Vietoris-Rips Complexes of Regular Polygons

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Topological Data Analysis

Definition

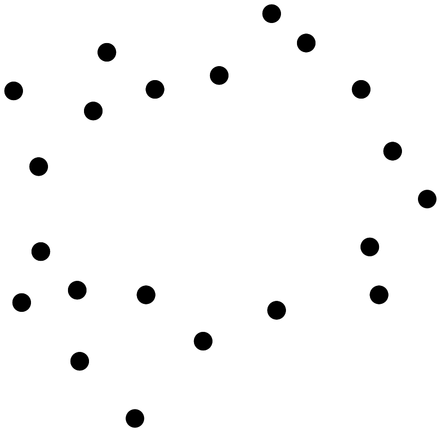
For metric space (X, d) and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathbf{VR}_{<}(X; r)$ is the set of all finite $\sigma \subseteq X$ with $\operatorname{diam}(\sigma) < r$.

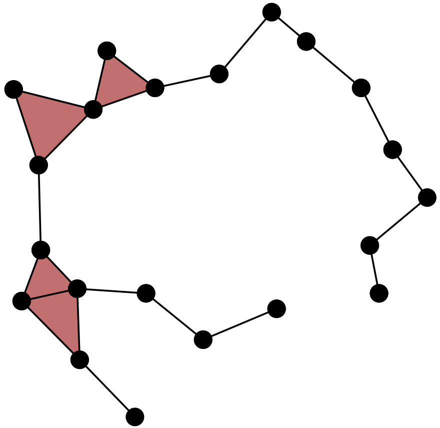
Definition

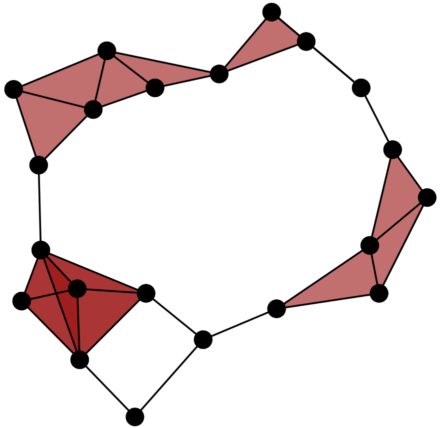
For metric space (X, d) and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathbf{VR}_{\leq}(X; r)$ is the set of all finite $\sigma \subseteq X$ with $\operatorname{diam}(\sigma) \leq r$.

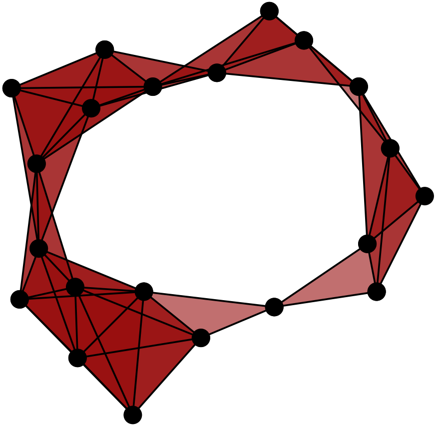
Remark

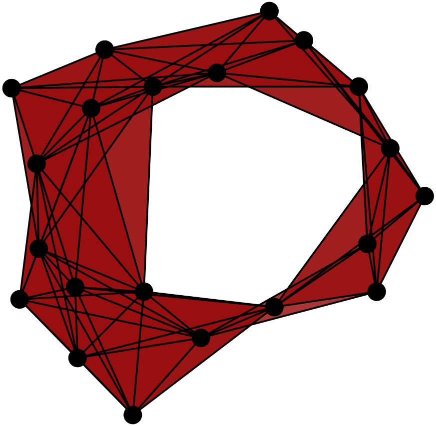
The *Vietoris–Rips simplicial complex* can be fully determined by the the underlying graph of its one skeleton, i.e the graph made by the zero and one dimensional simplices.

