \times PY_{n-1}^n \mid \alpha_i(s_{i+1}) = \alpha_{i+1}(s_{i+1})\} \\ &\simeq \{(\alpha_0, \dots, \alpha_{n-1}) \in \vec{P}Y_0^1 \times \dots \times \vec{P}Y_{n-1}^n \mid \alpha_i(s_{i+1}) = \alpha_{i+1}(s_{i+1})\} \\ &\cong \Gamma(Y). \end{aligned}$$

The diagram $|B(D \downarrow -)|$ in Top^D has at object i in D the space $|B(D \downarrow i)|$, which is the realization of the nerve of the overcategory $(D \downarrow i)$. The first homeomorphism is Proposition 18.2.6 of [Hir09] (see also Lemma 13). The second homeomorphism follows from the definition of $|B(D \downarrow -)|$ for our indexing category D generated by a quiver of type A_n . The homotopy equivalence comes from the deformation retractions of each PY_i^{i+1} onto $\vec{P}Y_i^{i+1}$. \square

4.2 Simplicial and cosimplicial objects

The material in this section can be found in [BK72a, GJ09]. Let Δ be the simplex category, whose objects are strings of ordinal numbers of the form

$$0 \rightarrow 1 \rightarrow \dots \rightarrow n-1 \rightarrow n,$$

and whose morphisms are order-preserving functions of sets. For a category \mathcal{C} , the category of simplicial objects over \mathcal{C} is the functor category with contravariant functors $\Delta^{op} \rightarrow \mathcal{C}$ as objects and with natural transformations as morphisms. Alternatively, a simplicial object X over \mathcal{C} is a collection of objects $X_n \in \mathcal{C}$ for $n \geq 0$, equipped with face and degeneracy maps

$$d_i: X_n \rightarrow X_{n-1}$$

$$s_i: X_n \rightarrow X_{n+1}$$

for $0 \leq i \leq n$. The face and degeneracy maps satisfy the following simplicial identities:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for } i < j \\ d_i s_j &= s_{j-1} d_i && \text{for } i < j \\ &= \mathbf{1} && \text{for } i = j, j+1 \\ &= s_j d_{i-1} && \text{for } i > j+1 \\ s_i s_j &= s_j s_{i-1} && \text{for } i > j. \end{aligned}$$

Similarly, let $c\mathcal{C}$ be the functor category of cosimplicial objects over \mathcal{C} . Its objects are covariant functors $\Delta \rightarrow \mathcal{C}$ and its morphisms are natural transformations. Alternatively, an object $X \in c\mathcal{C}$ is a collection of objects $X^n \in \mathcal{C}$ for $n \geq 0$, equipped with

coface and codegeneracy maps

$$\begin{aligned} d^i &: X^{n-1} \rightarrow X^n \\ s^i &: X^{n+1} \rightarrow X^n \end{aligned}$$

for $0 \leq i \leq n$. The coface and codegeneracy maps satisfy the following cosimplicial identities:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{for } i < j \\ s^j d^i &= d^i s^{j-1} && \text{for } i < j \\ &= \mathbf{1} && \text{for } i = j, j + 1 \\ &= d^{i-1} s^j && \text{for } i > j + 1 \\ s^j s^i &= s^{i-1} s^j && \text{for } i > j. \end{aligned}$$

For \mathcal{C} a model category there is an induced Reedy model category structure on $c\mathcal{C}$ [GJ09, Hov99].

Let \mathcal{S} be the category of simplicial sets and let \mathcal{S}_* be the category of pointed simplicial sets. There is an equivalence between the homotopy theories of simplicial sets and of topological spaces. For the remainder of this chapter and the next we work simplicially, meaning that a space is a pointed simplicial set. We refer to $c\mathcal{S}_*$ as the category of cosimplicial spaces.

4.3 Model category structure for diagrams of spaces

In this section we describe model category structures for pointed simplicial sets and for diagrams thereof. The category \mathcal{S}_* of pointed simplicial sets is a simplicial cofibrantly generated model category [Hir09]. A morphism is a weak equivalence if its induced map on geometric realizations is a weak equivalence of topological spaces, and a morphism is a fibration if it is a Kan fibration. The tensor of $X \in \mathcal{S}_*$ with $K \in \mathcal{S}$ is

given by $X \otimes K = X \wedge K^+$, where $K^+ \in \mathcal{S}_*$ is formed by adding a disjoint basepoint. For $W, X \in \mathcal{S}_*$ the pointed simplicial set $\text{hom}(W, X)$ has as its n -simplices the set of maps from $W \otimes \Delta^n$ to X .

Since D is a small category, Theorem 11.7.3 of [Hir09] implies that \mathcal{S}_*^D is also a simplicial cofibrantly generated model category. A morphism $W \rightarrow X$ in \mathcal{S}_*^D is a weak equivalence if it is a pointwise weak equivalence, meaning that each $W_i \rightarrow X_i$ is a weak equivalence in \mathcal{S}_* . Similarly, a morphism is a fibration if it is a pointwise fibration. Cofibrations consist of all free maps and their retracts. The tensor of $X \in \mathcal{S}_*^D$ with $K \in \mathcal{S}$, denoted $X \otimes K \in \mathcal{S}_*^D$, is given by $(X \otimes K)_i = X_i \otimes K$ and $p_{i,j}^{X \otimes K} = p_{i,j}^X \otimes \mathbb{1}_K$. For $W, X \in \mathcal{S}_*^D$ the pointed simplicial set $\text{hom}_D(W, X)$ has as its n -simplices the set of maps from $W \otimes \Delta^n$ to X .

Let $\Theta*$ be the constant diagram in \mathcal{S}_*^D with basepoint $*$ at each object in D . Note that \mathcal{S}_*^D is a pointed model category since the map $\Theta* \rightarrow \Theta*$ from the initial object to the terminal object is an isomorphism. Let Σ be the suspension functor for this pointed model category. Lemma 6.1.2 of [Hov99] and Proposition 9.5.24 of [Hir09] then imply the following.

Lemma 5. *Let $W \in \mathcal{S}_*^D$ be cofibrant and let $X \in \mathcal{S}_*^D$ be fibrant. We have natural isomorphisms*

$$\pi_t \text{hom}_D(W, X) \cong [\Sigma^t W, X]$$

for all $t \geq 0$, where $[\Sigma^t W, X]$ is the homotopy class of maps $\Sigma^t W \rightarrow X$.

4.4 Adjoints for forgetful functors on diagrams

Let \mathcal{C} be a category and let D be the free category generated by a quiver of type A_n . Let D^δ be the discrete category with the same objects as D but with no non-identity morphisms. The inclusion functor $D^\delta \rightarrow D$ induces a forgetful functor $F: \mathcal{C}^D \rightarrow \mathcal{C}^{D^\delta}$.

Also, for $1 \leq i \leq n$, let $F_i: \mathcal{C}^D \rightarrow \mathcal{C}$ be the forgetful functor mapping $C \in \mathcal{C}^D$ to its value C_i at the i -th object of D .

If category \mathcal{C} has finite products then the forgetful functors have right adjoints [Hir09].

Lemma 6. *If \mathcal{C} has finite products then we have an adjunction $F_i: \mathcal{C}^D \leftrightarrow \mathcal{C}: G_i$.*

Let $C \in \mathcal{C}$. If there is a morphism $j \rightarrow i$ in D then $(G_i C)_j = C$; otherwise $(G_i C)_j$ is the terminal object in \mathcal{C} . Morphism $p_{i,j}^{G_i C}$ is either the identity map on C or the unique map to the terminal object. For a morphism $f: C \rightarrow C'$ in \mathcal{C} , map $(G_i f)_j: (G_i C)_j \rightarrow (G_i C')_j$ is f if there is a morphism $j \rightarrow i$ in D , and is the identity map on the terminal object otherwise.

Lemma 7. *If \mathcal{C} has finite products then we have an adjunction $F: \mathcal{C}^D \leftrightarrow \mathcal{C}^{D^\delta}: G$.*

For $C \in \mathcal{C}^{D^\delta}$ we have $GC = \prod_i G_i C_i$. That is,

$$(GC)_j = \prod_{j \rightarrow i} C_i$$

where the product is over the morphisms in D with source j . Morphism $p_{i,j}^{GC}$ is given by canonical projection.

Dually, if category \mathcal{C} has finite coproducts then the forgetful functors have left adjoints.

Lemma 8. *If \mathcal{C} has finite coproducts then we have an adjunction $P_j: \mathcal{C} \leftrightarrow \mathcal{C}^D: F_j$.*

Let $C \in \mathcal{C}$. If there is a morphism $j \rightarrow i$ in D then $(P_j C)_i = C$; otherwise $(P_j C)_i$ is the initial object in \mathcal{C} . Morphism $p_{i,j}^{P_j C}$ is either the identity map on C or the unique map from the initial object. For a morphism $f: C \rightarrow C'$ in \mathcal{C} , map $(P_j f)_i: (P_j C)_i \rightarrow (P_j C')_i$ is f if there is a morphism $j \rightarrow i$ in D , and is the identity map on the initial object otherwise.

Lemma 9. *If \mathcal{C} has finite coproducts then we have an adjunction $P: \mathcal{C}^{D^\delta} \leftrightarrow \mathcal{C}^D: F$.*

For $C \in \mathcal{C}^{D^\delta}$ we have $PC = \coprod_j P_j C_j$. That is,

$$(PC)_i = \coprod_{j \rightarrow i} C_j$$

where the coproduct is over the morphisms in D with target i . Morphism $p_{i,j}^{PC}$ is given by canonical injection.

For example, suppose the quiver underlying D is

$$\bullet_1 \rightarrow \bullet_2 \leftarrow \bullet_3 \rightarrow \bullet_4 \leftarrow \bullet_5$$

and $C \in \mathcal{C}^{D^\delta}$ is

$$C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5.$$

Then diagram GC is

$$C_1 \times C_2 \rightarrow C_2 \leftarrow C_2 \times C_3 \times C_4 \rightarrow C_4 \leftarrow C_4 \times C_5,$$

with each map the canonical projection. Diagram PC is

$$C_1 \rightarrow C_1 \amalg C_2 \amalg C_3 \leftarrow C_3 \rightarrow C_3 \amalg C_4 \amalg C_5 \leftarrow C_5,$$

with each map the canonical injection.

