The Vietoris–Rips Complex of the Circle

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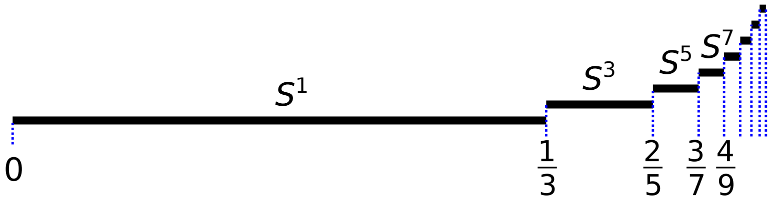
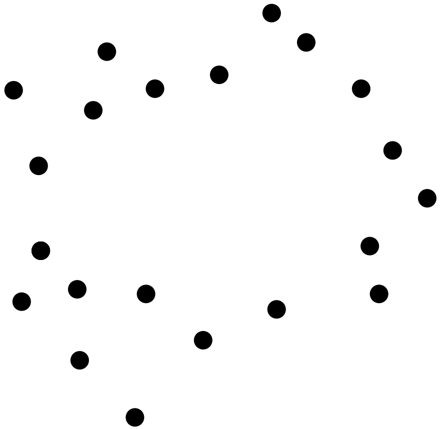
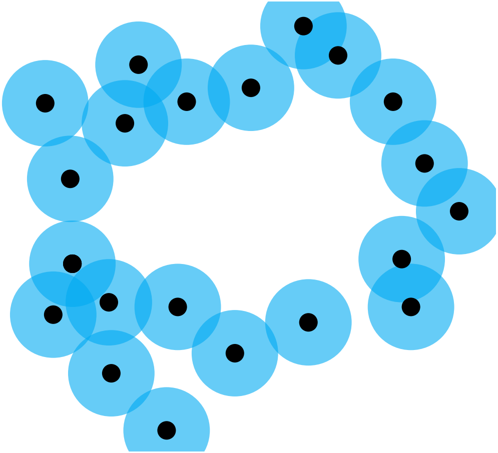
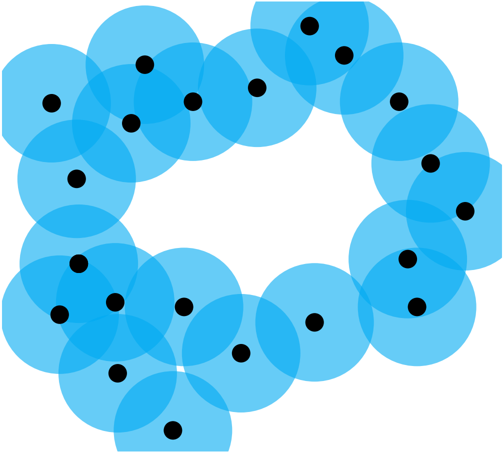
Duke University, Institute for Mathematics and its Applications

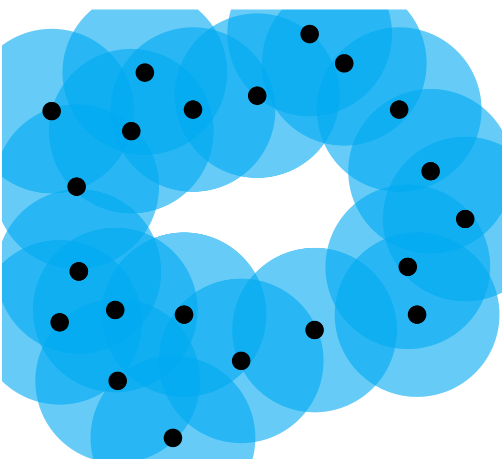
Joint with Michał Adamaszek
University of Copenhagen

