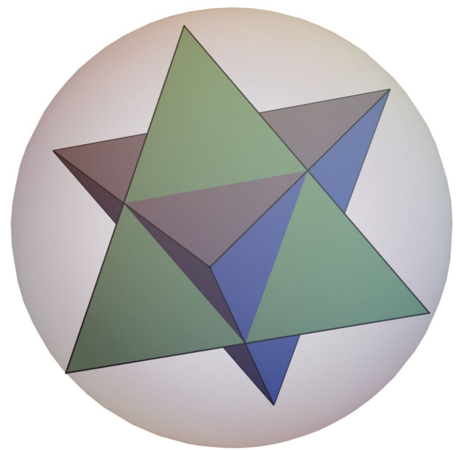
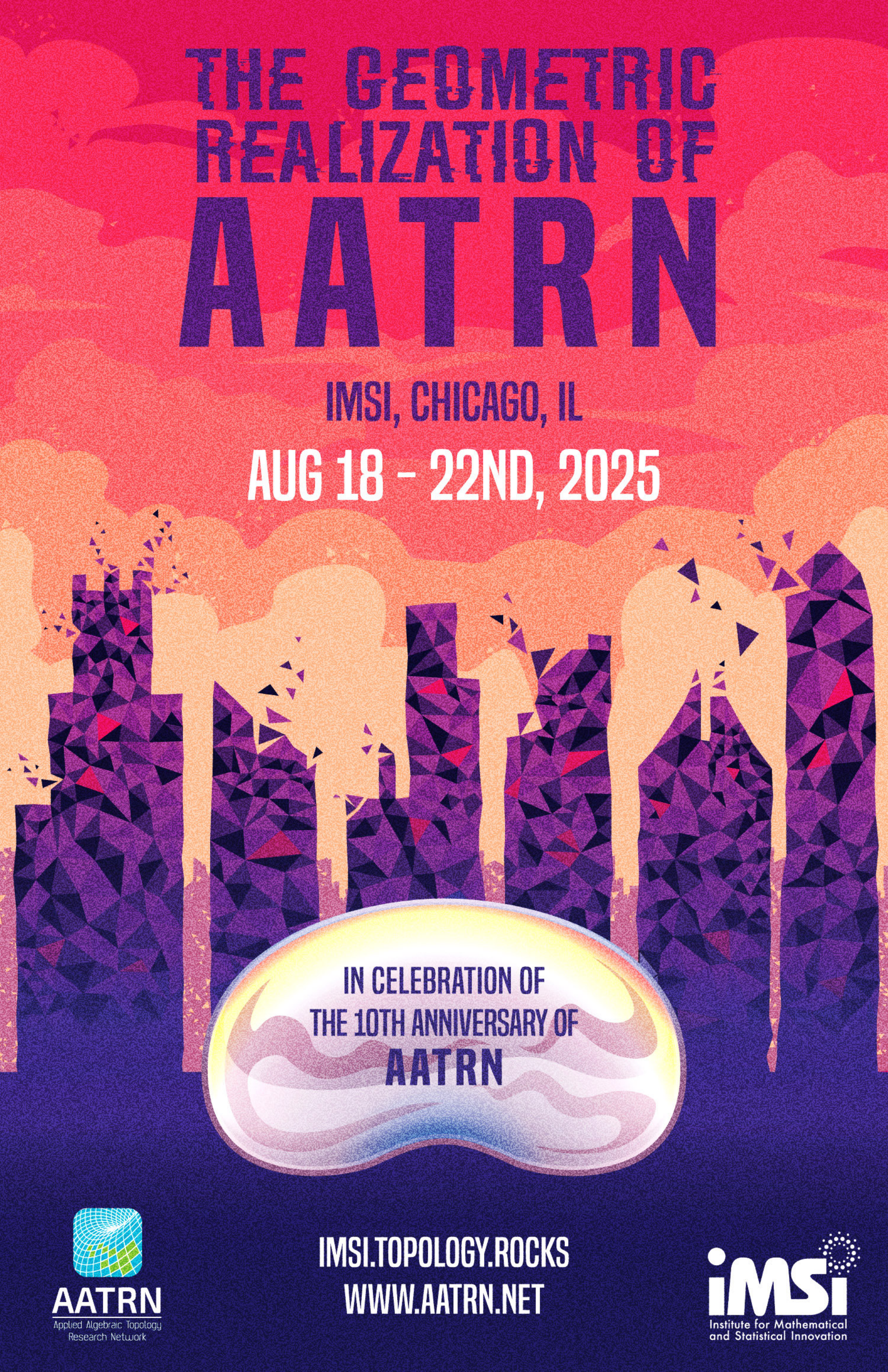
Vietoris-Rips complexes of manifolds