4.5 Projective and injective diagrams

Let D be the free category generated by a quiver of type A_n . Recall from Theorem 1 that the categories Vect_k^D and $kD\text{-Mod}$ are equivalent, where kD is the path algebra

over a field k . For a ring R we can analogously define the ring RD as the free left R -module with basis the morphisms in D and with multiplication given by composition. The categories $(R\text{-Mod})^D$ and $RD\text{-Mod}$ are equivalent [EE05].

Lemma 10. *If $M \in R\text{-Mod}$ is projective then so is $P_j M \in (R\text{-Mod})^D$, and if M is injective then so is $G_i M \in (R\text{-Mod})^D$.*

In particular, in FVect_k^D the set

$$\{P_j(k) \mid 1 \leq j \leq n\} \quad \left(\text{respectively } \{G_i(k) \mid 1 \leq i \leq n\} \right)$$

is a complete set of indecomposable projective (respectively injective) quiver representations that are pairwise non-isomorphic [CB92].

In order to study projective resolutions in $(R\text{-Mod})^D$ we first define a functor $\tilde{P}: (R\text{-Mod})^{D^\delta} \rightarrow (R\text{-Mod})^D$. For a morphism $j \rightarrow i$ in D , let its length $|j \rightarrow i|$ be the number of arrows in the corresponding path of the underlying quiver. The set of morphisms with $|j \rightarrow i| = 1$ is the set of arrows in the quiver. For $N \in (R\text{-Mod})^{D^\delta}$ we define $\tilde{P}N = \coprod_{|j \rightarrow i|=1} P_i N_j$, which gives

$$(\tilde{P}N)_i = \coprod_{|j \rightarrow i| \geq 1} N_j.$$

Moreover, given $M \in (R\text{-Mod})^D$, we define morphism $\phi^M: \tilde{P}FM \rightarrow PFM$ by defining each $\phi_i^M: (\tilde{P}FM)_i \rightarrow (PFM)_i$. If $|j \rightarrow i| \geq 1$ and $m \in M_j \subset (\tilde{P}FM)_i$, then

$$\phi_i^M(m) = \begin{cases} m & \text{in summand } M_j \text{ of } (PFM)_i \\ -p_{i,j}^M(m) & \text{in summand } M_i \text{ of } (PFM)_i \\ 0 & \text{in all other summands.} \end{cases}$$

The following lemma generalizes the standard projective resolution for quiver representations [CB92].

Lemma 11. *Suppose*

$$\dots \xrightarrow{d_{l+1}} S_l \xrightarrow{d_l} \dots \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \xrightarrow{d_0} M \rightarrow 0$$

is a resolution in $(R\text{-Mod})^D$ with each $(S_l)_i \in R\text{-Mod}$ projective. Then

$$\begin{array}{ccccccc} \dots & \rightarrow & PFS_3 & \xrightarrow{\begin{bmatrix} PFd_3 & \phi^{S_2} \\ 0 & -\tilde{P}Fd_2 \end{bmatrix}} & PFS_2 & \xrightarrow{\begin{bmatrix} PFd_2 & \phi^{S_1} \\ 0 & -\tilde{P}Fd_1 \end{bmatrix}} & PFS_1 & \xrightarrow{\begin{bmatrix} PFd_1 & \phi^{S_0} \end{bmatrix}} & PFS_0 & \rightarrow & M & \rightarrow & 0 \\ & & \oplus & & \oplus & & \oplus & & & & & & & \\ & & \tilde{P}FS_2 & & \tilde{P}FS_1 & & \tilde{P}FS_0 & & & & & & & \end{array}$$

is a projective resolution in $(R\text{-Mod})^D$.

Proof. The morphism $PFS_0 \rightarrow M$ is the adjoint of $Fd_0: FS_0 \rightarrow FM$ given by Lemma 9. Note each $PFS_l \oplus \tilde{P}FS_{l-1}$ is projective by Lemma 10. To show exactness at stage $l \geq 2$ in the resolution, we first compute

$$\begin{aligned} \begin{bmatrix} PFd_l & \phi^{S_{l-1}} \\ 0 & -\tilde{P}Fd_{l-1} \end{bmatrix} \begin{bmatrix} PFd_{l+1} & \phi^{S_l} \\ 0 & -\tilde{P}Fd_l \end{bmatrix} &= \begin{bmatrix} PF(d_l d_{l+1}) & PFd_l \phi^{S_l} - \phi^{S_{l-1}} \tilde{P}Fd_l \\ 0 & \tilde{P}F(d_{l-1} d_l) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Next, suppose

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \ker \left(\begin{bmatrix} PFd_l & \phi^{S_{l-1}} \\ 0 & -\tilde{P}Fd_{l-1} \end{bmatrix} \right).$$

This gives

$$\begin{aligned} PFd_l a + \phi^{S_{l-1}} b &= 0 \\ -\tilde{P}Fd_{l-1} b &= 0. \end{aligned} \tag{4.1}$$

Hence there exists some $b' \in \tilde{P}FS_l$ with $-\tilde{P}Fd_l b' = b$. Note

$$\begin{aligned} PFd_l(a - \phi^{S_l} b') &= PFd_l a - PFd_l \phi^{S_l} b' \\ &= PFd_l a - \phi^{S_{l-1}} PFd_l b' \\ &= PFd_l a - \phi^{S_{l-1}} \tilde{P}Fd_l b' && \text{since } b' \in \tilde{P}FS_l \\ &= PFd_l a + \phi^{S_{l-1}} b && \text{by definition of } b' \\ &= 0 && \text{by (4.1)}. \end{aligned}$$

Hence $a - \phi^{S_l} b' \in \ker(PFd_l) = \text{im}(PFd_{l+1})$, so there exists some $a' \in PFS_{l+1}$ with $PFd_{l+1} a' + \phi^{S_l} b' = a$. For $l \geq 2$, we have shown

$$\text{im} \left(\begin{bmatrix} PFd_{l+1} & \phi^{S_l} \\ 0 & -\tilde{P}Fd_l \end{bmatrix} \right) = \ker \left(\begin{bmatrix} PFd_l & \phi^{S_{l-1}} \\ 0 & -\tilde{P}Fd_{l-1} \end{bmatrix} \right).$$

The verifications for $l = 0$ and 1 are similar. \square

The projective dimension of a ring R is the minimum integer l (or ∞) such that every $M \in R\text{-Mod}$ has a resolution by projective modules

$$0 \rightarrow S_l \rightarrow \dots \rightarrow S_1 \rightarrow S_0 \rightarrow M \rightarrow 0$$

of length at most l . The injective dimension is defined dually. For any ring R the projective and injective dimensions are equal, and this common number is called the global dimension of R [Wei95].

Lemma 12. *If ring R has global dimension l then ring RD has global dimension at most $l + 1$.*

Proof. Let $M \in RD\text{-Mod}$. Since R has global dimension l there exists a projective resolution

$$0 \rightarrow \tilde{S}_l \xrightarrow{\tilde{d}_l} \dots \xrightarrow{\tilde{d}_2} \tilde{S}_1 \xrightarrow{\tilde{d}_1} \tilde{S}_0 \xrightarrow{\tilde{d}_0} FM \rightarrow 0$$

in $(R\text{-Mod})^{D^\delta}$. For each morphism $j \rightarrow i$ in D of length one, we use the fact that $(\tilde{S}_0)_j$ is projective to find a lift $p_{i,j}^{S_0}$ making the following diagram commute.

$$\begin{array}{ccc} (\tilde{S}_0)_j & \xrightarrow{(\tilde{d}_0)_j} & M_j \\ p_{i,j}^{S_0} \downarrow & & \downarrow p_{i,j}^M \\ (\tilde{S}_0)_i & \xrightarrow{(\tilde{d}_0)_i} & M_i \end{array}$$

For $j \rightarrow i$ of length greater than one we define $p_{i,j}^{S_0}$ by composition. Hence we have an exact sequence

$$S_0 \xrightarrow{d_0} M \rightarrow 0$$

in $AD\text{-Mod}$, with each $(S_0)_j = (\tilde{S}_0)_j$ and with each $(d_0)_j = (\tilde{d}_0)_j$. Continuing inductively, we produce a resolution

$$0 \rightarrow S_l \xrightarrow{d_l} \dots \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \xrightarrow{d_0} M \rightarrow 0$$

in $(R\text{-Mod})^D$ that is not necessarily projective, but in which each $(S_l)_i \in R\text{-Mod}$ is projective. By Lemma 11 we have a projective resolution

$$\begin{array}{ccccccc} \dots & \rightarrow & \tilde{PFS}_l & \xrightarrow{\begin{bmatrix} \phi^{S_l} \\ -\tilde{P}F d_l \end{bmatrix}} & PFS_l & \xrightarrow{\begin{bmatrix} P F d_l & \phi^{S_{l-1}} \\ 0 & -\tilde{P} F d_{l-1} \end{bmatrix}} & \dots & \xrightarrow{\begin{bmatrix} P F d_2 & \phi^{S_1} \\ 0 & -\tilde{P} F d_1 \end{bmatrix}} & PS_1 & \xrightarrow{\begin{bmatrix} P F d_1 & \phi^{S_0} \end{bmatrix}} & PS_0 & \rightarrow & M & \rightarrow & 0 \\ & & & & \oplus & & & & \oplus & & & & & & & \\ & & & & \tilde{PFS}_{l-1} & & & & \tilde{PFS}_0 & & & & & & & \end{array}$$

of length at most $l + 1$. □

It follows that if R is a semisimple ring then RD has global dimension at most one, and if R is a hereditary ring then RD has global dimension at most two. In particular, Vect_k^D has global dimension at most one and \mathbf{Ab}^D has global dimension at most two.