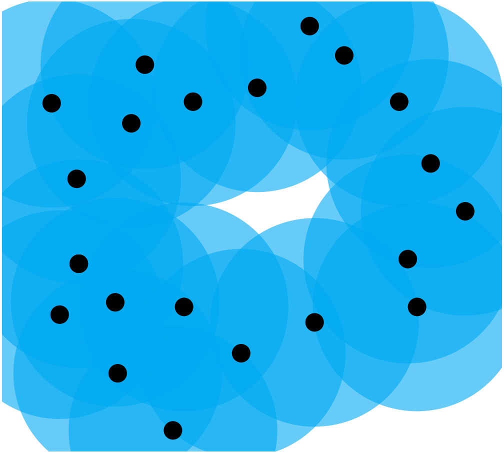


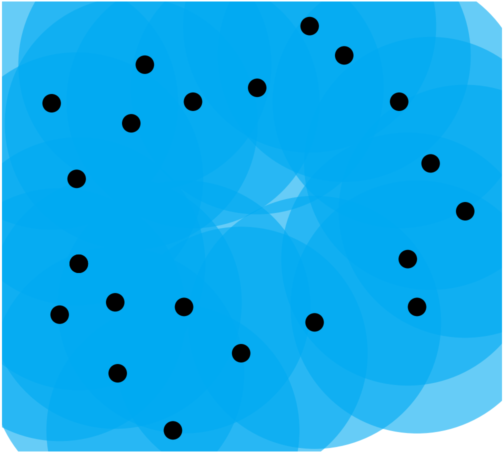


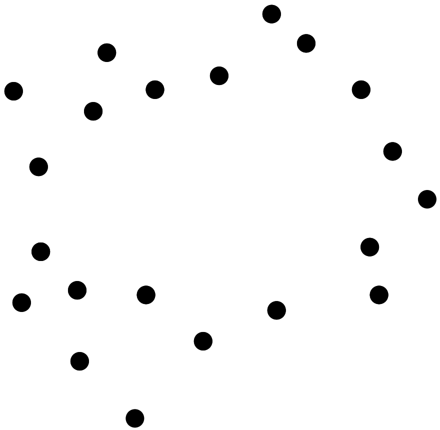








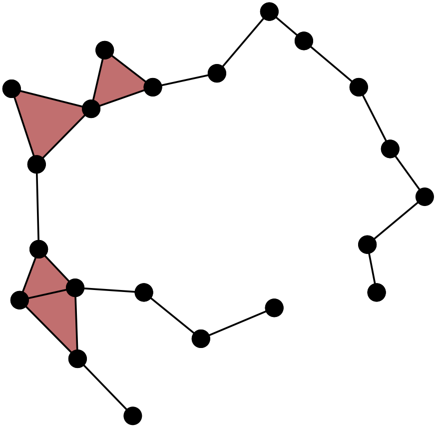




Definition

For metric space X and scale $r \geq 0$, the *Vietoris-Rips simplicial complex* $\mathrm{VR}(X, r)$ has

- vertex set X
- finite simplex σ when $\mathrm{diam}(\sigma) \leq r$.



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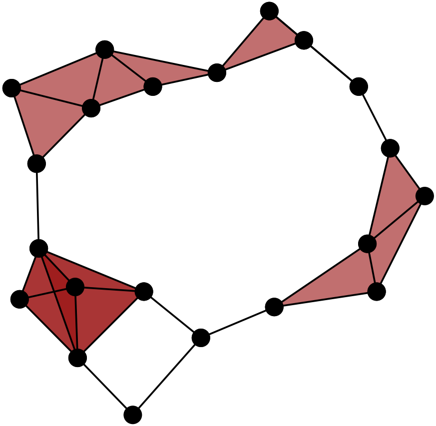

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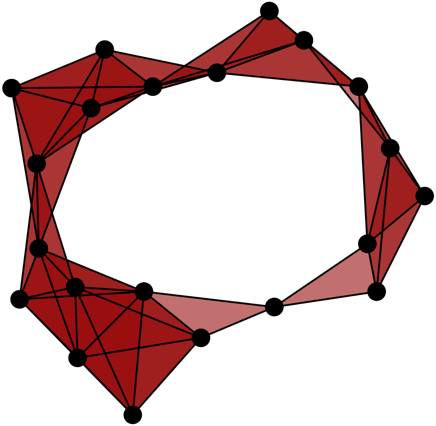

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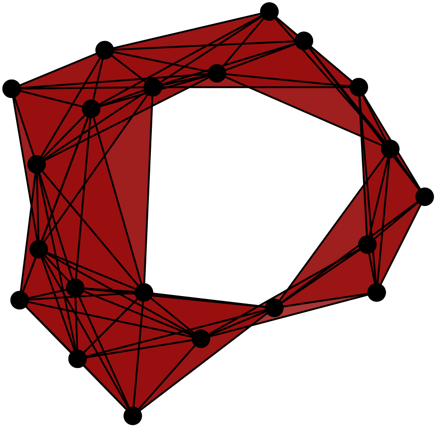

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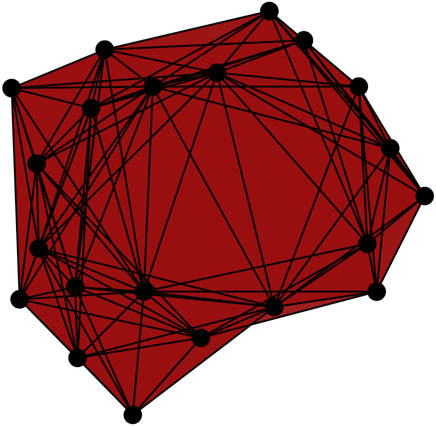