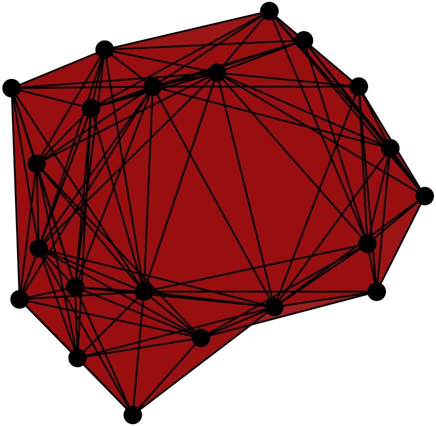










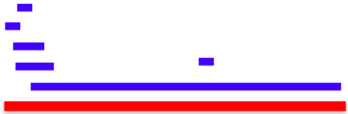


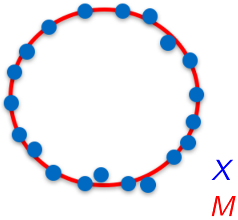
Problem Description

Theorem (Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot)

Suppose $X \subset M$ is a finite sampling of a manifold M. Then:

$$d_B(dgm_k^{VR}(X), dgm_k^{VR}(M)) \leq 2d_{GH}(X, M)$$





Regular Polygons

Definition

Given an integer $n \geq 3$, let the *regular n-gon* $P_n \subseteq \mathbb{R}^2$ be a set of n points equally spaced on S^1, with line segments connecting adjacent points together. We endow P_n with the Euclidean metric of $\mathbb{R}^2$.

We fix a homeomorphism $\phi : P_n \to S^1$ that we will sometimes use implicitly when discussing points on P_n.

Remark

A homeomorphism is an equivalence relation between metric spaces that preserves dimension.

Definition

We use a ternary relation to describe an order on P_n (actually S^1), writing $x \preceq y \preceq z$ when x, y, and z appear in clockwise order.

Definition

Let $\vec{d}$ represent the clockwise geodesic distance on the circle of circumference 1.

Main Result

Theorem

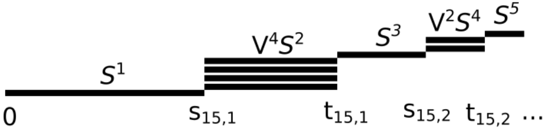
For fixed n, we have sequences of reals $\{s_{n,\ell}\}$ *and* $\{t_{n,\ell}\}$ *that correspond to the first and last scale parameters for which an equilateral* $(2\ell+1)$*-star can be inscribed within* P_n*. Then:*

$$\mathbf{VR}_{<}(P_n; r) \simeq \begin{cases} \bigvee^{q-1} S^{2\ell} & \text{when } s_{n,\ell} < r \leq t_{n,l} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r \leq s_{n,\ell+1} \end{cases}$$

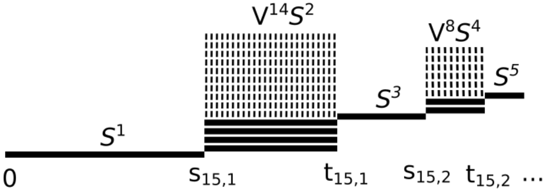
$$\mathbf{VR}_{\leq}(P_n; r) \simeq \begin{cases} \bigvee^{3q-1} S^{2\ell} & \text{when } s_{n,\ell} < r < t_{n,\ell} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r < s_{n,\ell+1}, \end{cases}$$

where $q = n/gcd(n, 2\ell+1)$*. Furthermore, all of the above homological features are persistent, except for* $2q$ *copies of* $S^{2\ell}$ *during the even sphere regimes for* $\leq$*.*

Main Result: Example



$\mathbf{VR}_<(P_{15}; r)$



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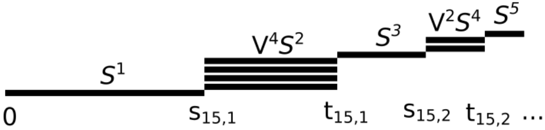
Main Result: Example



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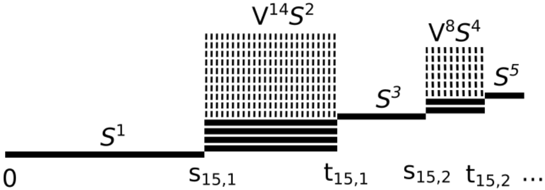


$\mathbf{VR}_{\leq}(P_{15}; r)$

Why do we get homology above dimension 1?