4.6 Monads

Let \mathcal{C} be a category and let $\mathbb{1}_{\mathcal{C}}$ be its identity functor. A monad (T, η, μ) on \mathcal{C} consists of an endofunctor $T: \mathcal{C} \rightarrow \mathcal{C}$ together with a unit natural transformation $\eta: \mathbb{1}_{\mathcal{C}} \rightarrow T$ and a multiplication natural transformation $\mu: T^2 \rightarrow T$ such that the following diagrams commute [Mac98].

$$\begin{array}{ccc}
 T^3 & \xrightarrow{T\mu} & T^2 \\
 \mu T \downarrow & & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}
 \qquad
 \begin{array}{ccc}
 T & \xrightarrow{T\eta} & T^2 \\
 \eta T \downarrow & \searrow & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}$$

Monads are also often called triples. Every adjunction $F: \mathcal{C} \leftrightarrow \mathcal{C}': F'$ gives a monad on the category \mathcal{C} with $T = F'F$.

From a monad (T, η, μ) on category \mathcal{C} and an object $C \in \mathcal{C}$ we can produce an augmented cosimplicial object $C \rightarrow T^\bullet C \in \mathcal{C}$ [EM65, BB69]. It is defined by setting $(T^\bullet C)^n = T^{n+1}C$ for $n \geq -1$.

$$C \xrightarrow{d^0} TC \xrightarrow{d^0, d^1} T^2C \xrightarrow{d^0, d^1, d^2} T^3C \xrightarrow{d^0, \dots, d^3} T^4C \longrightarrow \dots$$

$\xleftarrow{s^0} \quad \xleftarrow{s^0, s^1} \quad \xleftarrow{s^0, s^1, s^2} \quad \xleftarrow{\quad}$

The coface maps d^i and codegeneracy maps s^i are given by

$$\begin{aligned} d^i &= T^i \eta T^{n-i} : T^n C \rightarrow T^{n+1} C \\ s^i &= T^i \mu T^{n-i} : T^{n+2} C \rightarrow T^{n+1} C. \end{aligned}$$

4.7 The homotopy spectral sequence of a cosimplicial space

The material in this section can be found in [BK72a, GJ09]. Let $X^\bullet \in c\mathcal{S}_*$ be a cosimplicial space. If X^\bullet is not fibrant, then choose a weak equivalence $X^\bullet \rightarrow Y^\bullet$ with Y^\bullet fibrant and replace X^\bullet with Y^\bullet . The extended homotopy spectral sequence for cosimplicial space X^\bullet is the extended homotopy spectral sequence of the following tower of fibrations.

$$\dots \rightarrow \text{Tot}_2 X^\bullet \rightarrow \text{Tot}_1 X^\bullet \rightarrow \text{Tot}_0 X^\bullet \rightarrow *$$

Let $E_r^{s,t}$ for $t \geq s \geq 0$ denote the E_r page of this second quadrant spectral sequence, whose differentials are of the form $d_r : E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$. The E_r page consists of abelian groups in dimensions $t - s \geq 2$, possibly non-abelian groups in dimension $t - s = 1$, and pointed sets in dimension $t - s = 0$. Under favorable conditions the spectral sequence for cosimplicial space X^\bullet converges to $\pi_{t-s} \text{Tot} X^\bullet$. If $X \rightarrow X^\bullet$ is an augmented cosimplicial space then we have a map $X \rightarrow \text{Tot} X^\bullet$.

For $r = 2$ there are natural isomorphisms $E_2^{s,t} \cong \pi^s \pi_t X^\bullet$ for $t \geq s \geq 0$, where π^s is defined as follows. For a cosimplicial abelian group $A^\bullet \in c\mathbf{Ab}$, let $\pi^s A^\bullet$ be the cohomology of the cochain complex

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \dots$$

with coboundary map $\partial = \sum (-1)^i d^i$ the alternating sum of the coface maps. For $\mathcal{G}^\bullet \in c\mathbf{Grp}$ a cosimplicial group that is not necessarily abelian, group $\pi^0 \mathcal{G}^\bullet$ is the equalizer of d^0 and d^1 , giving

$$\pi^0 \mathcal{G}^\bullet = \{g \in \mathcal{G}^0 \mid d^0 g = d^1 g\}.$$

Pointed set $\pi^1 \mathcal{G}^\bullet$ is equal to $Z^1 \mathcal{G}^\bullet / \mathcal{G}^0$, where $Z^1 \mathcal{G}^\bullet$ is the pointed set of cocycles

$$Z^1 \mathcal{G}^\bullet = \{x \in \mathcal{G}^1 \mid (d^2 x)(d^1 x)^{-1}(d^0 x) = e\},$$

and where \mathcal{G}^0 acts on $Z^1 \mathcal{G}$ via $g \cdot x = (d^0 g)x(d^1 g)^{-1}$. For S^\bullet a cosimplicial pointed set we let pointed set $\pi^0 S^\bullet$ be the equalizer of d^0 and d^1 .

4.8 Homotopy limits

Since \mathcal{S}_* has finite products we have an adjunction $F: \mathcal{S}_*^D \leftrightarrow \mathcal{S}_*^{D^\delta}: G$. We also write G for the resulting monad $G = GF$ on category \mathcal{S}_*^D . For $X \in \mathcal{S}_*^D$ we have an augmented cosimplicial object $X \rightarrow G^\bullet X$, and applying \lim_D gives an augmented cosimplicial space $\lim_D X \rightarrow \lim_D G^\bullet X$ called the cosimplicial replacement of X . The homotopy limit of X is defined as $\text{holim}_D X = \text{Tot } \lim_D G^\bullet X$, and if $X \in \mathcal{S}_*^D$ is fibrant then so is $\text{holim}_D X \in \mathcal{S}_*$ [GJ09, Hir09].

We give a second description of the homotopy limit. For an object i in D , let $B(D \downarrow i)^+$ be the nerve of the overcategory $(D \downarrow i)$ with a disjoint basepoint added. Then diagram $B(D \downarrow -)^+$ in \mathcal{S}_*^D has at object i the space $B(D \downarrow i)^+$. Since D is a small category, Proposition 18.2.6 of [Hir09] gives the following lemma.

Lemma 13. *For $X \in \mathcal{S}_*^D$, the homotopy limit $\text{holim}_D X$ is naturally isomorphic to $\text{hom}_D(B(D \downarrow -)^+, X)$.*

We now consider a spectral sequence that, under favorable circumstances, converges to the homotopy groups of a homotopy limit [GJ09]. Let $E_r^{s,t}$ denote the homotopy spectral sequence for augmented cosimplicial space $\lim_D X \rightarrow \lim_D G^\bullet X$. We describe the E_2 term using the right derived functors \lim_D^s of \lim_D . Since the limit functor is left exact, $\lim_D^0 = \lim_D$. For a diagram of abelian groups $A \in \mathbf{Ab}^D$, we have a natural isomorphism $\lim_D^s A \cong \pi^s \lim_D G^\bullet A$. By analogy, for a diagram of possibly non-abelian groups $\mathcal{G} \in \mathbf{Grp}^D$, we define $\lim_D^1 \mathcal{G} = \pi^1 \lim_D G^\bullet \mathcal{G}$. Since

$$(G^{m+1}X)_i = \prod_{i \rightarrow i_0 \rightarrow \dots \rightarrow i_m} X_{i_m}$$

and $\lim_D GX \cong \prod_i X_i$, it follows that

$$\pi_t \lim_D G^{m+1}X \cong \pi_t \prod_{i_0 \rightarrow \dots \rightarrow i_m} X_{i_m} \cong \prod_{i_0 \rightarrow \dots \rightarrow i_m} \pi_t X_{i_m} \cong \lim_D G^{m+1} \pi_t X.$$

Since $(G^\bullet X)^m = G^{m+1}X$, the E_2 term of this spectral sequence is

$$E_2^{s,t} \cong \pi^s \pi_t \lim_D G^\bullet X \cong \pi^s \lim_D G^\bullet \pi_t X \cong \lim_D^s \pi_t X.$$

Under favorable conditions the spectral sequence converges to

$$\pi_{t-s} \text{Tot} \lim_D G^\bullet X = \pi_{t-s} \text{holim}_D X.$$

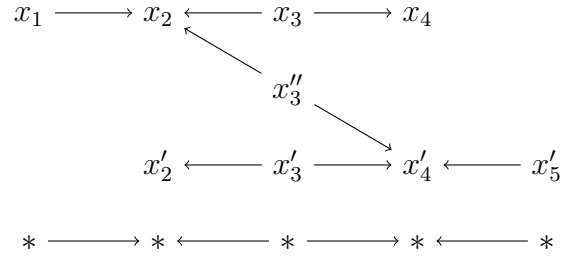
Since D is the free category generated from a quiver of type A_n , Lemma 12 implies $E_2^{s,t} = 0$ for $s > 2$.

Recall from Section 4.1 that we are interested in using homotopy limits to describe spaces of sections. The following two low-dimensional examples model the uncovered regions for the sensor networks in Figure 3.9, one with an evasion path and one without, before taking the disjoint union with a basepoint.

Example 3. Let D be the free category generated from quiver

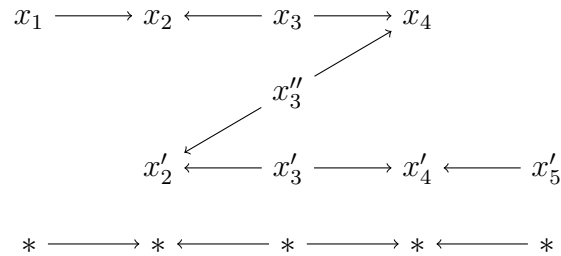


and let $X \in \mathcal{S}_*^D$ be the diagram below.



Each X_i is the union of a basepoint $*$ and some subset of the points x_i , x'_i , and x''_i . All terms on the E_2 page are trivial except for $E_2^{0,0} \cong \lim_D \pi_0 X = S^0$. By Corollary VI.2.21 of [GJ09], the spectral sequence converges completely to the homotopy groups of $\text{holim}_D X = S^0$. We recover the existence of an evasion path.

Example 4. Suppose $X \in \mathcal{S}_*^D$ is instead given by the following diagram.



In this case $E_2^{0,0} \cong \lim_D \pi_0 X = *$, and the spectral sequence converges completely to the homotopy groups of $\text{holim}_D X = *$. No evasion path exists.