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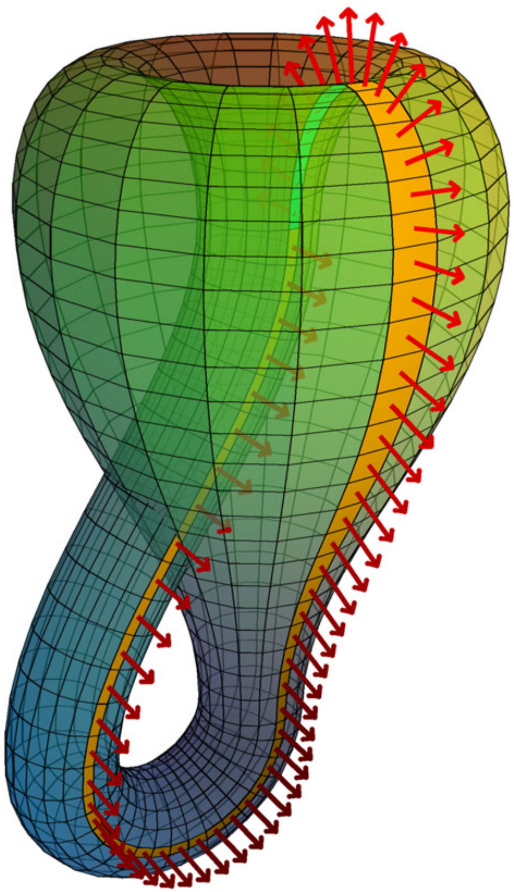
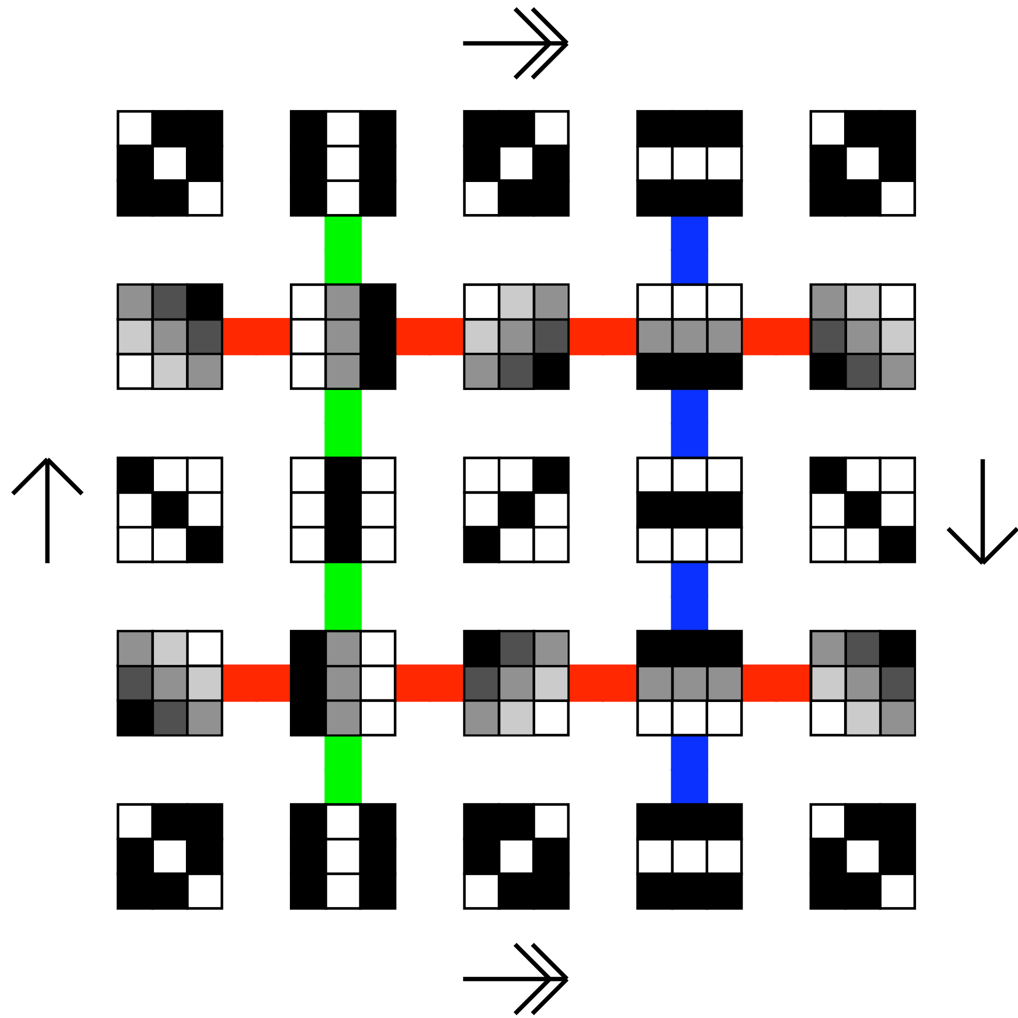
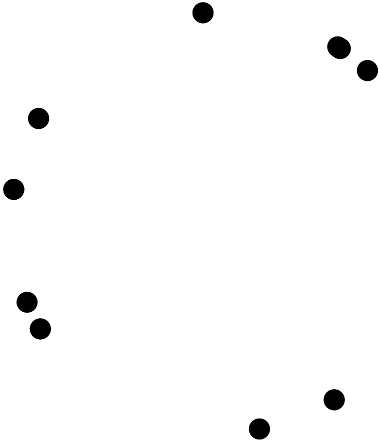