Intuition

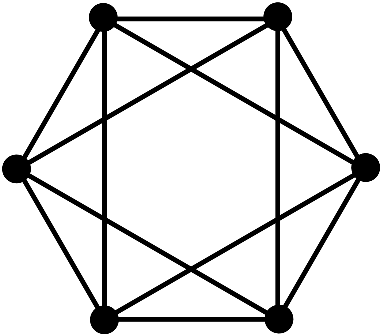
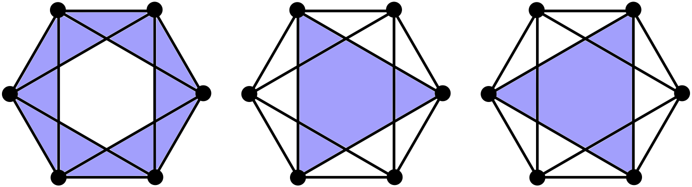
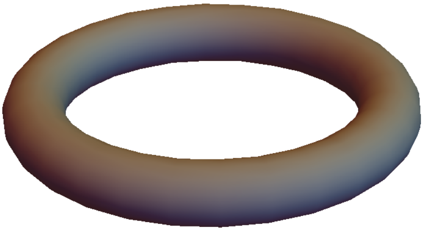
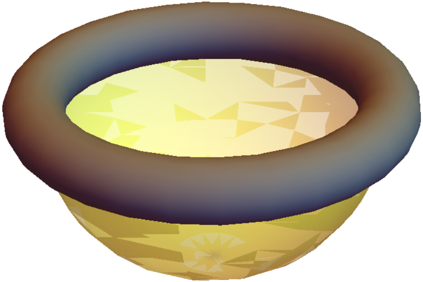


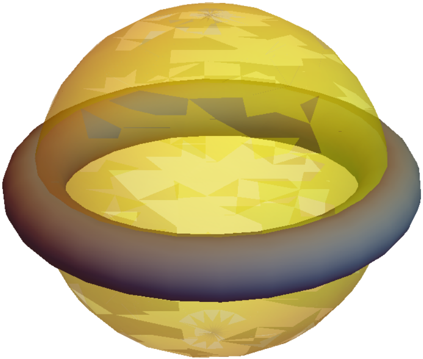
Figure: $\mathrm{VR}_{\leq}(6 \text{ points}; \frac{1}{3}) \simeq S^2$

Intuition









Intuition

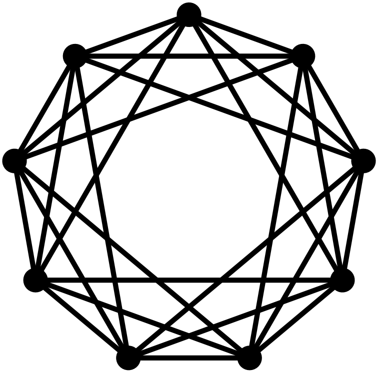
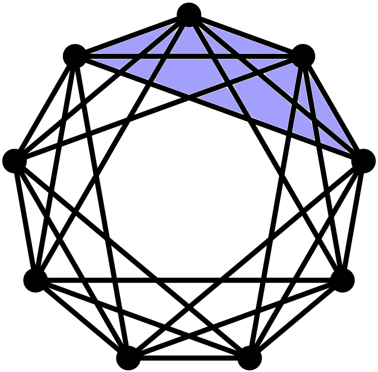