The spectral sequence for a homotopy limit converges to desirable information, but its input depends on homotopy groups and maps between homotopy groups which are

not easily accessible. For this reason, in Section 5.2 we consider a homotopy spectral sequence for diagrams of spaces whose input depends on homology information.

Chapter 5

The Homotopy Spectral Sequence for Diagrams

The homotopy spectral sequence for a space with coefficients in a ring R goes from R -homology to R -homotopy, roughly speaking [BK72a, BK72b]. A generalization for function complexes between diagrams of spaces, which we refer to as the homotopy spectral sequence for diagrams, is given in [DZ87]. In this chapter we apply the homotopy spectral sequence for diagrams to study the space of evasion paths of a sensor network. The resulting spectral sequence has input depending on unstable invariants of the uncovered region and under favorable circumstances converges to information about the space of evasion paths. It remains to obtain these unstable invariants of the uncovered region from embedding invariants of the sensor network.

The homotopy spectral sequence for diagrams is constructed using the composition of two monads. The first monad R maps a based simplicial set to the R -module generated by its simplices; this monad is used to construct the homotopy spectral sequence for a space in Section 5.1. The second monad is the right adjoint G to the forgetful functor on diagrams, which we studied in Section 4.8 on the spectral sequence for a homotopy limit. Composing monads G and R produces a monad on

the category of diagrams of spaces, whose cosimplicial object is used to construct the homotopy spectral sequence for diagrams.

The diagrams of spaces studied in [DZ87] are indexed by a small simplicial category. Our setting is the specific case when the indexing category is generated by a quiver of type A_n . This allows us to simplify the proofs of Theorems 6 and 7 describing the E_2 term algebraically for coefficients modulo a prime.

The authors in [DMN89] construct an unstable Adams spectral sequence which is adapted to studying the space of sections of a fibration. We do not use their spectral sequence because the fibers of a fibration are all homotopy equivalent, but in sensor network problems the homotopy types of the covered and uncovered regions vary with time.

In Sections 5.1 and 5.2 we describe the homotopy spectral sequences for spaces and for diagrams. We study several low-dimensional examples relevant for sensor networks. In Sections 5.3 and 5.4 we give algebraic descriptions, when $R = \mathbb{Z}/p\mathbb{Z}$, of the E_2 term for spaces and for diagrams. We consider rational coefficients for spaces in Section 5.5. In Section 5.6 we show that a diagram of modules over a diagram of algebras is equivalent to a module over a more complicated algebra, and in Section 5.7 we construct projective resolutions over a diagram of algebras. We use these projective resolutions in Section 5.8 to study the rational E_2 page for diagrams of spaces.

5.1 The homotopy spectral sequence for spaces

We describe the homotopy spectral sequence of a space with coefficients in a ring [BK72a, BK72b, BK73]. Other useful expositions of this sequence include [GJ09, Mil84]. Let R be a ring and let \mathcal{S}_R be the category of simplicial R -modules. We have

an adjunction

$$R: \mathcal{S}_* \leftrightarrow \mathcal{S}_R: S, \tag{5.1}$$

where S is the forgetful functor and where R takes a pointed simplicial set $X \in \mathcal{S}_*$ to the simplicial R -module generated by the simplices of X . That is, $(RX)_n$ is the free R -module on X_n with the basepoint of X put equal to 0. We also write R for the resulting monad $R = RS$ on category \mathcal{S}_* . This monad (R, η, μ) has $\eta: X \rightarrow RX$ and $\mu: R^2X \rightarrow RX$ via

$$\eta(x) = [x] \quad \text{and} \quad \mu\left(\sum r'[\sum r[x]]\right) = \sum r'r[x].$$

Consider the homotopy spectral sequence for the fibrant augmented cosimplicial space $X \rightarrow R^\bullet X$. The total complex $R_\infty X = \text{Tot } R^\bullet X$ is the Bousfield–Kan R -completion of X , and under favorable conditions the spectral sequence converges to $\pi_* R_\infty X$. When $R = \mathbb{Z}/p\mathbb{Z}$, this produces an unstable Adams spectral sequence converging, for X nilpotent and for each $\pi_n X$ finitely generated, to $\pi_* X$ modulo torsion prime to p . For $R = \mathbb{Q}$ and X connected, the spectral sequence consists of the primitive elements in the rational cobar spectral sequence and converges to $\pi_* X \otimes \mathbb{Q}$. There is also a function complex version, which for $W, X \in \mathcal{S}_*$ is the homotopy spectral sequence built from the cosimplicial space $\text{hom}(W, X) \rightarrow \text{hom}(W, R^\bullet X)$.

5.2 The homotopy spectral sequence for diagrams

The homotopy spectral sequence for a space is generalized by Dror Farjoun and Zabrodsky in [DZ87] for function complexes between diagrams of spaces. Though they consider diagrams indexed by any small simplicial category, we restrict attention to the special case when the indexing category is generated by a quiver of type A_n .

Let D be the free category generated by a quiver of type A_n . The adjunction in

(5.1) induces an adjunction $R: \mathcal{S}_*^D \leftrightarrow \mathcal{S}_R^D: S$ between functors which, by an abuse of notation, we denote with the same names. Since category \mathcal{S}_R has finite products, by Lemma 7 we have an adjoint pair $F: \mathcal{S}_R^D \leftrightarrow \mathcal{S}_R^{D^\delta}: G$. We combine these adjunctions in the diagram

$$\mathcal{S}_*^D \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{S} \end{array} \mathcal{S}_R^D \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{S}_R^{D^\delta}$$

to get the adjunction $FR: \mathcal{S}_*^D \leftrightarrow \mathcal{S}_R^{D^\delta}: SG$. We define monad $T = (SG)(FR)$ on category \mathcal{S}_*^D as the composition of these adjoint functors. Monad T is equal to that of [DZ87] but we have defined it differently in order to simplify the proofs in Section 5.4.

For an object $X \in \mathcal{S}_*^D$ we form the augmented cosimplicial object $X \rightarrow T^\bullet X$ in $\mathcal{C}\mathcal{S}_*^D$. Let $W \in \mathcal{S}_*^D$ be cofibrant¹. Applying $\text{hom}_D(W, _)$ produces the augmented cosimplicial space $\text{hom}_D(W, X) \rightarrow \text{hom}_D(W, T^\bullet X)$. Let $E_r^{s,t}$ denote the E_r page of the homotopy spectral sequence. We have

$$E_2^{s,t} \cong \pi^s \pi_t \text{hom}_D(W, T^\bullet X),$$

and under favorable conditions the spectral sequence converges to

$$\pi_* \text{Tot } \text{hom}_D(W, T^\bullet X) \cong \pi_* \text{hom}_D(W, R_\infty X).$$

Since D has a finite number of objects and morphisms, Theorem 1.2 of [DZ87] gives the following.

Theorem 5. *Let $R = \mathbb{Z}/p\mathbb{Z}$. If W is cofibrant and W and X have finite $\mathbb{Z}/p\mathbb{Z}$ homology in each dimension, then the homotopy spectral sequence for $\text{hom}_D(W, T^\bullet X)$ converges completely to $\pi_* \text{hom}_D(W, R_\infty X)$.*

¹By [Dro87a, Dro87b], the function complex $\text{hom}_D(W, X)$ has some homotopy meaning even for W not cofibrant.

We are interested in the case when $W = B(D \downarrow -)^+$. By Lemma 13, the E_2 term is then

$$E_2^{s,t} \cong \pi^s \pi_t \operatorname{holim}_D T^\bullet X.$$

The spectral sequence converges to $\pi_{t-s} \operatorname{holim}_D R_\infty X$, the homotopy groups of the homotopy limit of the R -completion of the diagram. We obtain information related to the homotopy limit of X , which as we saw in Section 4.1 can model a space of evasion paths. The advantage of this spectral sequence for diagrams over the one in Section 4.8 is that algebraic descriptions of the E_2 term are more accessible.

The following two low-dimensional computations are analogous to Examples 3 and 4 from Section 4.8.

Example 5. Let D be the free category generated from quiver

$$\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet,$$

let $W = B(D \downarrow -)^+$, and let $X \in \mathcal{S}_*^D$ be the diagram from Example 3.

$$\begin{array}{ccccccc} x_1 & \longrightarrow & x_2 & \longleftarrow & x_3 & \longrightarrow & x_4 \\ & & & \swarrow & & & \\ & & & & x_3'' & \searrow & \\ & & & & & & \\ & & x_2' & \longleftarrow & x_3' & \longrightarrow & x_4' & \longleftarrow & x_5' \\ & & & & & & \\ * & \longrightarrow & * & \longleftarrow & * & \longrightarrow & * & \longleftarrow & * \end{array}$$

Each X_i is the union of a basepoint $*$ and some subset of the points x_i , x_i' , and x_i'' . All terms on the E_2 page are trivial except for $E_2^{0,0} \cong \pi^0 \pi_0 \operatorname{holim}_D T^\bullet X$. We have

$$\operatorname{holim}_D TX \simeq RX_1 \times RX_2 \times RX_3 \times RX_4 \times RX_5$$

and

$$\begin{aligned} \text{holim}_D T^2 X &\simeq \\ R[RX_1 \times RX_2] \times R^2 X_2 \times R[RX_2 \times RX_3 \times RX_4] \times R^2 X_4 \times R[RX_4 \times RX_5]. \end{aligned}$$

Let

$$b = (r_1 x_1, r_2 x_2 + r'_2 x'_2, r_3 x_3 + r''_3 x''_3 + r'_3 x'_3, r_4 x_4 + r'_4 x'_4, r'_5 x'_5) \in \pi_0 \text{holim}_D TX.$$