Henry Adams, University of Florida


THE GEOMETRIC REALIZATION OF
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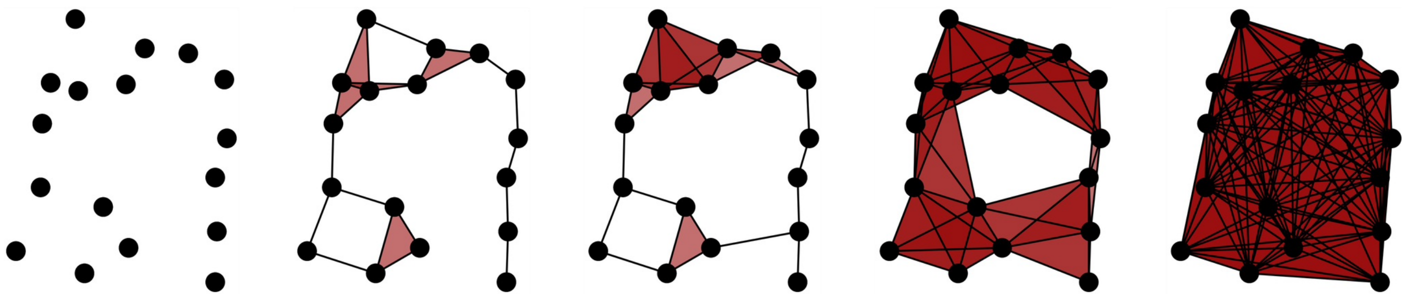
<u>Def</u> X a metric space, $r \geq 0$. The <u>Vietoris-Rips simplicial complex $VR(X;r)$</u> has

- vertex set X
- all simplices of diameter $< r$.



<u>Stability (Chazal, de Silva, Oudot '14)</u> $X \approx M \Rightarrow PH(VR(X;-)) \approx PH(VR(M;-))$.

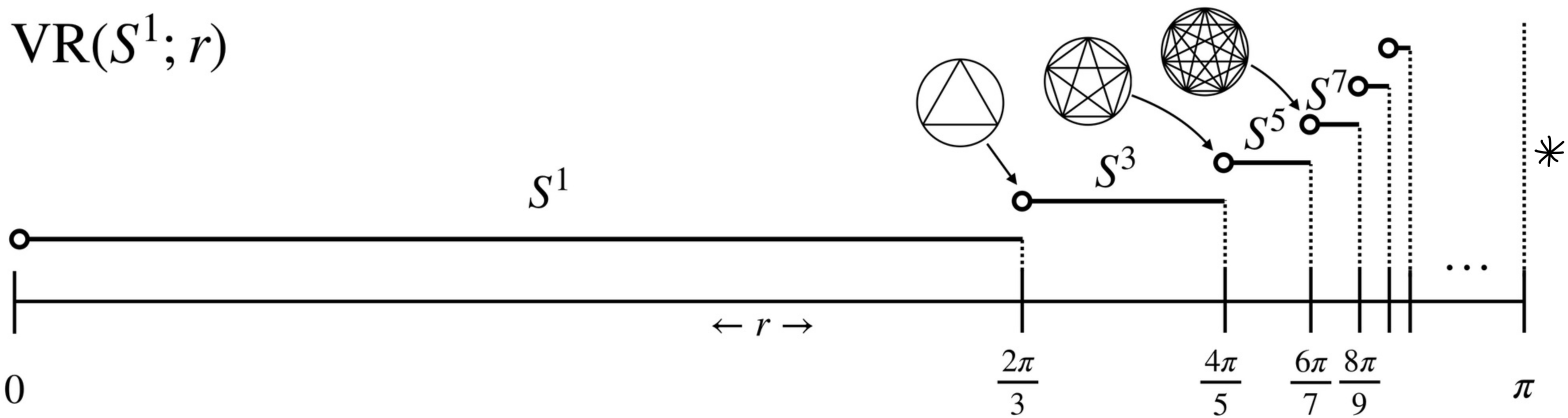
For M a manifold, what can we say about $VR(M;r)$?

<u>Hausmann '95</u> M compact Riemannian manifold $\Rightarrow VR(M;r) \simeq M$ for all r sufficiently small.