Nature prefers linear and
quadratic gradients at all angles.



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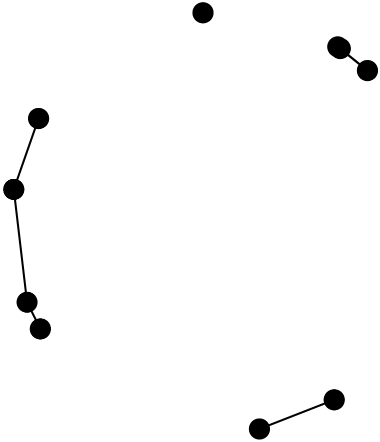

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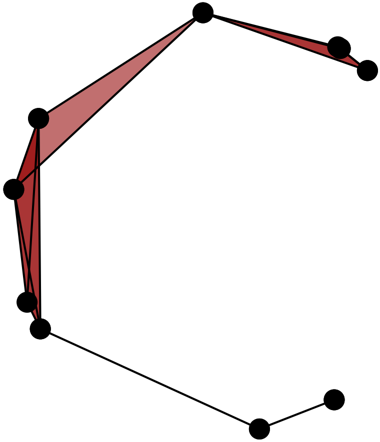

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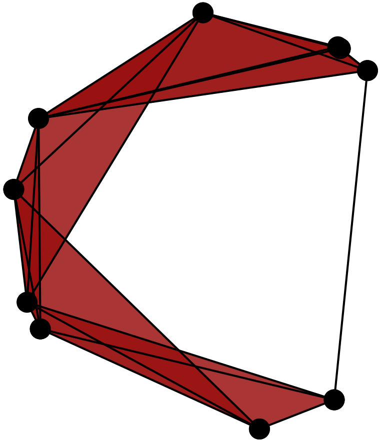

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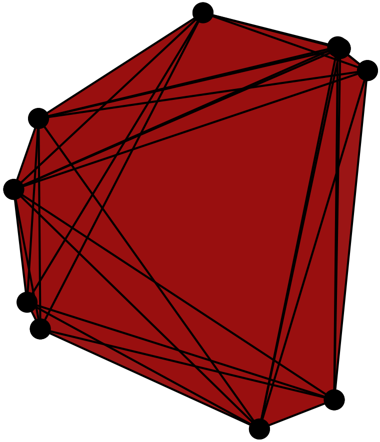

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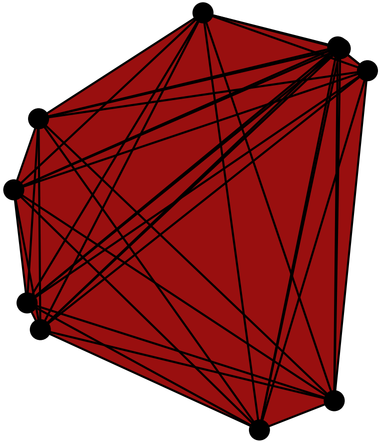

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Theorem (Hausmann, 1995)

For M a compact Riemannian manifold and r sufficiently small, $\mathrm{VR}(M, r) \simeq M$.

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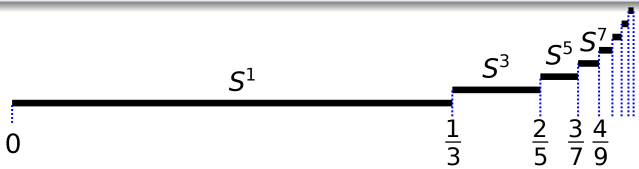
What happens when r is not small?

Let S^1 be the circle of unit circumference and $0 \leq r < \frac{1}{2}$.