We have $d^0, d^1: \pi_0 \text{holim}_D TX \rightarrow \pi_0 \text{holim}_D T^2 X$ via

$$d^0 b = \begin{pmatrix} 1[r_1 x_1, r_2 x_2 + r'_2 x'_2] \\ 1[r_2 x_2 + r'_2 x'_2] \\ 1[r_2 x_2 + r'_2 x'_2, r_3 x_3 + r''_3 x''_3 + r'_3 x'_3, r_4 x_4 + r'_4 x'_4] \\ 1[r_4 x_4 + r'_4 x'_4] \\ 1[r_4 x_4 + r'_4 x'_4, r'_5 x'_5] \end{pmatrix}$$

and

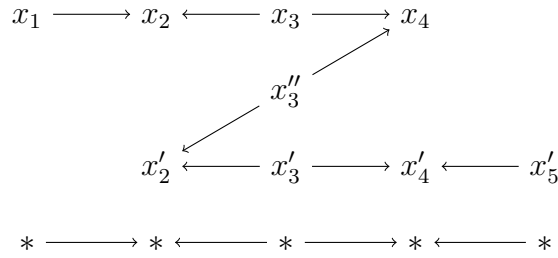
$$d^1 b = \begin{pmatrix} r_1[1x_1, 1x_2] \\ r_2[1x_2] + r'_2[1x'_2] \\ r_3[1x_2, 1x_3, 1x_4] + r''_3[1x_2, 1x''_3, 1x'_4] + r'_3[1x'_2, 1x'_3, 1x'_4] \\ r_4[1x_4] + r'_4[1x'_4] \\ r'_5[1x'_4, 1x'_5] \end{pmatrix}.$$

Hence

$$\begin{aligned}
 E_2^{0,0} &\cong \pi^0 \pi_0 \operatorname{holim}_D X \\
 &= \{b \in \pi_0 TX \mid d^0 b = d^1 b\} \\
 &= \{(*, *, *, *, *) \text{ and } (1x_1, 1x_2, 1x_3'', 1x_4', 1x_5')\} \\
 &= S^0.
 \end{aligned}$$

By Theorem 5 the spectral sequence converges completely to the homotopy groups of $\operatorname{holim}_D R_\infty X = S^0$. We recover the existence of an evasion path.

Example 6. Suppose $X \in \mathcal{S}_*^D$ is instead given by the diagram in Example 4.



In this case the third coordinate of $d^1 b$ changes to

$$r_3 [1x_2, 1x_3, 1x_4] + r_3'' [1x_2', 1x_3'', 1x_4] + r_3' [1x_2', 1x_3', 1x_4'].$$

Now $E_2^{0,0} \cong \{(*, *, *, *, *)\} = *$ and the spectral sequence converges completely to the homotopy groups of $\operatorname{holim}_D R_\infty X = *$. No evasion path exists.

5.3 The E_2 term for spaces when $R = \mathbb{Z}/p\mathbb{Z}$

Let p be prime and let $\mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements. For $R = \mathbb{Z}/p\mathbb{Z}$, the homotopy spectral sequence for spaces from Section 5.1 is an unstable Adams spectral

sequence. In this section we describe the E_2 term for $R = \mathbb{Z}/p\mathbb{Z}$ as an unstable Ext [BK72b]; other useful expositions include [GJ09, Mil84]. In the following section we consider the diagram case.

Let \mathcal{A} be the mod- p Steenrod algebra and let \mathcal{CA} be the category of unstable \mathcal{A} -coalgebras in [BK72b]. Let \mathcal{ML}_p be the category of connected graded $\mathbb{Z}/p\mathbb{Z}$ -modules, where by connected we mean trivial in nonpositive degrees. We have a pair of adjoint functors

$$J: \mathcal{CA} \leftrightarrow \mathcal{ML}_p: V. \tag{5.2}$$

Here J is the forgetful functor given by $(JC)_n = C_n$ for $C \in \mathcal{CA}$ and $n \geq 1$. For $M \in \mathcal{ML}_p$, functor V is given by

$$VM = \tilde{H}_* \left(\prod_{n=0}^{\infty} K(M_n, n); \mathbb{Z}/p\mathbb{Z} \right),$$

where each $K(M_n, n)$ is an Eilenberg-MacLane space. We also write V for the resulting monad $V = VJ$ on category \mathcal{CA} .

Let \mathcal{S}_{*c} be the category of connected pointed simplicial sets. Homology with coefficients in $\mathbb{Z}/p\mathbb{Z}$ is a functor

$$\tilde{H}_*(\ ; \mathbb{Z}/p\mathbb{Z}): \mathcal{S}_{*c} \rightarrow \mathcal{CA},$$

with comultiplication induced from the diagonal map $X \rightarrow X \times X$. Theorem 11.5 of [BK72b] states that for $X \in \mathcal{S}_{*c}$ and $t > s \geq 0$, we have a natural isomorphism

$$E_2^{s,t} \cong \pi^s \operatorname{Hom}_{\mathcal{CA}}(\tilde{H}_*(S^t; \mathbb{Z}/p\mathbb{Z}), V^\bullet \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})).$$

We can also describe the E_2 term as an unstable Ext. Let \mathcal{V} be the class of all $C \in \mathcal{CA}$ with $C \cong VM$ for some $M \in \mathcal{ML}_p$. By Proposition 7.8 of [Bou70], \mathcal{V}

is a class of injective models for \mathcal{CA} . If $B \in \mathcal{CA}$ has trivial comultiplication, for example if $B = \tilde{H}_*(S^t; \mathbb{Z}/p\mathbb{Z})$ for $t > 0$, then $\text{Hom}_{\mathcal{CA}}(B, _)$ is actually a functor

$$\text{Hom}_{\mathcal{CA}}(B, _): \mathcal{CA} \rightarrow \mathbb{Z}/p\mathbb{Z}\text{-Mod}.$$

For such B we define the unstable Ext functors $\text{Ext}_{\mathcal{CA}}^s(B, _)$ as the right derived functors

$$\text{Ext}_{\mathcal{CA}}^s(B, _) = R^s \text{Hom}_{\mathcal{CA}}(B, _): \mathcal{CA} \rightarrow \mathbb{Z}/p\mathbb{Z}\text{-Mod}$$

with respect to \mathcal{V} . Theorem 12.1 of [BK72b] states that for $X \in \mathcal{S}_{*c}$ and $t > s \geq 0$, we have a natural isomorphism

$$E_2^{s,t} \cong \text{Ext}_{\mathcal{CA}}^s(\tilde{H}_*(S^t; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})).$$

5.4 The E_2 term for diagrams when $R = \mathbb{Z}/p\mathbb{Z}$

The diagrams of spaces studied in [DZ87] are indexed by a small simplicial category, which is more general than our context in which the free category D is generated by a quiver of type A_n . By restricting to this special case, we simplify the proofs of Theorems 6 and 7 describing the E_2 term algebraically for $R = \mathbb{Z}/p\mathbb{Z}$. In addition, our proofs more closely follow [BK72b].

The adjunction in (5.2) induces an adjunction $J: \mathcal{CA}^D \leftrightarrow \mathcal{ML}_p^D: V$ between functors which, by an abuse of notation, we denote with the same names. Since category \mathcal{ML}_p has finite products given by degreewise product, Lemma 7 gives an adjoint pair $F: \mathcal{ML}_p^D \leftrightarrow \mathcal{ML}_p^{D^\delta}: G$. We combine these adjunctions in the diagram

$$\mathcal{CA}^D \begin{array}{c} \xrightarrow{J} \\ \xleftarrow{V} \end{array} \mathcal{ML}_p^D \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{ML}_p^{D^\delta}$$

to get the adjunction $FJ: \mathcal{CA}^D \leftrightarrow \mathcal{ML}_p^{D^\delta}: VG$. We define monad $U = (VG)(FJ)$

on category \mathcal{CA}^D as the composition of these adjoint functors. So for an object $C \in \mathcal{CA}^D$ we can form the cosimplicial object $U^\bullet C \in \mathcal{CA}^D$.

Recall \mathcal{S}_{*c} is the category of connected pointed simplicial sets, and let $\mathcal{S}_{p,c}$ be the category of connected simplicial $\mathbb{Z}/p\mathbb{Z}$ -modules. Monad U is closely related to the restriction of monad T to \mathcal{S}_{*c}^D .

Lemma 14. *The rectangles in the following diagram commute up to natural isomorphism.*

$$\begin{array}{ccc}
 \mathcal{S}_{*c}^D & \xrightarrow{\tilde{H}_*(\cdot; \mathbb{Z}/p\mathbb{Z})} & \mathcal{CA}^D \\
 \begin{array}{c} \uparrow S \\ R \downarrow \end{array} & & \begin{array}{c} \uparrow V \\ J \downarrow \end{array} \\
 \mathcal{S}_{p,c}^D & \xrightarrow{\pi_*} & \mathcal{ML}_p^D \\
 \begin{array}{c} \uparrow G \\ F \downarrow \end{array} & & \begin{array}{c} \uparrow G \\ F \downarrow \end{array} \\
 \mathcal{S}_{p,c}^{D^\delta} & \xrightarrow{\pi_*} & \mathcal{ML}_p^{D^\delta}
 \end{array}$$

Proof. The top rectangle commutes up to natural isomorphism by Lemma 11.6 of [BK72b]. For the bottom rectangle, note that $\pi_*F = F\pi_*$ is obvious and that $\pi_*G \cong G\pi_*$ follows from the fact that the homotopy groups of a finite product are the products of the homotopy groups. □

In [DZ87], the functor G is defined more generally and the diagram above does not fully commute. Hence the proof of their more general version of Theorem 6 requires additional steps.

Corollary 1. *Let $X \in \mathcal{S}_{*c}^D$. Then*

$$\tilde{H}_*(T^\bullet X; \mathbb{Z}/p\mathbb{Z}) \cong U^\bullet \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})$$

is a natural isomorphism of cosimplicial objects over \mathcal{CA}^D .

Proof. This follows from Lemma 14. Note

$$\begin{aligned}
 \tilde{H}_*(T^k X; \mathbb{Z}/p\mathbb{Z}) &= \tilde{H}_*((SGFR)^k X; \mathbb{Z}/p\mathbb{Z}) && \text{by definition of } T \\
 &\cong (VGFJ)^k \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z}) && \text{by Lemma 14} \\
 &= U^k \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z}) && \text{by definition of } U.
 \end{aligned}$$

□

Lemma 15. *Let $W \in \mathcal{S}_*^D$ be cofibrant and let $Y \in \mathcal{S}_{p,c}^D$. Then for $t \geq 1$, the functor $\tilde{H}_*(\ ; \mathbb{Z}/p\mathbb{Z})$ induces isomorphisms*

$$\pi_t \operatorname{hom}_D(W, Y) \cong [\Sigma^t W, Y] \cong \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(Y; \mathbb{Z}/p\mathbb{Z})).$$

Proof. The first isomorphism is Lemma 5. The second isomorphism follows from the naturality of Lemma 11.8 of [BK72b]. □

We obtain the following analogue of Theorem 11.5 from [BK72b].