The circle

Adamaszek, A. '17

$$VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{2\pi k}{2k+1} < r \le \frac{2\pi(k+1)}{2k+3} \\ * & \text{if } r \ge \pi. \end{cases}$$



May '23 $VR^m_{\le}(S^1;r)$

The circle

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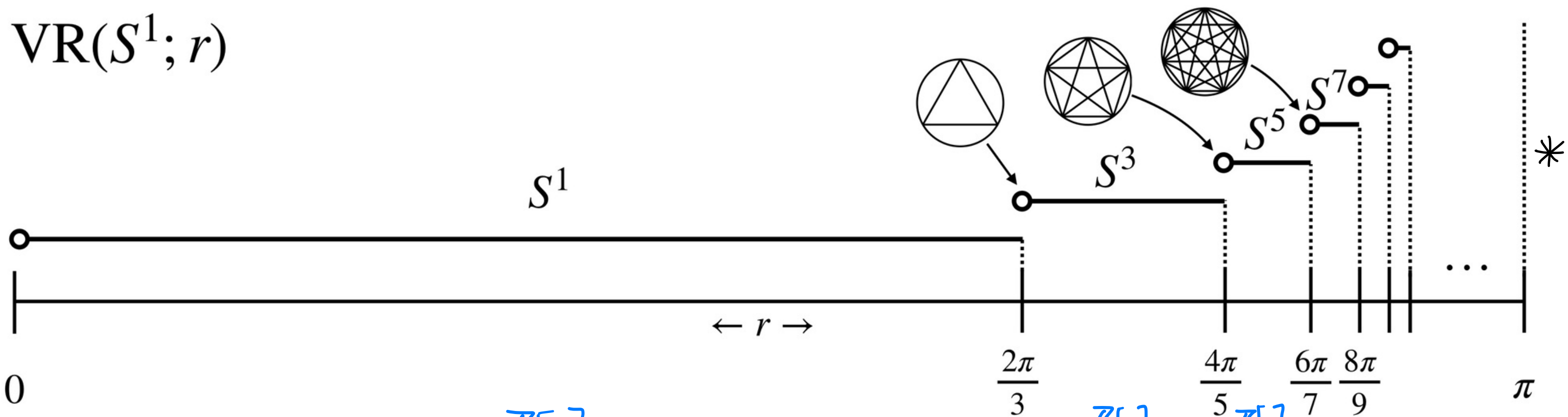


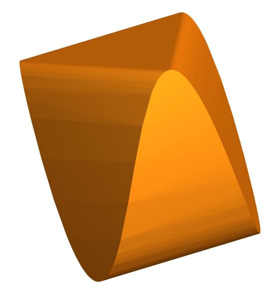
$$\mathbb{Z} = \frac{\mathbb{Z}[u]}{1\cdot u} \longleftarrow \frac{\mathbb{Z}[u]}{1\cdot 3\cdot u^2} \longleftarrow \frac{\mathbb{Z}[u]}{1\cdot 3\cdot 5\, u^3} \longleftarrow \cdots \longleftarrow \mathbb{Z}[u]$$

A., Lagoda, Moy, Sadovek, de Saha '24

$$\mathrm{VR}^m_{\le}(S^1;r) \simeq_{S^1} S^1 * S^1_3 * S^1_5 * \dots * S^1_{2k+1}$$

Stronger connections to Barvinok-Novik orbitope
Convex Hull$\left(\{(e^{i\theta}, e^{i3\theta}, e^{i5\theta}, \dots, e^{i(2k+1)\theta}) \in \mathbb{C}^{2k+2}\}\right)$?