Figure: $\mathrm{VR}_{\leq}(9 \text{ points}; \frac{1}{3}) \simeq \bigvee^2 S^2$

Intuition



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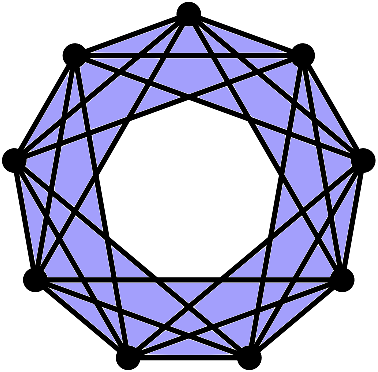

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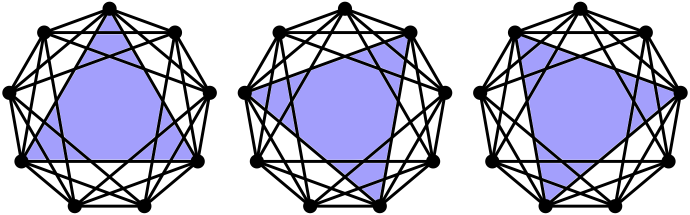
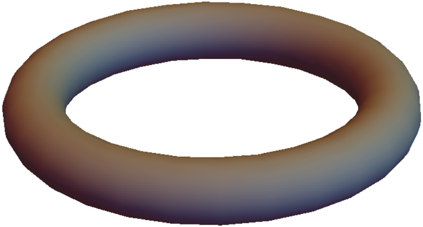
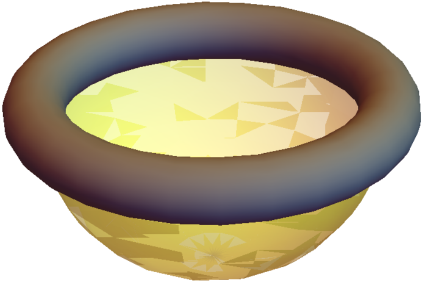
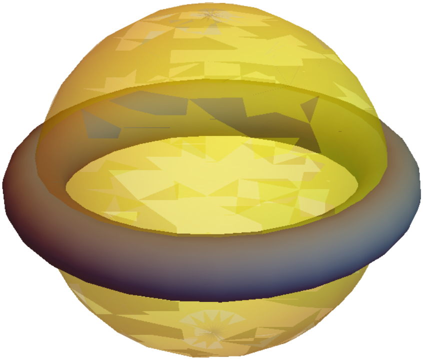
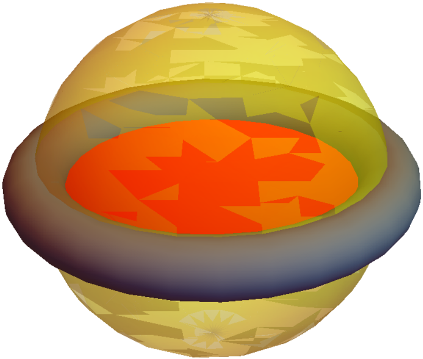










Cyclic Graphs

Definition

A *directed graph* is a pair $G = (V, E)$ with V the set of vertices and $E \subseteq V \times V$ the set of directed edges, where no edge is oriented in both directions.

Definition

A directed graph G is *cyclic* if its vertices can be placed in a cyclic order such that, whenever there is a directed edge $v \to u$, then there are also edges $v \to w \to u$ for all $v \prec w \prec u \prec v$.

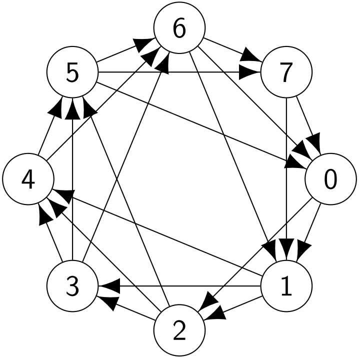


Figure: a cyclic graph

Dynamical Systems

Definition

For a cyclic graph G and a vertex v, define $f(v)$ to be the clockwise-most point u such that there exists a directed edge $v \to u$.