Theorem 6. *Let $W \in \mathcal{S}_*^D$ be cofibrant and let $X \in \mathcal{S}_{*c}^D$. Then for $t > s \geq 0$, we have a natural isomorphism*

$$E_2^{s,t} \cong \pi^s \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), U^\bullet \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})).$$

Proof.

$$\begin{aligned}
 E_2^{s,t} &\cong \pi^s \pi_t \operatorname{hom}_D(W, T^\bullet X) \\
 &\cong \pi^s \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(T^\bullet X; \mathbb{Z}/p\mathbb{Z})) && \text{by Lemma 15} \\
 &\cong \pi^s \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), U^\bullet \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})) && \text{by Corollary 1.}
 \end{aligned}$$

□

This theorem is a special case of the results in [DZ87], but for D the free category generated by a quiver of type A_n we have given a simpler proof that more closely follows [BK72b].

We can also describe the E_2 term as an unstable Ext. Let \mathcal{U} be the class of all $C \in \mathcal{C}\mathcal{A}^D$ with $C \cong UM$ for some $M \in \mathcal{M}\mathcal{L}_p^{D\delta}$. By Proposition 7.8 of [Bou70], \mathcal{U} is a class of injective models for $\mathcal{C}\mathcal{A}^D$. If $B \in \mathcal{C}\mathcal{A}^D$ has trivial comultiplication at each object in D , for example if $B = \tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z})$ for some cofibrant $W \in \mathcal{S}_*^D$ and $t > 0$, then functor $\text{Hom}_{\mathcal{C}\mathcal{A}^D}(B, \quad)$ is actually a functor

$$\text{Hom}_{\mathcal{C}\mathcal{A}^D}(B, \quad): \mathcal{C}\mathcal{A}^D \rightarrow \mathbb{Z}/p\mathbb{Z}\text{-Mod}.$$

By Remark 7.9 of [Bou70], the right derived functors with respect to \mathcal{U} satisfy

$$R^s \text{Hom}_{\mathcal{C}\mathcal{A}^D}(B, C) \cong \pi^s \text{Hom}_{\mathcal{C}\mathcal{A}^D}(B, U^\bullet C)$$

for $C \in \mathcal{C}\mathcal{A}^D$. For such B , we define the unstable Ext functors $\text{Ext}_{\mathcal{C}\mathcal{A}^D}^s(B, \quad)$ by

$$\text{Ext}_{\mathcal{C}\mathcal{A}^D}^s(B, \quad) = R^s \text{Hom}_{\mathcal{C}\mathcal{A}^D}(B, \quad): \mathcal{C}\mathcal{A}^D \rightarrow \mathbb{Z}/p\mathbb{Z}\text{-Mod}.$$

We have the following analogue of Theorem 12.1 from [BK72b].

Theorem 7. *Let $X \in \mathcal{S}_{*c}^D$. Then for $t > s \geq 0$, we have a natural isomorphism*

$$E_2^{s,t} \cong \text{Ext}_{\mathcal{C}\mathcal{A}^D}^s(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})).$$

Proof.

$$\begin{aligned}
 E_2^{s,t} &\cong \pi^s \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), U^\bullet \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})) && \text{by Theorem 6} \\
 &\cong R^s \operatorname{Hom}_{\mathcal{C}\mathcal{A}^D}(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})) \\
 &= \operatorname{Ext}_{\mathcal{C}\mathcal{A}^D}^s(\tilde{H}_*(\Sigma^t W; \mathbb{Z}/p\mathbb{Z}), \tilde{H}_*(X; \mathbb{Z}/p\mathbb{Z})).
 \end{aligned}$$

□

5.5 The E_2 term for spaces when $R = \mathbb{Q}$

To make computations more accessible, for the remainder of the chapter we consider rational coefficients. In this section we return to the homotopy spectral sequence for spaces from Section 5.1 and discuss the E_2 page when $R = \mathbb{Q}$. In the following sections we consider the diagram case.

Let $X \in \mathcal{S}_{*c}$ with $H_*(X; \mathbb{Q})$ of finite type. By Corollary 15.6 of [BK73] we have

$$E_2 \cong P \operatorname{Cotor}^{H_*(X; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}),$$

where P denotes the set of primitive elements. When $H_*(X; \mathbb{Q})$ is finite in each dimension, the Hopf algebra $\operatorname{Cotor}^{H_*(X; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q})$ is equivalent to the classical cohomology $\operatorname{Ext}_{H^*(X; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q})$. Under favorable circumstances this spectral sequence converges to $\pi_* \mathbb{Q}_\infty X \cong \pi_* X \otimes \mathbb{Q}$.

The rational homotopy spectral sequence for spaces is closely related to the Eilenberg–Moore spectral sequence for the fibration $\Omega X \rightarrow \mathcal{P}X \rightarrow X$, where path space $\mathcal{P}X$ is contractible. This second quadrant spectral sequence has E_2 page given by $\operatorname{Cotor}^{H_*(X; k)}(\mathbb{Q}, \mathbb{Q})$ and under favorable circumstances converges to $H_*(\Omega X; \mathbb{Q})$. Note

that upon taking primitives we obtain

$$PH_*(\Omega X; \mathbb{Q}) \cong \pi_*(\Omega X) \otimes \mathbb{Q} \cong \pi_{*+1}(X) \otimes \mathbb{Q}.$$

5.6 Diagrams of algebras

In order to study the rational E_2 page for diagrams of spaces, in this section we define a category \mathcal{M} of diagrams of modules over a diagram of algebras. We show that \mathcal{M} is equivalent to the category of modules over a single algebra AD . This equivalence allows us, in Section 5.8, to define the derived functors Ext over a diagram of algebras.

Let D be the free category generated from a quiver of type A_n . Let $A \in (k\text{-Alg})^D$ be a diagram of k -algebras

$$A_1 \leftrightarrow A_2 \leftrightarrow \dots \leftrightarrow A_n,$$

where each arrow points either to the left or to the right. We define the category \mathcal{M} as follows. An object $M \in \mathcal{M}$ is a diagram

$$M_1 \leftrightarrow M_2 \leftrightarrow \dots \leftrightarrow M_n$$

in Vect_k^D such that

- each $M_i \in A_i\text{-Mod}$, and
- if there is a morphism in D from object j to i , then $p_{i,j}^M(a_j m_j) = p_{i,j}^A(a_j) p_{i,j}^M(m_j)$ for each $a_j \in A_j$ and $m_j \in M_j$.

The second condition is equivalent to saying that $p_{i,j}^M$ is a morphism of A_j -modules, where M_i is an A_j -module via restriction of scalars. A morphism $h: M \rightarrow N$ in \mathcal{M} is a morphism

$$\begin{array}{ccccccc}
 M_1 & \longleftrightarrow & M_2 & \longleftrightarrow & \dots & \longleftrightarrow & M_n \\
 \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n \\
 N_1 & \longleftrightarrow & N_2 & \longleftrightarrow & \dots & \longleftrightarrow & N_n
 \end{array}$$

in Vect_k^D where each h_i is also a morphism in $A_i\text{-Mod}$.

Recall from Theorem 1 that Vect_k^D is equivalent to the category of modules over the path algebra kD . In Theorem 8 we prove an analogous result for the category \mathcal{M} . First we must define the k -algebra AD .

Definition 3. Let D be the free category generated from a quiver of type A_n , let k be a field, and let $A \in (k\text{-Alg})^D$ be a diagram of algebras. The k -algebra AD is defined as follows. An element of AD is a matrix

$$a = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix},$$

where each $a_{i,j} \in A_i$ and where $a_{i,j} = 0$ if there is no morphism in D from object j to i . For $a, b \in AD$, their product ab is given by

$$(ab)_{i,j} = a_{i,1}p_{i,1}^A(b_{1,j}) + \dots + a_{i,n}p_{i,n}^A(b_{n,j}).$$

The algebra structure comes from the embedding of k into the center of AD that sends $r \in k$ to the diagonal matrix with r in each diagonal entry.

Suppose A is a diagram of graded k -algebras with A_i^m the grading of k -algebra A_i at level m . Then AD is a graded k -algebra with $AD^m = \{a \in AD \mid \text{each } a_{i,j} \in A_i^m\}$. Let $AD\text{-Mod}$ be the category of left AD -modules.

Theorem 8. *The categories \mathcal{M} and $AD\text{-Mod}$ are equivalent.*

Proof. We only give the correspondence defining the equivalence. Given $M \in \mathcal{M}$, we define module $\tilde{M} \in AD\text{-Mod}$ by $\tilde{M} = \bigoplus_{i=1}^n M_i$. For

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \in AD \quad \text{and} \quad m = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \in \tilde{M},$$

we define am by letting its i -th component in \tilde{M} be

$$a_{i,1}p_{i,1}^M(m_1) + \cdots + a_{i,n}p_{i,n}^M(m_n).$$

For the reverse direction, let $e_{i,j}$ be the matrix with a 1 in entry i, j and with zeros elsewhere. Given $\tilde{M} \in AD\text{-Mod}$, we define a diagram $M \in \mathcal{M}$ by setting $M_i = e_{i,i}\tilde{M}$. For $m_j \in M_j$ we set

$$p_{i,j}^{\tilde{M}}(m_j) = e_{i,j}m_j = e_{i,i}e_{i,j}m_j \in M_i.$$

It is possible to check that the correspondences above are inverses and that morphisms behave appropriately. □

We work interchangeably with the equivalent categories \mathcal{M} and $AD\text{-Mod}$. As the the category of modules over a ring, $AD\text{-Mod}$ is an abelian category with enough projectives. A morphism $h: M \rightarrow N$ in $AD\text{-Mod}$ is a monomorphism if each h_i is a monomorphism in $A_i\text{-Mod}$, and h is an epimorphism if each h_i is an epimorphism. The kernel $\ker(h)$ has $\ker(h)_i = \ker(h_i)$, with the morphisms $p_{i,j}^{\ker(h)}$ defined by restricting the morphisms $p_{i,j}^M$. The cokernel $\text{coker}(h)$ has $\text{coker}(h)_i = \text{coker}(h_i)$, with the morphisms $p_{i,j}^{\text{coker}(h)}$ induced from the morphisms $p_{i,j}^N$.