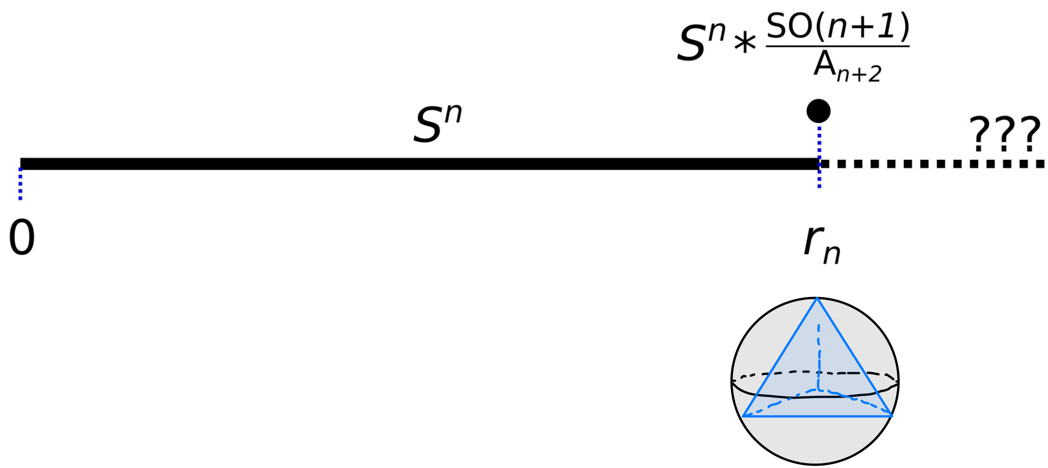
Barvinok, Novik '08 A., Bush, Frick '20



The n-sphere

Adamaszek, A., Frick '18

$$VR^{m}_{\le}(S^n; r) \simeq \begin{cases} S^n & \text{if } \quad r < r_n = \arccos\left(\frac{-1}{n+1}\right) \\ S^n * \frac{SO(n+1)}{A_{n+2}} & \text{if } \quad r = r_n. \end{cases}$$



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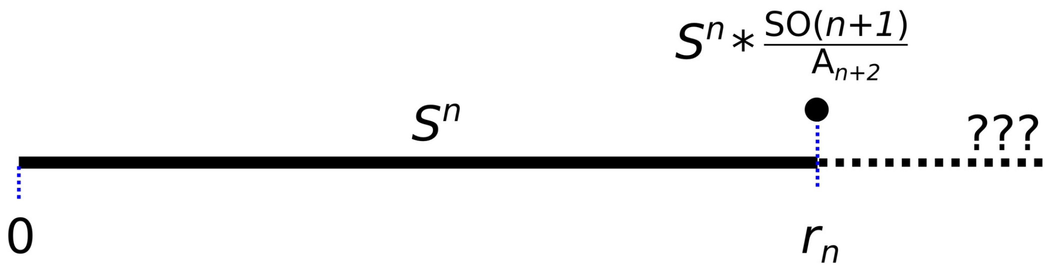
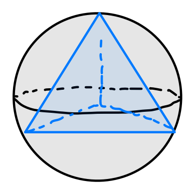
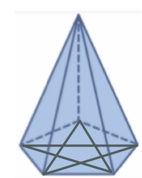
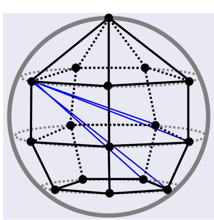


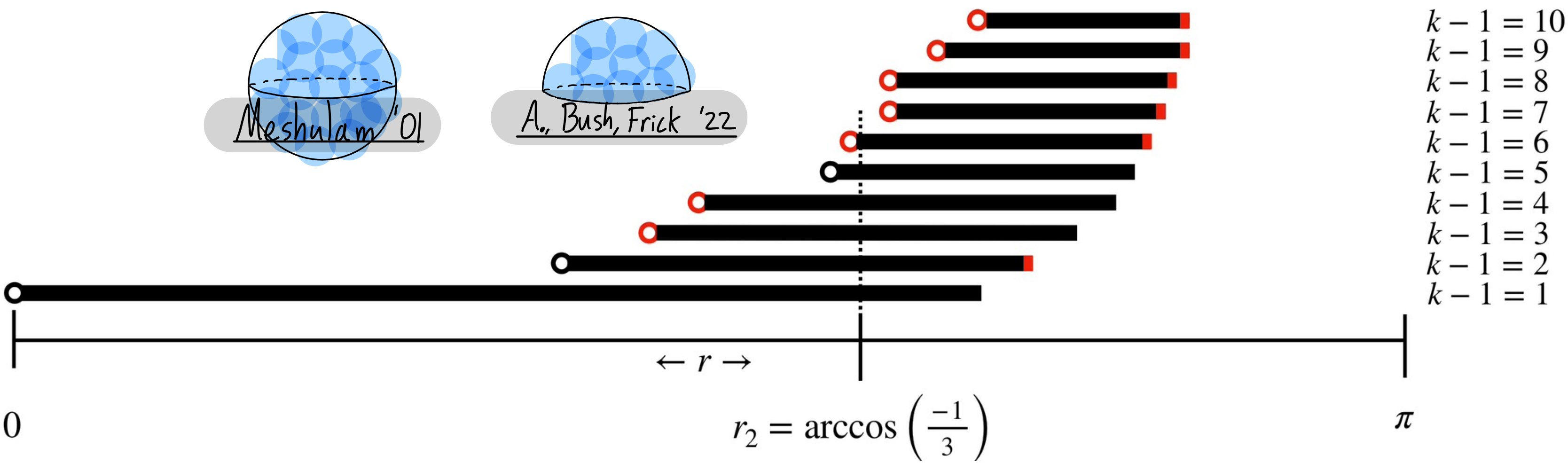






Conjecture: Critical scales are diameters of strongly self-dual polytopes Lovász '83

__A., Bush, Virk '24__ If $\mathrm{conn}(VR(S^n;\pi-\delta)) = k-1$, then $\mathrm{cov}_{S^n}(2k+2) \leq \delta < 2\cdot \mathrm{cov}_{\mathbb{RP}^n}(k)$.