Theorem (Adamaszek, HA, 2014)

$$\mathrm{VR}(S^1, r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ & \end{cases} \quad \text{for some } \ell \in \mathbb{N}.$$



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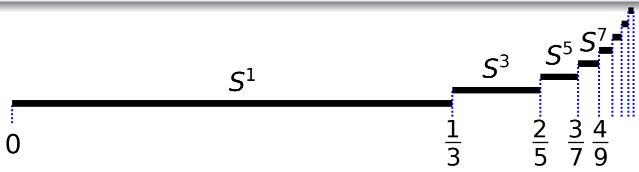

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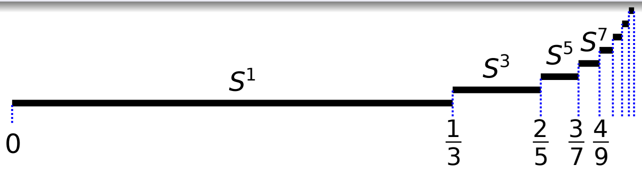
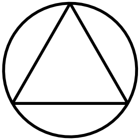
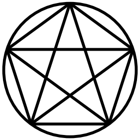
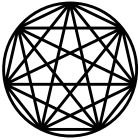


$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

$r = \frac{3}{7}$ 

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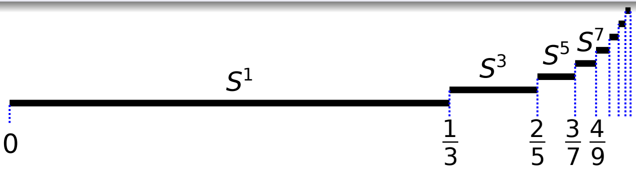
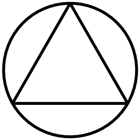
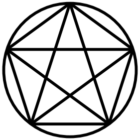
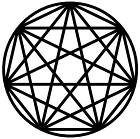


$r = \frac{1}{3}$  $r = \frac{2}{5}$  $r = \frac{3}{7}$ 

Intuition

$$S^3 = S^1 \times D^2 \bigcup_{S^1 \times S^1} D^2 \times S^1 \qquad S^{2\ell+1} = S^{2\ell-1} \times D^2 \bigcup_{S^{2\ell-1} \times S^1} D^{2\ell} \times S^1$$

Let S^1 be the circle of unit circumference and $0 \leq r < \frac{1}{2}$.

Theorem (Adamaszek, HA, 2014)

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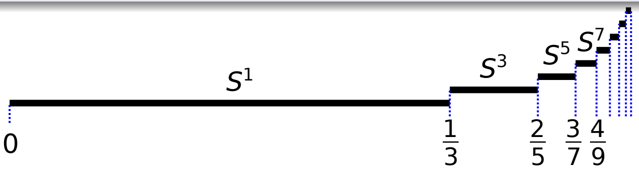
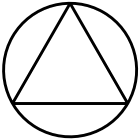
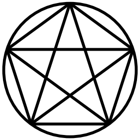
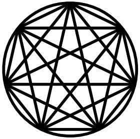


$r = \frac{1}{3}$  $r = \frac{2}{5}$  $r = \frac{3}{7}$ 

Corollary

The persistent homology of $\mathrm{VR}(S^1, r)$ *has a single interval* $\left(\frac{\ell}{2\ell+1}, \frac{\ell+1}{2\ell+3}\right)$ *in each dimension* $2\ell + 1$.

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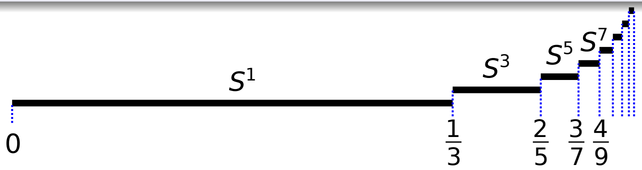

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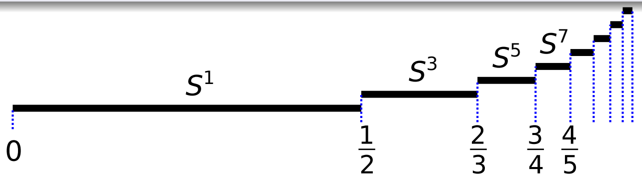

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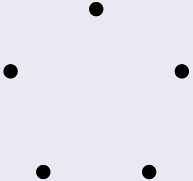
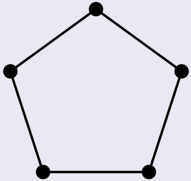
Evenly-spaced subsets

Let $X_n \subset S^1$ be n evenly-spaced points.



