The Vietoris–Rips Complexes of a Circle

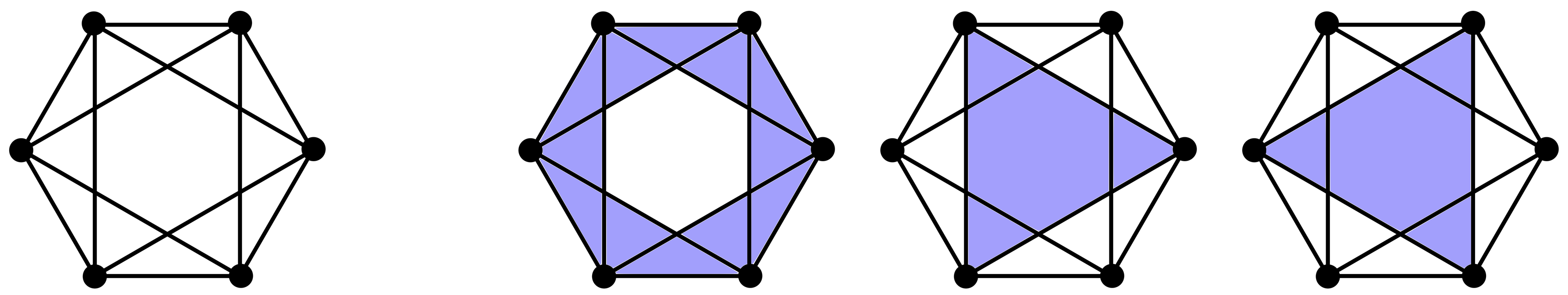
Michał Adamaszek and Henry Adams
University of Copenhagen and Colorado State University

Vietoris–Rips simplicial complexes

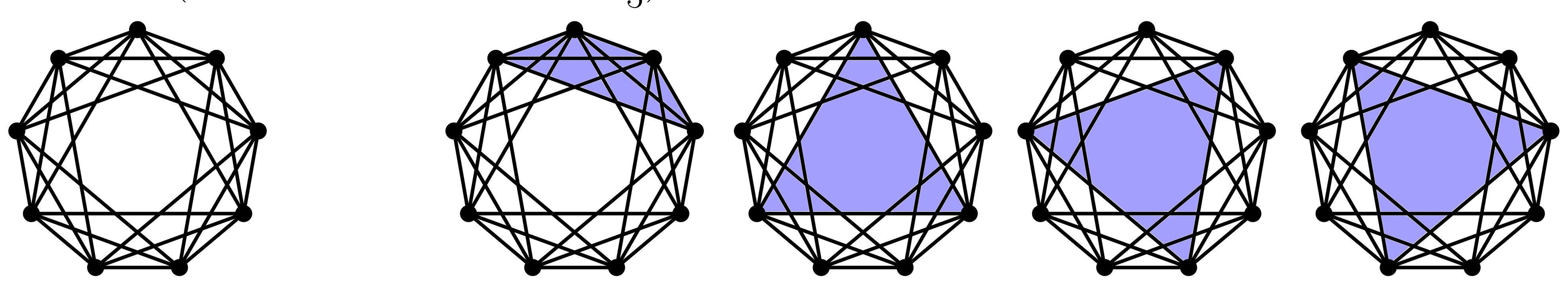
Definition. For X a metric space and $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X, r)$ has vertex set X. A simplex $\sigma \subset X$ is in $\mathrm{VR}(X, r)$ when $\mathrm{diam}(\sigma) \leq r$.

Let S^1 be the circle of unit circumference with the geodesic metric.

Example. $\mathrm{VR}(6 \text{ even points on } S^1, \frac{1}{3}) = S^2$.