__Corollary__ The homotopy type of $VR(S^n;r)$ changes infinitely many times.

__Conjecture__ Only countably many times.

__Conjecture__ $\mathrm{conn}(VR(S^n;r))$ is nondecreasing in r.

__Conjecture__ $VR(S^n;r) \simeq$ finite dim'l CW complex for all r.

__Question__ Bounds on $h\text{-}\dim(VR(S^n;r))$?

Proof sketch that $\delta \geq 2 \cdot \mathrm{cov}_{\mathbb{RP}^n}(k) \Rightarrow \mathrm{conn}(VR(S^n;\pi-\delta)) \leq k-2$.

A., Bush, Frick '22

If the $\mathbb{Z}/2$ space $VR(S^n;\pi-\delta)$ were $(k-1)$ connected, we'd have $S^k \xrightarrow{\mathbb{Z}/2} VR(S^n;\pi-\delta)$.

This would contradict the Borsuk-Ulam theorem since if $\exists k$ closed balls $\{B([x_i];\delta/2)\}_{i=1}^k$ covering $\mathbb{RP}^n$, then $\exists\ VR(S^n;\pi-\delta) \xrightarrow{\mathbb{Z}/2} S^{k-1}$.

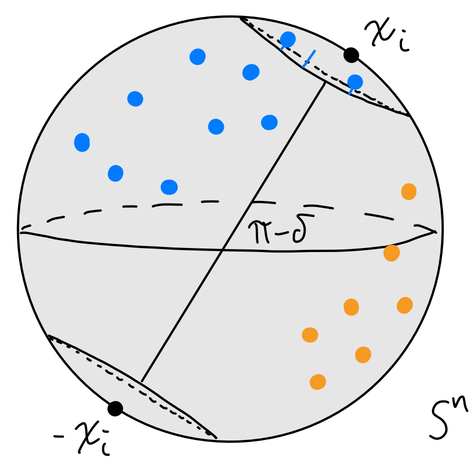
Indeed, no $\sum_j \lambda_j y_j \in |VR(S^n;\pi-\delta)|$ has vertex set intersecting both $B(x_i;\delta/2)$ and $B(-x_i;\delta/2)$.

Define $f_i : VR(S^n;\pi-\delta) \to \mathbb{R}$ by

$$f_i\left(\textstyle\sum_j \lambda_j y_j\right) = \begin{cases} \sum_j \lambda_j d(y_j, S^n \setminus B(x_i;\delta/2)) & \text{if no } y_j \text{ in } B(-x_i;\delta/2) \\ -\sum_j \lambda_j d(y_j, S^n \setminus B(-x_i;\delta/2)) & \quad\ldots\quad B(x_i;\delta/2). \end{cases}$$

Hence $(f_1,\ldots,f_k) : VR(S^n;\pi-\delta) \xrightarrow{\mathbb{Z}/2} \mathbb{R}^k \setminus \{\vec{0}\}$.

Radially project to S^{k-1}.



Application Gromov-Hausdorff distances between spheres

Lim, Mémoli, Smith, 2021

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

A., Bush, Clause, Frick, Gómez, Harrison, Jeffs, Lagoda, Lim, Mémoli, May, Sadovek, Superdock, Vargas, Wang, Zhou 2023

Harrison, Jeffs 2024

Main Theorem For $n<k$,

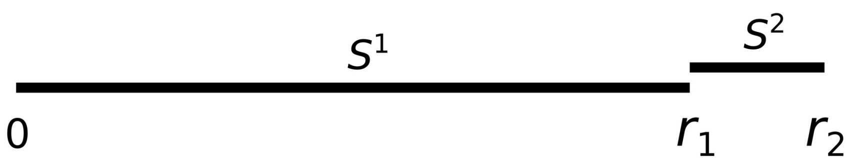
$$2 \cdot d_{GH}(S^n, S^k) \geq \inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\} \geq \pi - \mathrm{cov}_{\mathbb{RP}^n}(k).$$