Definition

If v is such that $f^i(v) = v$ for some integer $i \geq 1$, then we call v a *periodic* vertex.

Definition

If v is periodic, then we call the set $\{v, f(v), f^2(v), \ldots\}$ a *periodic orbit* and its *length* is the size of the set.

Remark

Every finite cyclic graph has at least one periodic orbit.

Definition

The *winding number* ω of a periodic orbit of length k is the value

$$\sum_{i=0}^{k-1} \vec{d}(f^i(v), f^{i+1}(v)).$$

Definition

The *winding fraction* of a cyclic graph G is

$$\mathrm{wf}(G) = \sup\left\{ \frac{\omega}{k} \;\middle|\; \begin{array}{l} G \text{ contains a periodic orbit of} \\ \text{length } k \text{ and winding number } \omega. \end{array} \right\}.$$

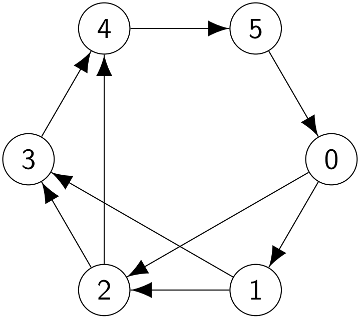


Figure: Left: $0 \to 2 \to 4 \to 5$ is a periodic orbit of length 4. Winding number$=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{2}{5}=1 \Rightarrow$ wf$=\frac{1}{4}$
Right: $0 \to 2 \to 3 \to 4 \to 5$ is a periodic orbit of length 5. wf$=\frac{1}{5}$

Dynamical Systems

Definition

For G a cyclic graph with $\mathrm{wf}(G) = \frac{p}{q}$, we call a non-periodic vertex v *fast* if

$$\sum_{i=0}^{q-1} \vec{d}(f^i(v), f^{i+1}(v)) > p$$

and *slow* if

$$\sum_{i=0}^{q-1} \vec{d}(f^i(v), f^{i+1}(v)) < p$$

That is, does v run "faster" or "slower" than periodic points?

Dynamical Systems

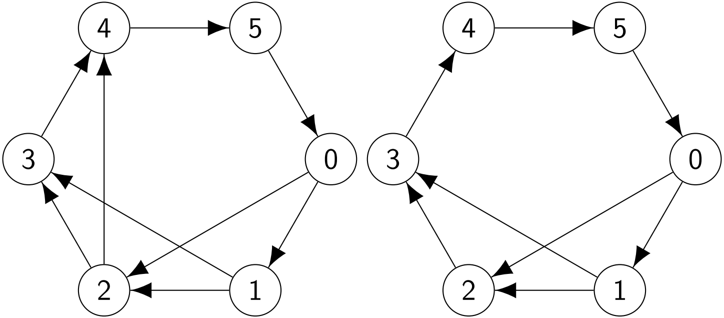


Figure: Left: 0, 2, 4, and 5 are periodic vertices. 1 and 3 are slow points. Right: 0, 2, 3, 4, and 5 are periodic vertices. 1 is a fast point.

Clique Complexes of Cyclic Graphs

Definition

For any graph G (not necessarily directed), the *clique complex* of G is the simplicial complex containing an n-simplex $[v_1, \ldots v_{n+1}]$ whenever the set $\{v_1, \ldots v_{n+1}\}$ is pairwise connected.

Theorem (Adamaszek, Adams, Reddy)

Let G be a cyclic graph with P periodic orbits and F "fast orbits".

- *If* $\frac{\ell}{2\ell+1} < \mathrm{wf}(G) \leq \frac{\ell+1}{2\ell+3}$ *for some integer* $\ell \geq 0$*, then* $\mathrm{Cl}(G) \simeq S^{2\ell+1}$.
- *If* $\mathrm{wf}(G) = \frac{\ell}{2\ell+1}$*, then* $\mathrm{Cl}(G) \simeq \bigvee^{P+F-1} S^{2\ell}$.