$\mathrm{VR}(X_5, \frac{0}{5}) \cong \bigvee^4 S^0$



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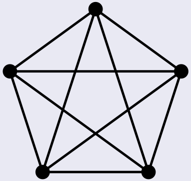

$\mathrm{VR}(X_5, \frac{2}{5}) \simeq *$

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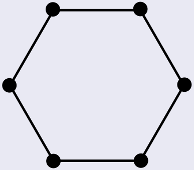
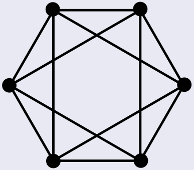


$\mathrm{VR}(X_6, \frac{1}{6}) \cong S^1$



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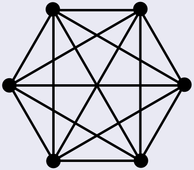

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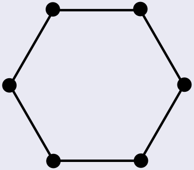

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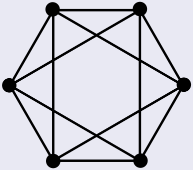

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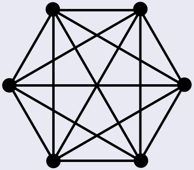

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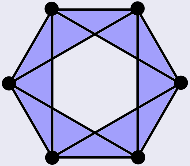

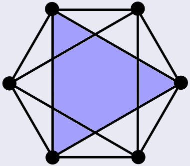

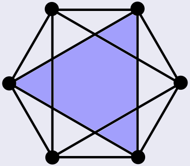

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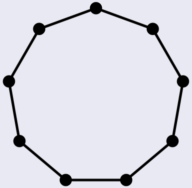
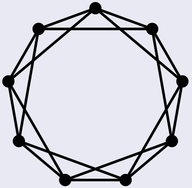
Evenly-spaced subsets

Let $X_n \subset S^1$ be n evenly-spaced points.



$\mathrm{VR}(X_9, \frac{1}{9}) \cong S^1$



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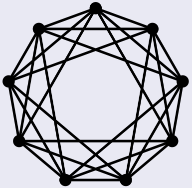

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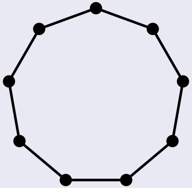

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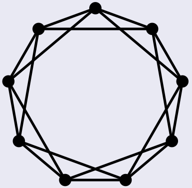

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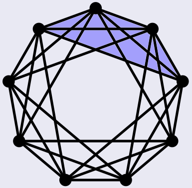

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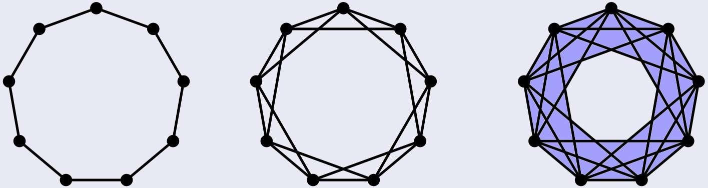

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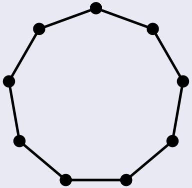

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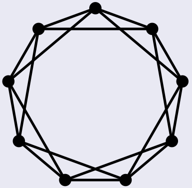

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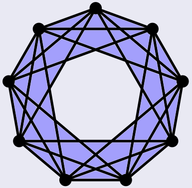

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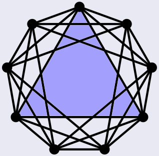

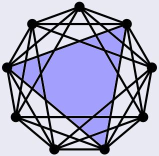

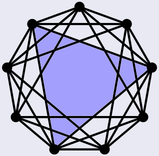

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	$k=0$	1	2	3	4	5	6	7	8	9	10
$n=5$	$\bigvee^4 S^0$	S^1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
6	$\bigvee^5 S^0$	S^1	S^2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
7	$\bigvee^6 S^0$	S^1	S^1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
8	$\bigvee^7 S^0$	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
10	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	$*$	$*$	$*$	$*$	$*$	$*$
11	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$
12	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	$*$	$*$	$*$	$*$	$*$
13	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$
14	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	$*$	$*$	$*$	$*$
15	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	$*$	$*$	$*$	$*$
16	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	$*$	$*$	$*$
17	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$
18	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$
19	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$

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	$k=7$	8	9	10	11	12	13	14	15	16
$n=20$	S^3	$\vee^3 S^4$	S^9	$*$	$*$	$*$	$*$	$*$	$*$	$*$
21	$\vee^6 S^2$	S^3	$\vee^2 S^6$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
22	S^1	S^3	S^5	S^{10}	$*$	$*$	$*$	$*$	$*$	$*$
23	S^1	S^3	S^3	S^7	$*$	$*$	$*$	$*$	$*$	$*$
24	S^1	$\vee^7 S^2$	S^3	S^5	S^{11}	$*$	$*$	$*$	$*$	$*$
25	S^1	S^1	S^3	$\vee^4 S^4$	S^7	$*$	$*$	$*$	$*$	$*$
26	S^1	S^1	S^3	S^3	S^5	S^{12}	$*$	$*$	$*$	$*$
27	S^1	S^1	$\vee^8 S^2$	S^3	S^5	$\vee^2 S^8$	$*$	$*$	$*$	$*$
28	S^1	S^1	S^1	S^3	S^3	$\vee^3 S^6$	S^{13}	$*$	$*$	$*$
29	S^1	S^1	S^1	S^3	S^3	S^5	S^9	$*$	$*$	$*$
30	S^1	S^1	S^1	$\vee^9 S^2$	S^3	$\vee^5 S^4$	S^7	S^{14}	$*$	$*$
31	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^9	$*$	$*$
32	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^7	S^{15}	$*$
33	S^1	S^1	S^1	S^1	$\vee^{10} S^2$	S^3	S^3	S^5	$\vee^2 S^{10}$	$*$
34	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^7	S^{16}