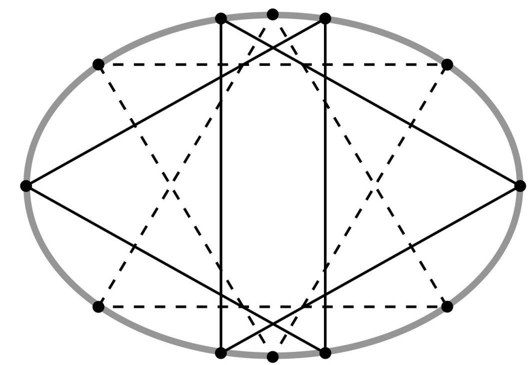
A., Bush, Frick, 2021

Open Questions

- $VR(M; r)$ for $M =$ spheres S^n, tori $(S^1)^n$, $\mathbb{CP}^n$, $\mathbb{RP}^n$, $Gr(n,k)$, Bolza surface $\mathbb{T}^2 \# \mathbb{T}^2$, ellipsoids?

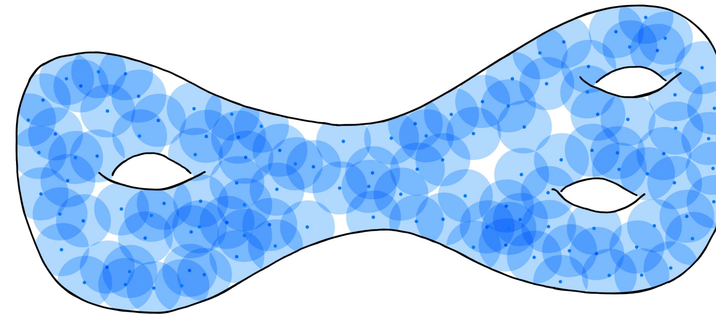
$VR(\text{ellipse}; r)$

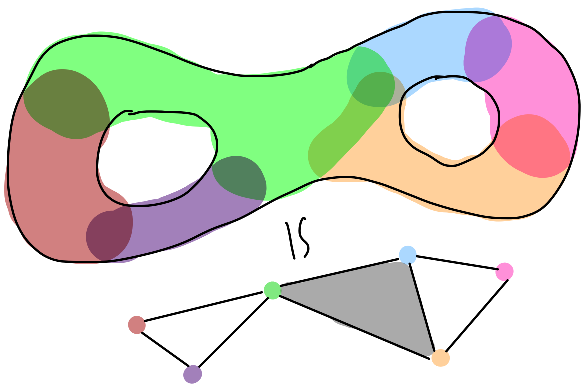


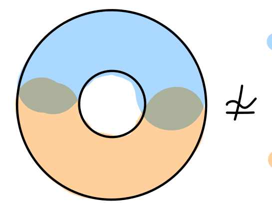


- A Morse (or Morse-Bott) theory for $VR(M; r)$?
- (Intrinsic) Čech complexes of manifolds ?

<u>Def</u> M a Riemannian manifold, $r \geq 0$. The <u>(intrinsic) Čech simplicial complex $\check{C}(M;r)$</u> is the nerve of the balls $\{B(x;r)\}_{x \in M}$







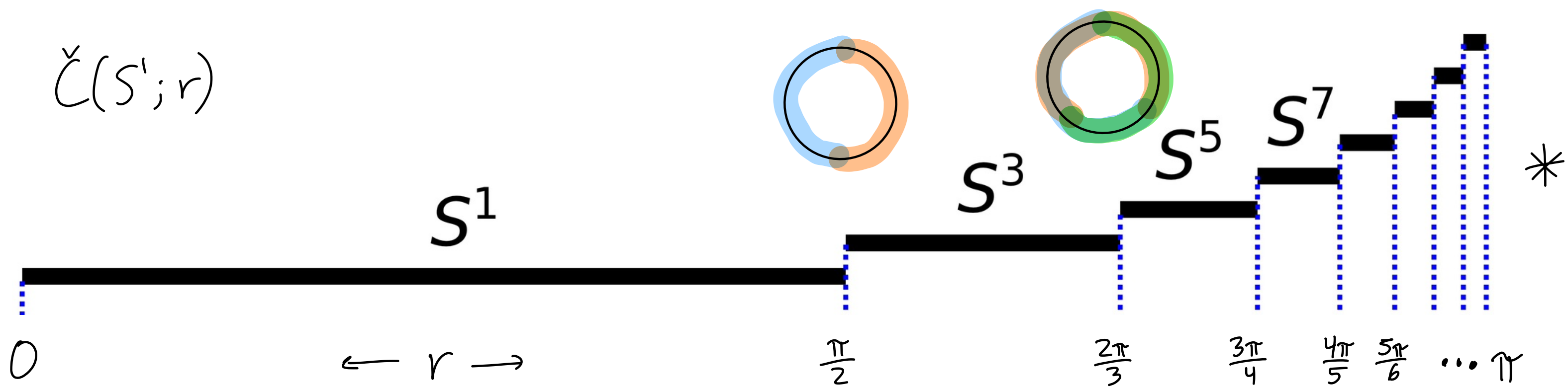
For $r <$ injectivity radius, the Nerve Lemma (Borsuk '48) implies $\check{C}(M;r) \simeq M$.