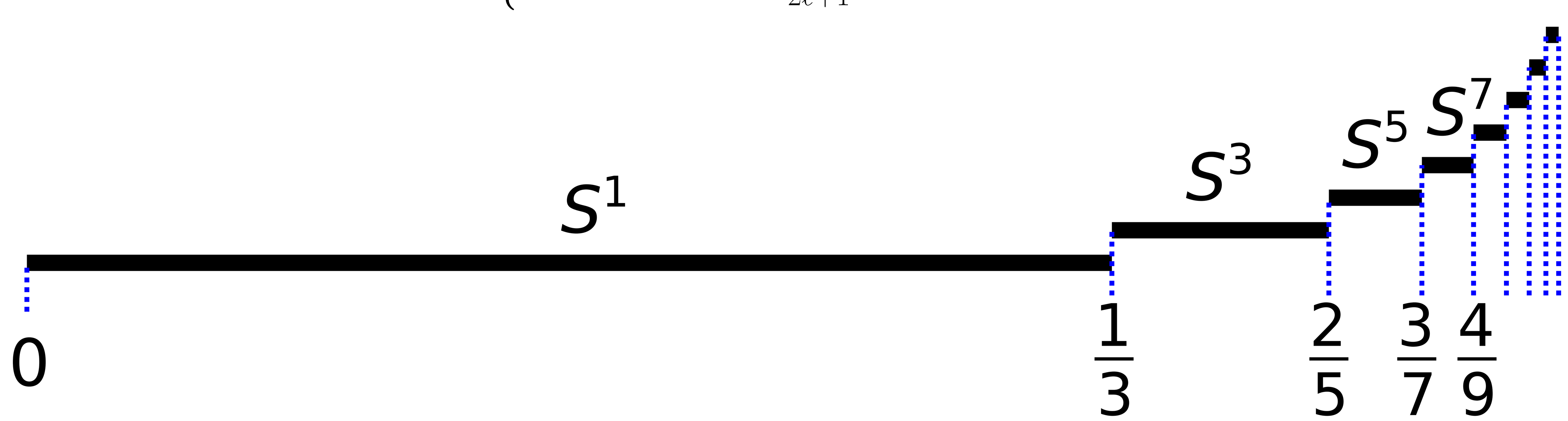
Example. $\mathrm{VR}(9 \text{ even points on } S^1, \frac{1}{3}) \simeq \vee^2 S^2$.

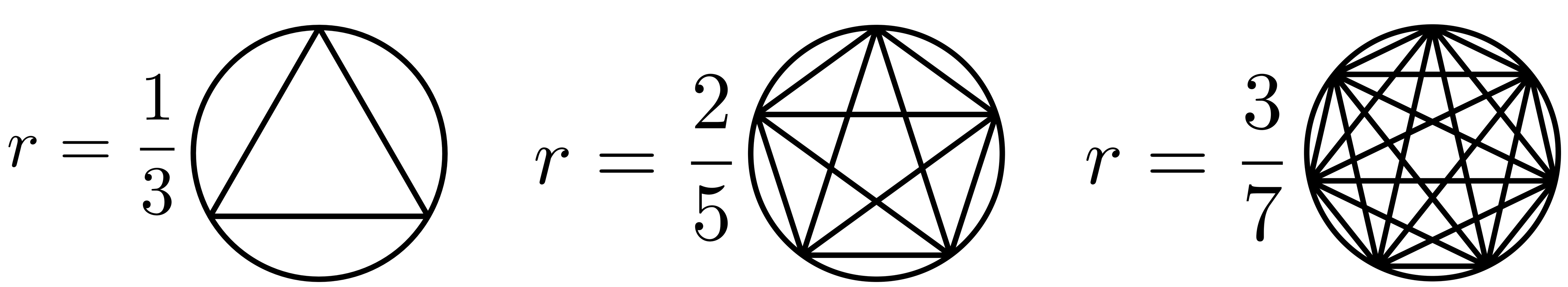


Homotopy type of $\mathrm{VR}(S^1, r)$

Theorem.