Geometric Lemmas for Regular Polygons

Question

For which scale parameters r are $\mathbf{VR}_<(P_n; r)$ and $\mathbf{VR}_\leq(P_n; r)$ cyclic graphs?

Let's denote the maximal such r by r_n.

It is equivalent to find values of r such that $B_r(c) \cap P_n$ is connected for all $c \in P_n$.

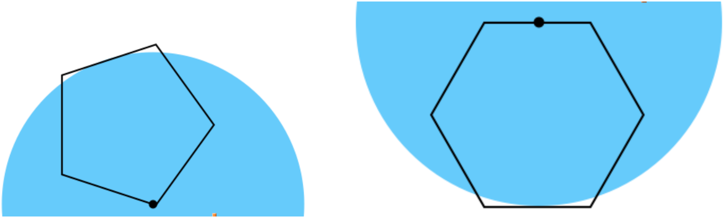


Figure: Some disconnected intersections in P_5(left) and P_6(right)

Geometric Lemmas for Regular Polygons

Lemma

If $n \geq 4$ and $r < r_n$ then both $\mathbf{VR}_{<}(P_n; r)$ and $\mathbf{VR}_{\leq}(P_n; r)$ are cyclic graphs.

Geometric Lemmas for Regular Polygons

Lemma

If $n \geq 4$ and $r < r_n$ then both $\mathbf{VR}_{<}(P_n; r)$ and $\mathbf{VR}_{\leq}(P_n; r)$ are cyclic graphs.

Proof

Let c be an arbitrary point on P_n. Suppose that c is in edge $[x, y]$ and that edge $[a, b]$ is on the opposite side, contained in line L.

Now parity considerations:

Geometric Lemmas for Regular Polygons

Proof (Cont.)

If n is odd:

Let q be the unique point on $[x, y]$ such that $proj_L(q) = a$. Let m be the midpoint of $[a, b]$ and check that $proj_L(x) = m$.

Observe that the distance function from c is monotonic on $[a, b]$ if and only if $proj_L(c) \notin [a, m] \leftrightarrow c \notin [x, q]$.

Moreover, the shortest distance from c to L approaches $|a - q|$ as c approaches q.

Hence, $r_n = |a - q| = 1 + \frac{\cos(\frac{2\pi}{n})}{\cos(\frac{\pi}{n})}$.

Geometric Lemmas for Regular Polygons

Proof (Cont.)

If n is even:

Observe that $proj_L(c) \in [a, b]$, so the distance function from c is not monotonic on $[a, b]$.

Moreover, the shortest distance from c to L is constant for all $c \in [x, y]$, since $[x, y]$ and $[a, b]$ are parallel.

Hence, $r_n = |a - y| = 2\cos(\frac{2\pi}{n})$. □

Geometric Lemmas for Regular Polygons

What about P_3?

Geometric Lemmas for Regular Polygons

Question

For which scale parameters r are $\mathbf{VR}_<(P_n; r)$ and $\mathbf{VR}_\leq(P_n; r)$ cyclic graphs?

Answer

For $n \geq 4$, we conclude that $\mathbf{VR}_<(P_n; r)$ and $\mathbf{VR}_\leq(P_n; r)$ are cyclic graphs for $r \in (0, r_n)$, where

$$r_n = \begin{cases} 2\cos(\frac{2\pi}{n}) & \text{if } n \text{ even} \\ 1 + \frac{\cos(\frac{2\pi}{n})}{\cos(\frac{\pi}{n})} & \text{if } n \text{ odd} \end{cases}.$$

Moreover, $\mathbf{VR}_<(P_3; r)$ and $\mathbf{VR}_\leq(P_3; r)$ are not cyclic graphs for any $r > 0$.