What about larger r?

The circle

Adamaszek, A. 2017

$$\check{C}(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k\pi}{k+1} < r \le \frac{(k+1)\pi}{k+2} \\ * & \text{if } r \ge \pi. \end{cases}$$



Conjecture $\check{C}(S^1;r) \simeq_{S^1} S^1 * S^1_2 * S^1_3 * \dots * S^1_{k+1}$

Stronger connections to the Carathéodory orbitope
Convex Hull $(\{(e^{i\theta}, e^{i2\theta}, e^{i3\theta}, \dots, e^{i(k+1)\theta}) \in \mathbb{C}^{2k+2}\})$?

Carathéodory 1907

Adamaszek, A., Frick, Peterson, Previte-Johnson 2018

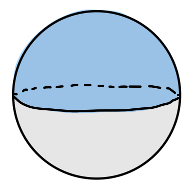
Bush 2021

The n-sphere

$$\check{C}(S^n;r) \simeq \begin{cases} S^n & r \le \frac{\pi}{2} \\ ? & \frac{\pi}{2} < r < \pi \\ * & r \ge \pi \end{cases}$$

$\check{C}(S^n;r)$ $\quad S^n \quad ? \quad *$

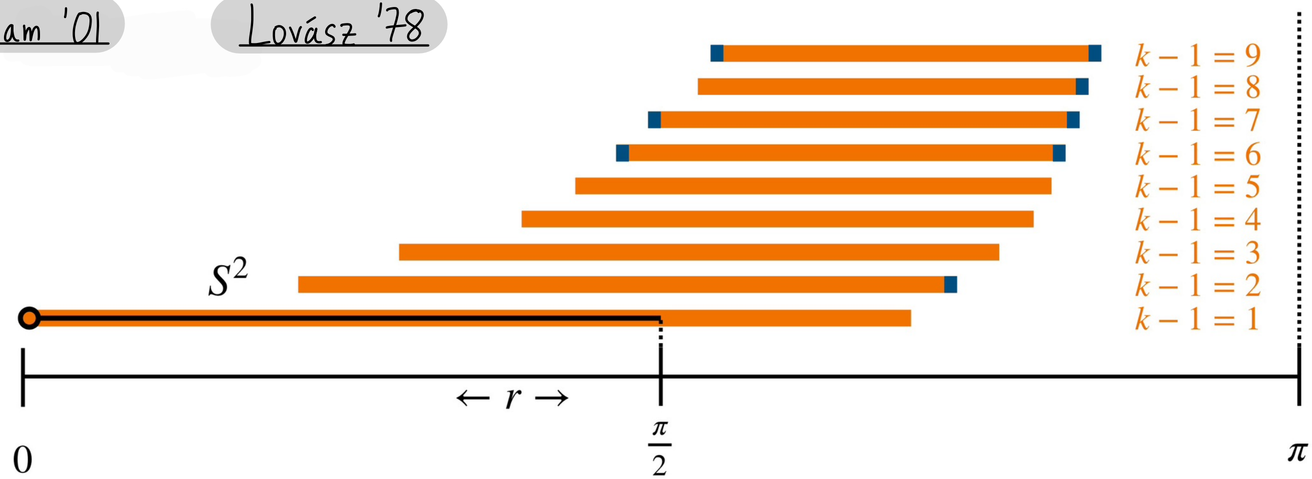
$0 \quad \leftarrow r \rightarrow \quad \frac{\pi}{2} \quad \pi$



Is $\check{C}(S^n;\frac{\pi}{2}+\varepsilon)$ homotopy equivalent to a subset of $S^n * \mathbb{R}P^n$?