5.7 Projective resolutions over a diagram of algebras

In this section we construct convenient projective resolutions over a diagram of algebras. In Section 5.8 we use these resolutions to compute the derived functors Ext over a diagram of algebras, which is relevant for the rational E_2 term of the homotopy spectral sequence for diagrams of spaces.

Let D be the free category generated by a quiver of type A_n . We have forgetful functors

$$F_j: AD\text{-Mod} \rightarrow A_j\text{-Mod}$$

and

$$F: AD\text{-Mod} \rightarrow \bigoplus_{j=1}^n (A_j\text{-Mod}).$$

The following lemmas are analogous to those in Sections 4.4 and 4.5.

Lemma 16. *There is an adjunction $Q_j: A_j\text{-Mod} \leftrightarrow AD\text{-Mod}: F_j$.*

For $N \in A_j\text{-Mod}$ we define $Q_j N$ using extension of scalars. If there is a morphism $j \rightarrow i$ in D then $(Q_j N)_i = A_i \otimes_{A_j} N$ and $p_{i,j}^{Q_j N} = p_{i,j}^A \otimes \mathbb{1}$; otherwise both $(Q_j N)_i$ and $p_{i,j}^{Q_j N}$ are zero.

Lemma 17. *If $N \in A_j\text{-Mod}$ is projective then so is $Q_j N \in AD\text{-Mod}$.*

Lemma 18. *There is an adjunction $Q: \bigoplus_{j=1}^n (A_j\text{-Mod}) \leftrightarrow AD\text{-Mod}: F$.*

For $N \in \bigoplus_{j=1}^n (A_j\text{-Mod})$ we have $QN = \bigoplus_j Q_j N_j$. That is,

$$(QN)_i = \bigoplus_{j \rightarrow i} A_i \otimes_{A_j} N_j,$$

where the coproduct is over the morphisms in D with target i . For example, suppose

the quiver underlying D is

$$\bullet_1 \rightarrow \bullet_2 \leftarrow \bullet_3 \rightarrow \bullet_4 \leftarrow \bullet_5$$

and $N \in \oplus_{j=1}^n (A_j\text{-Mod})$ is

$$N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5.$$

Then diagram QN is

$$N_1 \rightarrow (A_2 \otimes_{A_1} N_1) \oplus N_2 \oplus (A_2 \otimes_{A_3} N_3) \leftarrow N_3 \rightarrow (A_4 \otimes_{A_3} N_3) \oplus N_4 \oplus (A_4 \otimes_{A_5} N_5) \leftarrow N_5.$$

To construct projective resolutions in $AD\text{-Mod}$ we define a functor

$$\tilde{Q}: \oplus_{j=1}^n (A_j\text{-Mod}) \rightarrow AD\text{-Mod}.$$

Recall the length $|j \rightarrow i|$ of a morphism $j \rightarrow i$ in D is the number of arrows in the corresponding path in the underlying quiver. For $N \in \oplus_{j=1}^n (A_j\text{-Mod})$, we define $\tilde{Q}N = \oplus_{|j \rightarrow i|=1} Q_i N_j$. Moreover, given $M \in AD\text{-Mod}$, we define morphism $\phi^M: \tilde{Q}FM \rightarrow QFM$ by defining each $\phi_i^M: (\tilde{Q}FM)_i \rightarrow (QFM)_i$. If $|j \rightarrow i| \geq 1$ and $(a, m) \in A_i \otimes M_j \subset (\tilde{Q}FM)_i$, then

$$\phi_i^M(a, m) = \begin{cases} a \otimes m & \text{in summand } A_i \otimes M_j \text{ of } (QFM)_i \\ -p_{i,j}^A(a) \otimes p_{i,j}^M(m) & \text{in summand } A_i \otimes M_i \text{ of } (QFM)_i \\ 0 & \text{in all other summands.} \end{cases}$$

Lemma 19. *Suppose*

$$\dots \xrightarrow{d_{l+1}} S_l \xrightarrow{d_l} \dots \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \xrightarrow{d_0} M \rightarrow 0$$

is a resolution in $AD\text{-Mod}$ with each $(S_l)_j \in A_j\text{-Mod}$ projective. Then

$$\begin{array}{ccccccc} \dots & \rightarrow & QFS_3 & \xrightarrow{\begin{bmatrix} QFd_3 & \phi^{S_2} \\ 0 & -\tilde{Q}Fd_2 \end{bmatrix}} & QFS_2 & \xrightarrow{\begin{bmatrix} QFd_2 & \phi^{S_1} \\ 0 & -\tilde{Q}Fd_1 \end{bmatrix}} & QFS_1 & \xrightarrow{\begin{bmatrix} QFd_1 & \phi^{S_0} \end{bmatrix}} & QFS_0 & \rightarrow & M & \rightarrow & 0 \\ & & \oplus & & \oplus & & \oplus & & & & & & & \\ & & \tilde{Q}FS_2 & & \tilde{Q}FS_1 & & \tilde{Q}FS_0 & & & & & & & \end{array}$$

is a projective resolution in $AD\text{-Mod}$.

Proof. The proof is analogous to that of Lemma 11. □

Let $M \in AD\text{-Mod}$, and suppose we have a projective resolution

$$\dots \xrightarrow{\tilde{d}_{l+1}} \tilde{S}_l \xrightarrow{\tilde{d}_l} \dots \xrightarrow{\tilde{d}_2} \tilde{S}_1 \xrightarrow{\tilde{d}_1} \tilde{S}_0 \xrightarrow{\tilde{d}_0} FM \rightarrow 0$$

of FM in $\bigoplus_{j=1}^n(A_j\text{-Mod})$. This is exactly the data of a projective resolution for each M_j in $A_j\text{-Mod}$. We show how to combine the resolutions for each M_j into a projective resolution for M in $AD\text{-Mod}$. For each morphism $j \rightarrow i$ in D of length one, we use the fact that $(\tilde{S}_0)_j$ is projective to find a lift $p_{i,j}^{S_0}$ making the following diagram in $A_j\text{-Mod}$ commute.

$$\begin{array}{ccc} (\tilde{S}_0)_j & \xrightarrow{(\tilde{d}_0)_j} & M_j \\ p_{i,j}^{S_0} \downarrow & & \downarrow p_{i,j}^M \\ (\tilde{S}_0)_i & \xrightarrow{(\tilde{d}_0)_i} & M_i \end{array}$$

The bottom row lives in $A_j\text{-Mod}$ via restriction of scalars. For $j \rightarrow i$ of length greater than one we define $p_{i,j}^{S_0}$ by composition. Hence we have an exact sequence

$$S_0 \xrightarrow{d_0} M \rightarrow 0$$

in $AD\text{-Mod}$, with each $(S_0)_j = (\tilde{S}_0)_j$ and with each $(d_0)_j = (\tilde{d}_0)_j$. Continuing inductively, we produce a resolution

$$\dots \xrightarrow{d_{l+1}} S_l \xrightarrow{d_l} \dots \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \xrightarrow{d_0} M \rightarrow 0$$

that is not necessarily projective, but in which each $(S_l)_j \in A_j\text{-Mod}$ is projective. Lastly, we use Lemma 19 to produce a projective resolution of M in $AD\text{-Mod}$. In the following section, we use this method for producing projective resolutions to compute the derived functor Ext over a diagram of algebras.

5.8 The E_2 term for diagrams when $R = \mathbb{Q}$

In this section we illustrate a computation of the derived functor Ext over a diagram of algebras. Such computations are relevant for the rational E_2 term of the homotopy spectral sequence for diagrams.

We follow [McC01] in our exposition of the derived functor Ext . Let $A \in (\mathbb{Q}\text{-Alg})^D$ be a diagram of graded algebras over \mathbb{Q} ; thus AD is a graded algebra over \mathbb{Q} . The suspension functor $s: AD\text{-Mod} \rightarrow AD\text{-Mod}$ acts on $M \in AD\text{-Mod}$ via $(sM)_n = M_{n-1}$, where for $\gamma \in AD$ of degree $|\gamma|$ and $m \in M$ we have $\gamma \cdot (sm) = (-1)^{|\gamma|} s(\gamma \cdot m)$. For $M, N \in AD\text{-Mod}$ we define

$$\text{Hom}_{AD}^t(M, N) = \text{Hom}_{AD}(M, s^t N).$$

Then $\text{Ext}_{AD}(\quad, N)$ is the right derived functor of $\text{Hom}_{AD}^*(\quad, N)$.

We describe the case we are most interested in, $\text{Ext}_{AD}(\Theta\mathbb{Q}, \Theta\mathbb{Q})$, in more detail. Let $\Theta\mathbb{Q} \in AD\text{-Mod}$ be the constant diagram with \mathbb{Q} at each object in D and with identity maps in-between. We build a projective resolution

$$\dots \xrightarrow{d_3} S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \xrightarrow{d_0} \Theta\mathbb{Q} \rightarrow 0$$

of $\Theta\mathbb{Q}$ in $AD\text{-Mod}$. We apply $\text{Hom}_{AD}^t(\quad, \Theta\mathbb{Q})$ and leave out the first nontrivial term to get

$$\dots \xleftarrow{(d_3)^*} \text{Hom}_{AD}^t(S_2, \Theta\mathbb{Q}) \xleftarrow{(d_2)^*} \text{Hom}_{AD}^t(S_1, \Theta\mathbb{Q}) \xleftarrow{(d_1)^*} \text{Hom}_{AD}^t(S_0, \Theta\mathbb{Q}) \leftarrow 0.$$

The cohomology of this sequence is the derived functor

$$\text{Ext}_{AD}^{s,t}(\Theta\mathbb{Q}, \Theta\mathbb{Q}) = (R^s \text{Hom}_{AD}^t(\quad, \Theta\mathbb{Q}))(\Theta\mathbb{Q}).$$

In the homotopy spectral sequence for a space $X \in \mathcal{S}_{*c}$, the rational E_2 term is given by the primitive elements in $\text{Ext}_{H^*(X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q})$. Now consider a diagram of spaces $X \in \mathcal{S}_{*c}^{Dop}$. We apply rational cohomology to get $A = H^*(X; \mathbb{Q}) \in (\mathbb{Q}\text{-Alg})^D$. Though we have not derived the rational E_2 term for the homotopy spectral sequence for diagrams, we expect it will be related to the primitives in $\text{Ext}_{AD}(\Theta\mathbb{Q}, \Theta\mathbb{Q})$. This is motivated by Example 7, in which we perform a computation of $\text{Ext}_{AD}(\Theta\mathbb{Q}, \Theta\mathbb{Q})$.