Geometric Lemmas for Regular Polygons

Definition

In a cyclic graph, a periodic orbit which has length $2\ell + 1$ and winding number ℓ is called a $(2\ell + 1)$-*star*. If all the adjacent distances are equal, then we call the star *equilateral*.

Remark

The winding fraction of G becomes $\frac{\ell}{2\ell+1}$ when the first equilateral $(2\ell + 1)$-star can be inscribed, and it stays $\frac{\ell}{2\ell+1}$ until the last equilateral $(2\ell + 1)$-star can be inscribed.

Geometric Lemmas for Regular Polygons

Lemma

For any point $x \in P_n$, there exists a unique equilateral $(2\ell + 1)$-star which contains x as one of its vertices.

Definition

For $x \in P_n$ and an integer $\ell \geq 1$, denote the unique inscribed $(2\ell + 1)$-star containing x by $S_{2\ell+1}(x)$, and its side length by $s_{2\ell+1}(x)$.

Lemma

The function $s_{2\ell+1} : P_n \to \mathbb{R}$ is continuous.

Geometric Lemmas for Regular Polygons

Question

How many distinct equilateral $(2\ell + 1)$-stars of side length r can be inscribed into P_n?

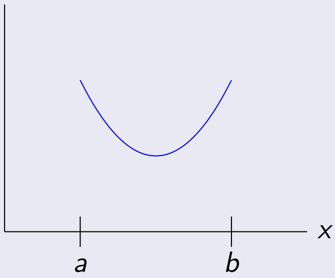
Geometric Lemmas for Regular Polygons

Definition

A "crossing" is a value $x \in P_n$ such that at least one vertex of $S_{2\ell+1}(x)$ falls on a vertex of P_n.

Lemma

If $[a, b)$ is a maximal half-open interval of P_n such that no $x \in [a, b)$ is a crossing, then the graph of $s_{2\ell+1}$ must look like:



Geometric Lemmas for Regular Polygons

Lemma

For any $(2\ell+1)$-star S inscribed in P_n, the number of vertices of S coinciding with vertices of P_n is equal to either 0 or gcd$(n, 2\ell+1)$.

Geometric Lemmas for Regular Polygons

Lemma

For any $(2\ell+1)$-star S inscribed in P_n, the number of vertices of S coinciding with vertices of P_n is equal to either 0 or gcd$(n, 2\ell+1)$.

Proof

Suppose $x \in P_n$ is a vertex, and consider the $(2\ell+1)$-pointed star $S = S_{2\ell+1}(x)$.

Let $x, y \in P_n$ be the closest together vertices of S which are also vertices of P_n. Let m denote the number of steps between them via S.

Observe $m|(2\ell+1)$, so we can write $m = (2\ell+1)/d$ for some $d|(2\ell+1)$.

Proof (Cont.)

Now d is exactly equal to the number of vertices of S which coincide with vertices of P_n.

Observe that we also have $d|n$.

Since m is defined to be minimal, d must be maximal, hence $d = \gcd(n, 2\ell + 1)$. □

Geometric Lemmas for Regular Polygons

Corollary

The total number of crossings in P_n is equal to $n(2\ell+1)/gcd(n, 2\ell+1)$.

Corollary

The number of local minima/maxima on $s_{2\ell+1}$ is equal to $n(2\ell+1)/gcd(n, 2\ell+1)$.

Lemma

All local minima/maxima of $s_{2\ell+1}$ are global minima/maxima.

Geometric Lemmas for Regular Polygons

We let $s_{n,\ell}$ be the value of the global min and $t_{n,\ell}$ be the value of the global max of $s_{2\ell+1}$ on P_n.

Corollary

The number of equilateral $(2\ell+1)$-stars inscribed into P_n which have minimal side length is equal to $n/gcd(n, 2\ell+1)$.

The number of equilateral $(2\ell+1)$-stars inscribed into P_n which have maximal side length is equal to $n/gcd(n, 2\ell+1)$.

The number of equilateral $(2\ell+1)$-stars inscribed into P_n which have side length r satisfying $s_{n,\ell} < r < t_{n,\ell}$ is equal to $2n/gcd(n, 2\ell+1)$.

