# The Vietoris-Rips Complex of the Circle

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## Notation

Let $S^1$ be the circle of unit circumference with the geodesic metric.

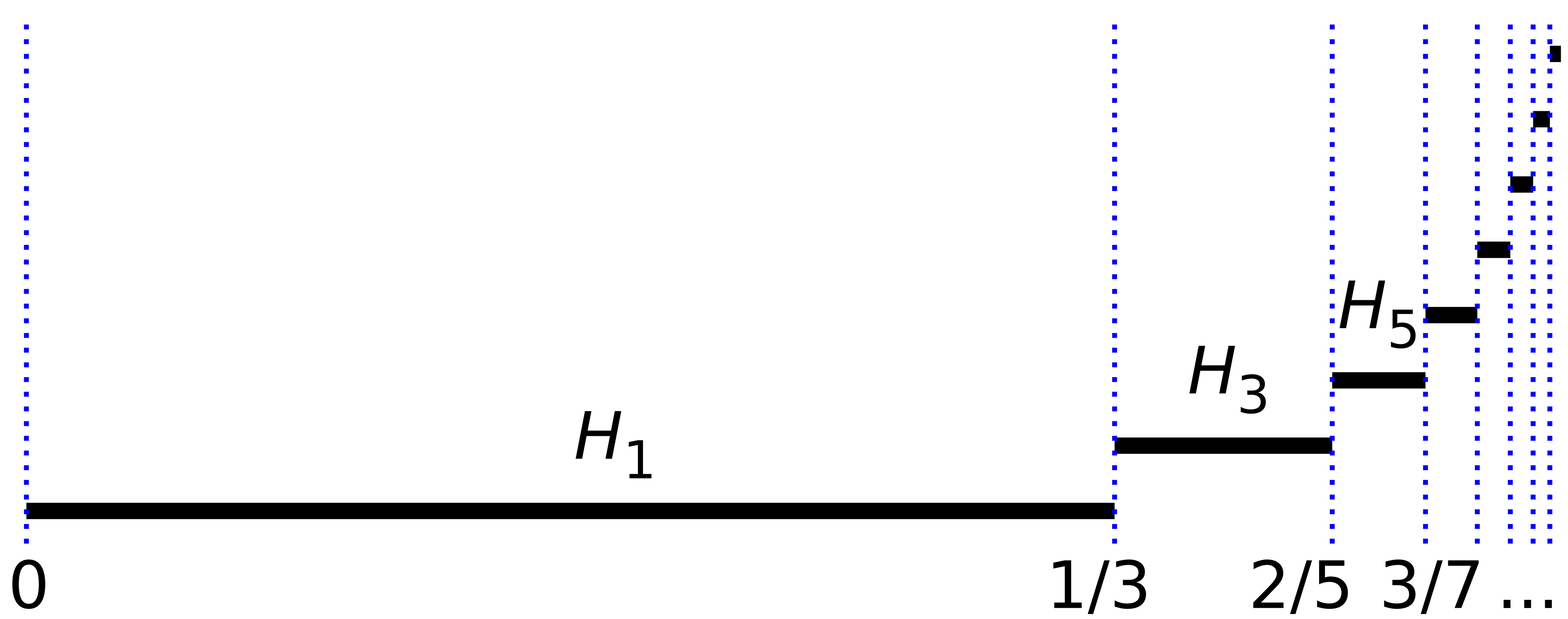
**Definition.** For $X \subset S^1$ and $r \geq 0$, let $\mathrm{VR}(X, r)$ be the Vietoris–Rips complex with vertex set $X$ and connectivity parameter $r$. That is, simplex $\sigma \subset X$ is in $\mathrm{VR}(X, r)$ when $\mathrm{diam}(\sigma) \leq r$.

## Persistent Homology of $\mathrm{VR}(S^1, r)$

**Theorem 1.** *The odd-dimensional persistent homology of* $\mathrm{VR}(S^1, r)$ *is*

$$\mathrm{dgm}\Big(H_{2l+1}\big(\mathrm{VR}(S^1)\big)\Big) = \Big\{\Big(\frac{l}{2l+1}, \frac{l+1}{2l+3}\Big)\Big\},$$

*and within this interval* $\mathrm{VR}(S^1, r) \simeq S^{2l+1}$. *The even-dimensional persistent homology has no intervals of positive length.*




## References

[1] Michał Adamaszek, *Clique complexes and graph powers*, Israel Journal of Mathematics 196 (2013), 295–319.