Example 7. Let $Y \in \mathcal{S}_{*c}$ have finite-dimensional rational cohomology. Let $X \in \mathcal{S}_{*c}^{Dop}$ and $A = H^*(X; \mathbb{Q}) \in (\mathbb{Q}\text{-Alg})^D$ be as below.

$$\begin{aligned} X &= \quad * \longrightarrow Y \longleftarrow * \\ A &= \quad \mathbb{Q} \longleftarrow H^*(Y; \mathbb{Q}) \longrightarrow \mathbb{Q} \end{aligned}$$

The particular case when $Y = S^1$ is Example 1. Let S_\bullet be a minimal projective resolution of \mathbb{Q} in $H^*(Y; \mathbb{Q})\text{-Mod}$.

$$\dots \xrightarrow{d_4} S_3 \xrightarrow{d_3} S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} H^*(Y; \mathbb{Q}) \xrightarrow{d_0} \mathbb{Q} \rightarrow 0$$

We build the following projective resolution of $\Theta\mathbb{Q}$ in $AD\text{-Mod}$, which is modified from the resolution in Lemma 19 to be smaller.

$$\begin{array}{ccccc}
 & \vdots & & \vdots & & \vdots \\
 \mathbf{1} \otimes \begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix} & \downarrow & & \downarrow d_4 & & \downarrow \mathbf{1} \otimes \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix} \\
 \mathbb{Q} \otimes (S_2 \oplus S_3) & \longleftarrow & S_3 & \longrightarrow & \mathbb{Q} \otimes (S_3 \oplus S_2) \\
 \mathbf{1} \otimes \begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix} & \downarrow & & \downarrow d_3 & & \downarrow \mathbf{1} \otimes \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix} \\
 \mathbb{Q} \otimes (S_1 \oplus S_2) & \longleftarrow & S_2 & \longrightarrow & \mathbb{Q} \otimes (S_2 \oplus S_1) \\
 \mathbf{1} \otimes \begin{bmatrix} \mathbf{1} & 0 \end{bmatrix} & \downarrow & & \downarrow d_2 & & \downarrow \mathbf{1} \otimes \begin{bmatrix} 0 & \mathbf{1} \end{bmatrix} \\
 \mathbb{Q} \otimes S_1 & \longleftarrow & S_1 & \longrightarrow & \mathbb{Q} \otimes S_1 \\
 0 & \downarrow & & \downarrow d_1 & & \downarrow 0 \\
 \mathbb{Q} & \longleftarrow & H^*(Y; \mathbb{Q}) & \longrightarrow & \mathbb{Q} \\
 \mathbf{1} & \downarrow & & \downarrow d_0 & & \downarrow \mathbf{1} \\
 \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} \\
 & \downarrow & & \downarrow & & \downarrow \\
 0 & \longleftarrow & 0 & \longrightarrow & 0
 \end{array}$$

This resolution contains the projective chain complex in $AD\text{-Mod}$ generated by S_\bullet , and shifted copies of $\mathbb{Q} \otimes S_\bullet$ have been added to the left and right columns in order to preserve exactness. After applying $\text{Hom}_{AD}^*(\ , \Theta\mathbb{Q})$ and taking cohomology, cancellation leaves one shifted copy of the cohomology of $\text{Hom}_{H^*(Y; \mathbb{Q})}^*(S_\bullet, \Theta\mathbb{Q})$. That is,

we have

$$\mathrm{Ext}_{AD}^{s,t}(\Theta\mathbb{Q}, \Theta\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & (s, t) = (0, 0) \\ \mathrm{Ext}_{H^*(Y; \mathbb{Q})}^{s-1, t}(\mathbb{Q}, \mathbb{Q}) & s \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

If we define the primitives in $\mathrm{Ext}_{AD}(\Theta\mathbb{Q}, \Theta\mathbb{Q})$ to be the shifted copies of the primitives in $\mathrm{Ext}_{H^*(Y; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q})$, then the E_2 page for diagram X is congruent to a shifted copy of the E_2 page for space Y , and converges as expected to

$$\pi_{t-s+1}(\mathbb{Q}_\infty Y) \cong \pi_{t-s}(\Omega\mathbb{Q}_\infty Y) \cong \pi_{t-s} \mathrm{holim}_{D^{op}} \mathbb{Q}_\infty X.$$

It remains to define an appropriate notion of the primitives in $\mathrm{Ext}_{AD}(\Theta\mathbb{Q}, \Theta\mathbb{Q})$ for A the cohomology of an arbitrary diagram of spaces over D .

Chapter 6

Conclusions

In this thesis we study an evasion problem for mobile sensor networks in which the sensors don't know their locations and instead measure only local connectivity data. De Silva and Ghrist in [dG06] provide a homological criterion, depending on this limited input, which rules out the existence of an evasion path in many sensor networks. We use zigzag persistence to produce a criterion of equivalent discriminatory power that is more amenable to streaming computation, which is an important feature for sensor networks moving over a long period of time.

It turns out that no method relying on connectivity data alone can determine in all cases if an evasion path exists. Indeed, we provide examples showing that the fibrewise homotopy type of the sensor network does not determine the existence of an evasion path; the embedding of the sensor network in spacetime also matters. We therefore consider a stronger model of sensors which measure cyclic orders and weak distance information, and given this model we provide necessary and sufficient conditions for the existence of an evasion path. We are quite interested in the open question stated in Section 3.5: can one determine the existence of an evasion path using only the additional cyclic order data but no additional weak distance data?

The evasion problem motivates a more theoretical question: what is the entire

space of evasion paths? This is the space of sections in the uncovered region of the sensor network, which we show can be modeled as the homotopy limit of a corresponding diagram of spaces. We apply the homotopy spectral sequence for function complexes between diagrams of spaces, which in our context converges to the homotopy groups of the homotopy limit of the R -completion of the diagram. Our setting is a specific case of the spectral sequence due to [DZ87], and for coefficients modulo a prime we provide a simpler derivation of their algebraic description of the E_2 term. We show that a diagram of modules over a diagram of algebras is equivalent to a single module over a more complicated algebra, and this allows us to perform Ext calculations related to the E_2 term for rational coefficients. The spectral sequence for diagrams has input depending on unstable invariants of the uncovered region of the sensor network, but it remains to obtain these unstable invariants of the uncovered region from embedding invariants of the sensor network. One idea is to try to use the tools of embedding calculus [Wei99] in a fibrewise setting.

Appendix A

The Vietoris–Rips complex

In this appendix we explain the Vietoris–Rips approximation to the Čech complex. The Vietoris–Rips complex is the maximal simplicial complex built on top of the connectivity graph, meaning a simplex is present in the complex if all its edges are in the connectivity graph [Vie27]. See Figure A.1 for an example. Hence the Vietoris–Rips complex can be constructed from the data measured by our sensors.

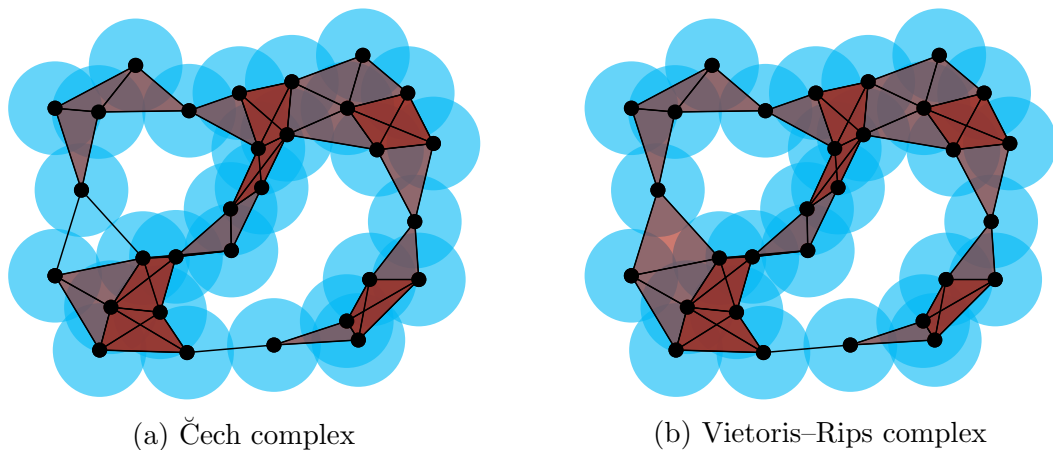


Figure A.1: Note the 2-simplex that is absent from the Čech complex but is present in the Vietoris–Rips complex.

By changing the radii of the sensor balls, we can approximate the Čech complex

from either direction using Vietoris–Rips complexes. Let $S \subset \mathbb{R}^d$ be a set of points, and let $C[\epsilon]$ denote the Čech complex built from balls of radius ϵ . That is, $C[\epsilon]$ is the nerve of the balls of radius ϵ centered at the points of S . Let $VR[2\epsilon]$ be the Vietoris–Rips complex built from balls of radius ϵ . So a simplex is in $VR[2\epsilon]$ if its diameter is at most 2ϵ . Then

$$VR[\epsilon] \subset C[\epsilon] \subset VR[2\epsilon].$$

Moreover, if $S \subset \mathbb{R}^2$ then we have

$$VR[\sqrt{3}\epsilon] \subset C[\epsilon] \subset VR[2\epsilon].$$

Hence the Čech complex is approximated on either side by a Vietoris–Rips complex measured by our sensors, perhaps after changing the radii of the sensor balls.

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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