Geometric Lemmas for Regular Polygons

Question

How many distinct equilateral $(2\ell + 1)$-stars of side length r can be inscribed into P_n?

Answer

The number of equilateral $(2\ell + 1)$-stars of side length r that can be inscribed into P_n is equal to:

$$\begin{cases} n/gcd(n, 2\ell + 1) & \text{if } r = s_{n,\ell} \text{ or } t_{n,\ell} \\ 2n/gcd(n, 2\ell + 1) & \text{if } s_{n,\ell} < r < t_{n,\ell} \\ 0 & \text{otherwise} \end{cases}$$

Geometric Lemmas for Regular Polygons

From this result we can determine P and F for a fixed r.

Let $q = n/\gcd(n, 2\ell + 1)$. We remark that:

For $\leq$, we have $P = 2q$ and $F = q$.

For $<$, we have $P = 0$ and $F = q$.

(Take, $n = 6, \ell = 1$ as an example.)

Recall

If $\mathrm{wf}(G) = \frac{\ell}{2\ell+1}$, then $\mathrm{Cl}(G) \simeq \bigvee^{P+F-1} S^{2\ell}$.

Main Result

Theorem

For $r \in (0, r_n)$ we have:

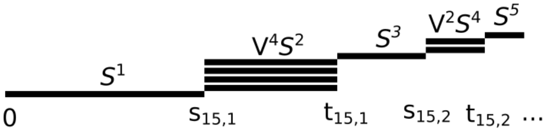
$$\mathbf{VR}_<(P_n; r) \simeq \begin{cases} \bigvee^{q-1} S^{2\ell} & \text{when } s_{n,\ell} < r \leq t_{n,l} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r \leq s_{n,\ell+1} \end{cases}$$

$$\mathbf{VR}_\leq(P_n; r) \simeq \begin{cases} \bigvee^{3q-1} S^{2\ell} & \text{when } s_{n,\ell} < r < t_{n,\ell} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r < s_{n,\ell+1}, \end{cases}$$

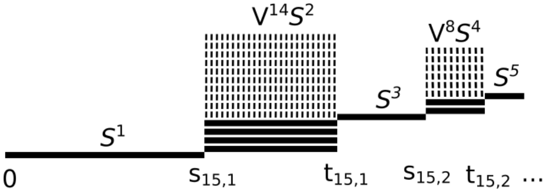
where $q = n/gcd(n, 2\ell + 1)$. Furthermore,

- *For $s_{n,\ell} < r < \tilde{r} \leq t_{n,\ell}$ or $t_{n,\ell} < r < \tilde{r} \leq s_{n,\ell+1}$, inclusion $\mathbf{VR}_<(P_n; r) \hookrightarrow \mathbf{VR}_<(P_n; \tilde{r})$ is a homotopy equivalence.*
- *For $t_{n,\ell} < r < \tilde{r} < s_{n,\ell+1}$, inclusion $\mathbf{VR}_\leq(P_n; r) \hookrightarrow \mathbf{VR}_\leq(P_n; \tilde{r})$ is a homotopy equivalence.*
- *For $s_{n,\ell} \leq r < \tilde{r} \leq t_{n,\ell}$, inclusion $\mathbf{VR}_\leq(P_n; r) \hookrightarrow \mathbf{VR}_\leq(P_n; \tilde{r})$ induces a rank $q - 1$ map on 2ℓ-dimensional homology $H_{2\ell}(-; \mathbb{F})$ for any field $\mathbb{F}$.*

Main Result: Example



$\mathbf{VR}_<(P_{15}; r)$



$\mathbf{VR}_\leq(P_{15}; r)$

Future Work

- Find analytical formulas when $2\ell + 1$ does not divide n
- Given n, find maximal ℓ such that a $(2\ell + 1)$-star can be inscribed into P_n
- Finish paper and post to arXiv

Thank you!