A., Jauhari, Mallick '25+ If $\mathrm{conn}(\check{C}(S^n; \pi-\delta)) = k-1$, then $\mathrm{cov}_{S^n}(2k+2) \leq \delta < 2 \cdot \mathrm{cov}_{S^n}(k+1)$.

Meshulam '01 Lovász '78



Corollary The homotopy type of $\check{C}(S^n; r)$ changes infinitely many times.

Conjecture Only countably many times.

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Question Bounds on $\text{h-dim}(\check{C}(S^n; r))$?

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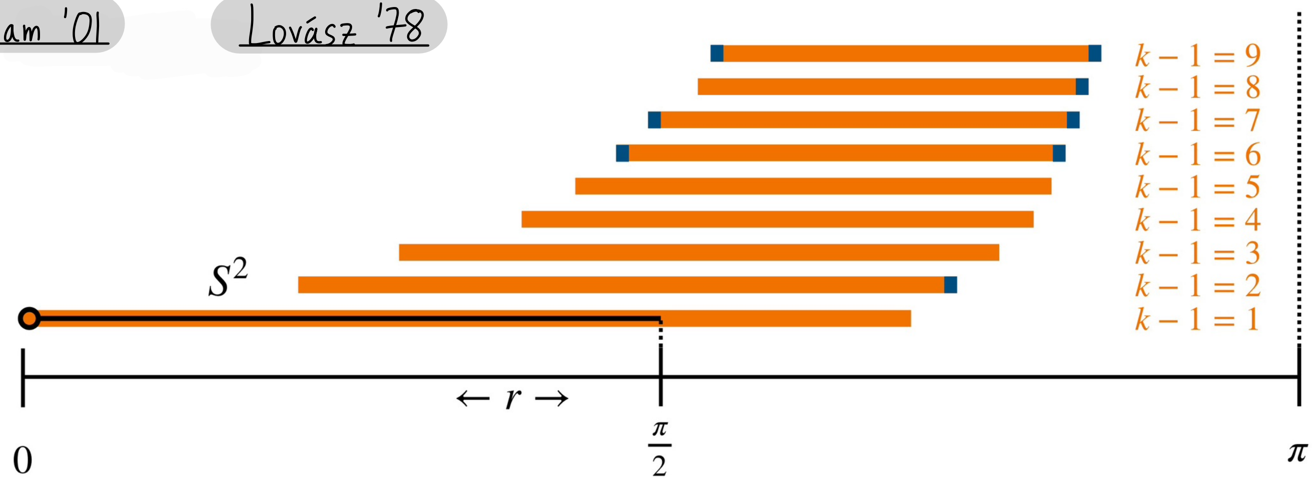
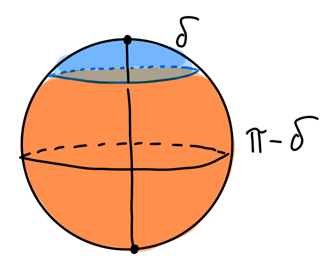


$$\chi(\operatorname{Bor}(S^n;\delta)) \geq \operatorname{conn}(N(\operatorname{Bor}(S^n;\delta))) + 3$$
$$= \operatorname{conn}(\check{C}(S^n;\pi-\delta)) + 3.$$



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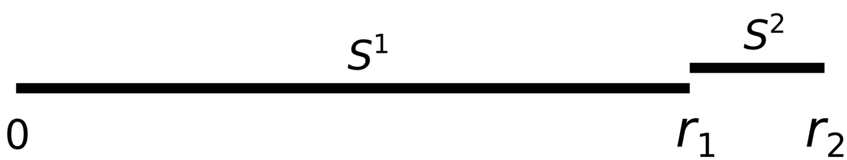

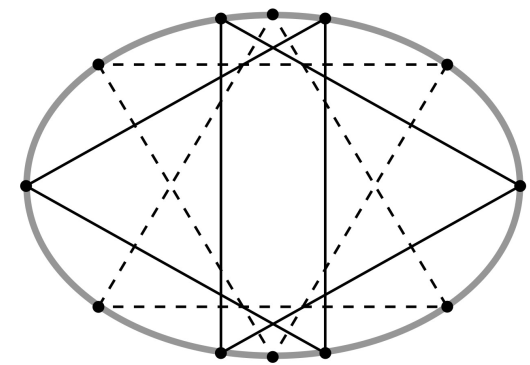

- A Morse (or Morse-Bott) theory for $VR(M;r)$ and $\check{C}(M;r)$?
- (Intrinsic) Čech complexes of manifolds? Spheres, tori, ...?
- Chromatic numbers of Borsuk graphs for scales $0<\delta<\pi$.