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Let $k < \frac{n}{2}$. Then

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	$k = 13$	14	15	16	17	18	19	20	21	22
$n = 35$	S^3	$\bigvee^6 S^4$	$\bigvee^4 S^6$	S^{11}	$*$	$*$	$*$	$*$	$*$	$*$
36	S^3	S^3	S^5	$\bigvee^3 S^8$	S^{17}	$*$	$*$	$*$	$*$	$*$
37	S^3	S^3	S^5	S^7	S^{11}	$*$	$*$	$*$	$*$	$*$
38	S^3	S^3	S^3	S^5	S^9	S^{18}	$*$	$*$	$*$	$*$
39	$\bigvee^{12} S^2$	S^3	S^3	S^5	S^7	$\bigvee^2 S^{12}$	$*$	$*$	$*$	$*$
40	S^1	S^3	S^3	$\bigvee^7 S^4$	S^5	S^9	S^{19}	$*$	$*$	$*$
41	S^1	S^3	S^3	S^3	S^5	S^7	S^{13}	$*$	$*$	$*$
42	S^1	$\bigvee^{13} S^2$	S^3	S^3	S^5	$\bigvee^5 S^6$	S^9	S^{20}	$*$	$*$
43	S^1	S^1	S^3	S^3	S^3	S^5	S^7	S^{13}	$*$	$*$
44	S^1	S^1	S^3	S^3	S^3	S^5	S^7	$\bigvee^3 S^{10}$	S^{21}	$*$
45	S^1	S^1	$\bigvee^{14} S^2$	S^3	S^3	$\bigvee^8 S^4$	S^5	$\bigvee^4 S^8$	$\bigvee^2 S^{14}$	$*$
46	S^1	S^1	S^1	S^3	S^3	S^3	S^5	S^7	S^{11}	S^{22}
47	S^1	S^1	S^1	S^3	S^3	S^3	S^5	S^5	S^9	S^{15}
48	S^1	S^1	S^1	$\bigvee^{15} S^2$	S^3	S^3	S^3	S^5	S^7	S^{11}
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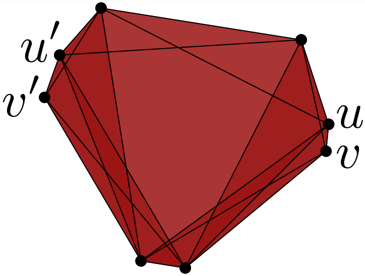
	$k=19$	20	21	22	23	24	25	26	27	28
$n=50$	S^3	$\bigvee^9 S^4$	S^5	S^7	S^{11}	S^{24}	$*$	$*$	$*$	$*$
51	S^3	S^3	S^5	S^7	S^9	$\bigvee^2 S^{16}$	$*$	$*$	$*$	$*$
52	S^3	S^3	S^5	S^5	S^7	$\bigvee^3 S^{12}$	S^{25}	$*$	$*$	$*$
53	S^3	S^3	S^3	S^5	S^7	S^9	S^{17}	$*$	$*$	$*$
54	S^3	S^3	S^3	S^5	S^5	$\bigvee^5 S^8$	S^{13}	S^{26}	$*$	$*$
55	S^3	S^3	S^3	$\bigvee^{10} S^4$	S^5	S^7	$\bigvee^4 S^{10}$	S^{17}	$*$	$*$
56	S^3	S^3	S^3	S^3	S^5	$\bigvee^7 S^6$	S^9	S^{13}	S^{27}	$*$
57	$\bigvee^{18} S^2$	S^3	S^3	S^3	S^5	S^5	S^7	S^{11}	$\bigvee^2 S^{18}$	$*$
58	S^1	S^3	S^3	S^3	S^3	S^5	S^7	S^9	S^{13}	S^{28}
59	S^1	S^3	S^3	S^3	S^3	S^5	S^5	S^7	S^{11}	S^{19}
60	S^1	$\bigvee^{19} S^2$	S^3	S^3	S^3	$\bigvee^{11} S^4$	S^5	S^7	S^9	$\bigvee^3 S^{14}$
61	S^1	S^1	S^3	S^3	S^3	S^3	S^5	S^5	S^7	S^{11}
62	S^1	S^1	S^3	S^3	S^3	S^3	S^5	S^5	S^7	S^9
63	S^1	S^1	$\bigvee^{20} S^2$	S^3	S^3	S^3	S^3	S^5	$\bigvee^8 S^6$	$\bigvee^6 S^8$
64	S^1	S^1	S^1	S^3	S^3	S^3	S^3	S^5	S^5	S^7