[2] Frédéric Chazal, Vin de Silva, Steve Oudot, *Persistence stability for geometric complexes*, Geometriae Dedicata (2013), 1–22.

[3] Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.

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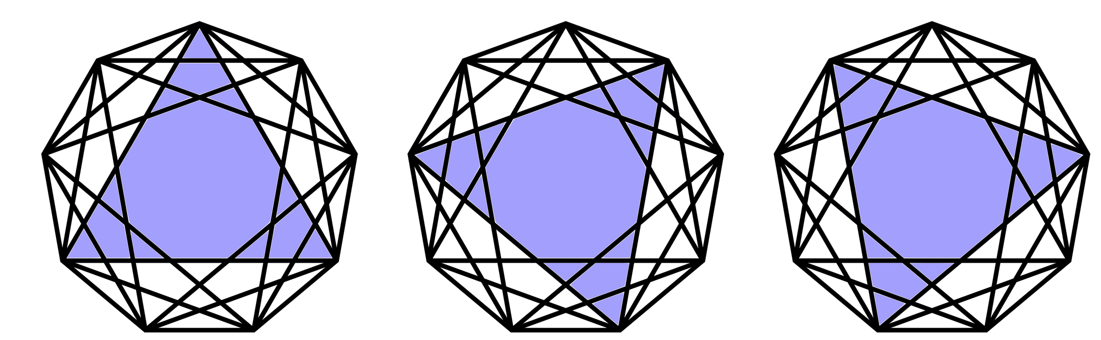
## Homotopy Types of $\mathrm{VR}(X, r)$ for $X \subset S^1$

**Theorem 2.** *For* $X \subset S^1$ *finite,* $\mathrm{VR}(X, r)$ *is homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.*

For example, let $X_n \subset S^1$ consist of $n$ evenly-spaced points. This case is given in [1].

**Example.** We have $\mathrm{VR}(X_9, 1/3) \simeq \vee_2 S^2$.



**Corollary 6.7 from [1].** *Let* $k < n/2$. *Then*

$$\mathrm{VR}(X_n, k/n) \simeq \begin{cases} \vee_{n-2k-1} S^{2l} & \text{if } \frac{k}{n} = \frac{l}{2l+1} \\ S^{2l+1} & \text{if } \frac{l}{2l+1} < \frac{k}{n} < \frac{l+1}{2l+3} \end{cases} \quad \text{for some } l \geq 0.$$

| | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=4$ | $S^1$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 5 | $S^1$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 6 | $S^1$ | $S^2$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 7 | $S^1$ | $S^1$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 8 | $S^1$ | $S^1$ | $S^3$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 9 | $S^1$ | $S^1$ | $\vee_2 S^2$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 10 | $S^1$ | $S^1$ | $S^1$ | $S^4$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 11 | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 12 | $S^1$ | $S^1$ | $S^1$ | $\vee_3 S^2$ | $S^5$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 13 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 14 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^6$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 15 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $\vee_4 S^2$ | $\vee_2 S^4$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 16 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^7$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 17 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^5$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 18 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $\vee_5 S^2$ | $S^3$ | $S^8$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 19 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^5$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| 20 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $\vee_3 S^4$ | $S^9$ | $*$ | $*$ | $*$ | $*$ |
| 21 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $\vee_6 S^2$ | $S^3$ | $\vee_2 S^6$ | $*$ | $*$ | $*$ | $*$ |
| 22 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^5$ | $S^{10}$ | $*$ | $*$ | $*$ |
| 23 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^3$ | $S^7$ | $*$ | $*$ | $*$ |
| 24 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $\vee_7 S^2$ | $S^3$ | $S^5$ | $S^{11}$ | $*$ | $*$ |
| 25 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $\vee_4 S^4$ | $S^7$ | $*$ | $*$ |
| 26 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^3$ | $S^5$ | $S^{12}$ | $*$ |
| 27 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $\vee_8 S^2$ | $S^3$ | $S^5$ | $\vee_2 S^8$ | $*$ |
| 28 | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^1$ | $S^3$ | $S^3$ | $\vee_3 S^6$ | $S^{13}$ |