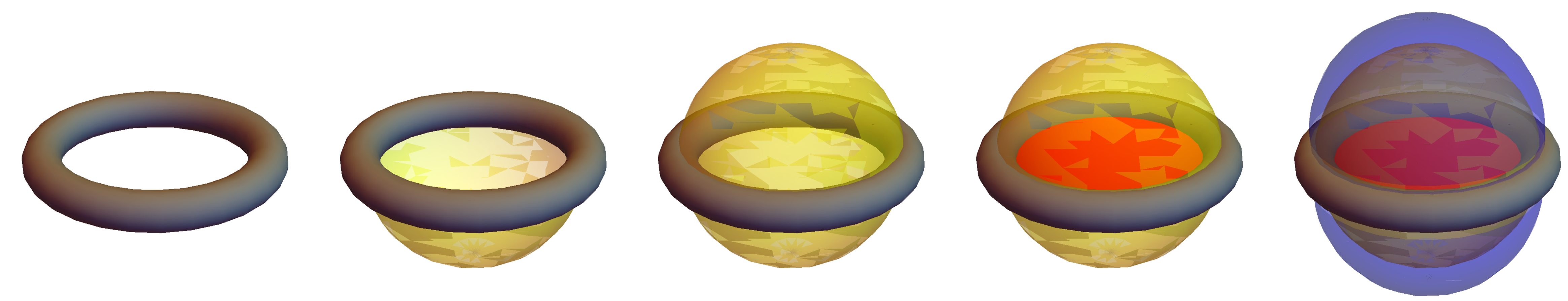
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \vee^{\infty} S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N}.$$





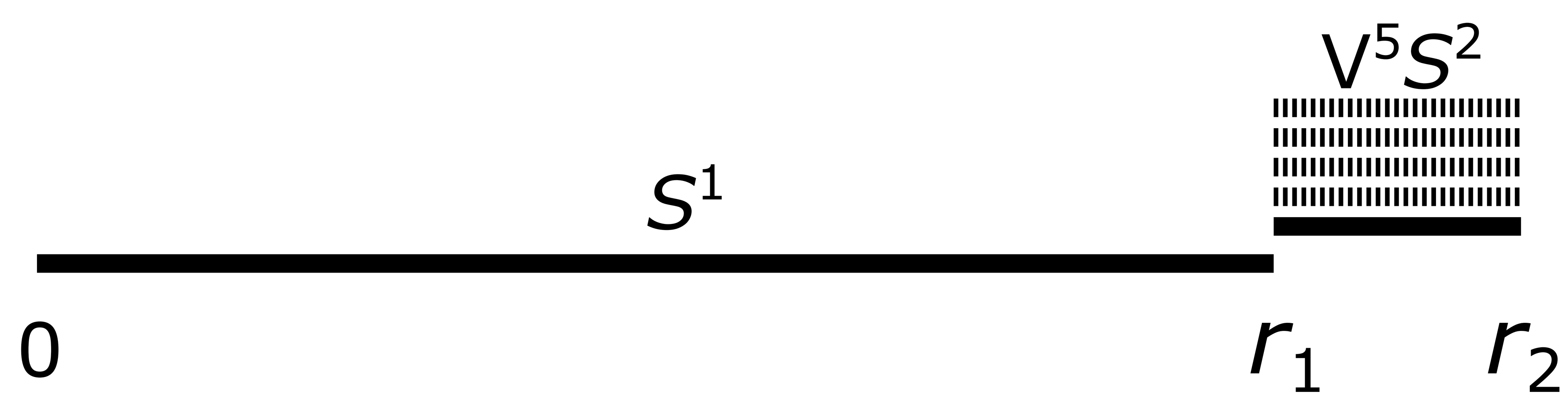
Intuition behind the theorem

$$\begin{aligned} \mathrm{VR}(S^1, \tfrac{1}{3} + \epsilon) &\simeq \mathrm{VR}(S^1, \tfrac{1}{3} - \epsilon) \cup (D^2 \times S^1) \\ &\simeq (S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1) \\ &= S^3 \end{aligned}$$



Other examples?

- $\mathrm{VR}(S^n, r)$? Relation to strongly self-dual polytopes [6]?
- For M a Riemannian manifold, is $\mathrm{conn}(\mathrm{VR}(M, r))$ a non-decreasing function of r [4]?
- $\mathrm{VR}(\text{ellipse}, r)$:



References

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[2] Michał Adamaszek, Henry Adams, Florian Frick, Chris Peterson, and Corrine Previte–Johnson, *Nerve and clique complexes of circular arcs*, Discrete & Computational Geometry 56 (2016), 251–273.

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[4] Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.

[5] Janko Latschev, *Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold*, Archiv der Mathematik 77 (2001), 522–528.

[6] Lásló Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Scientiarum Mathematicarum 45 (1983), 317–323.