Theorem for arbitrary subsets (Adamaszek, HA, Frick, Peterson, Previte–Johnson, 2014)

For $X \subset S^1$ finite, $\mathrm{VR}(X, r) \simeq \begin{cases} S^{2\ell+1} & \text{for some } \ell \in \mathbb{N}, \text{ or} \\ \bigvee^m S^{2\ell} & \text{for some } \ell, m \in \mathbb{N}. \end{cases}$

Computable in time $O(|X| \log |X|)$.



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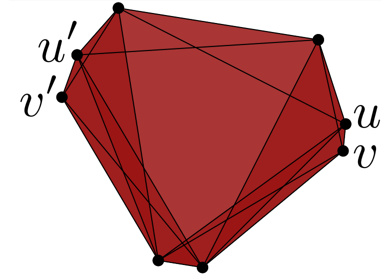

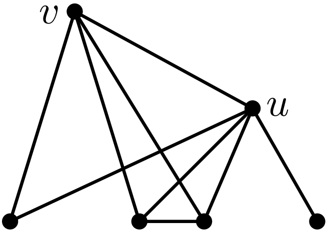

Proof Sketch

If $N[v] \subseteq N[u]$ (v is *dominated* by u), then $\mathrm{lk}(v)$ is a cone and

$$\mathrm{VR}(X, r) \simeq \mathrm{VR}(X \setminus v, r).$$

Remove dominated vertices. Result isomorphic to some $\mathrm{VR}(X_n, r)$.

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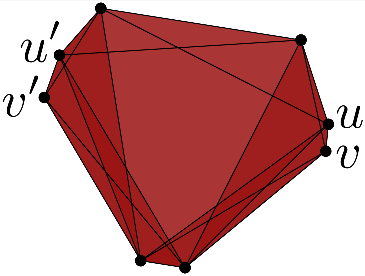

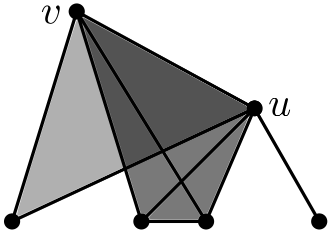

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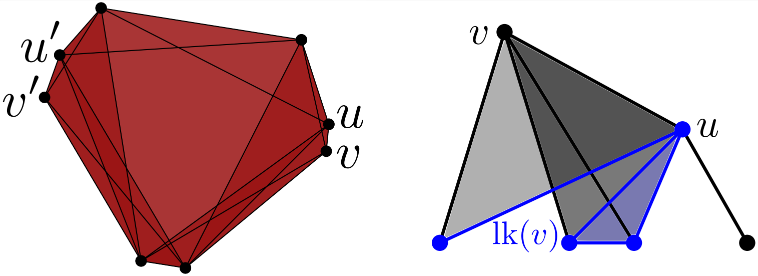

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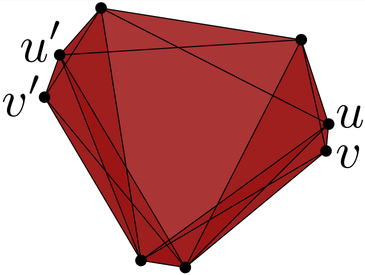

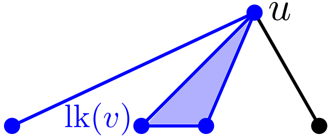

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If $N[v] \subseteq N[u]$ (v is *dominated* by u), then $\mathrm{lk}(v)$ is a cone and

$$\mathrm{VR}(X, r) \simeq \mathrm{VR}(X \setminus v, r).$$

Remove dominated vertices. Result isomorphic to some $\mathrm{VR}(X_n, r)$.

Theorem for arbitrary subsets (Adamaszek, HA, Frick, Peterson, Previte–Johnson, 2014)

For $X \subset S^1$ finite, $\mathrm{VR}(X, r) \simeq \begin{cases} S^{2\ell+1} & \text{for some } \ell \in \mathbb{N}, \text{ or} \\ \bigvee^m S^{2\ell} & \text{for some } \ell, m \in \mathbb{N}. \end{cases}$

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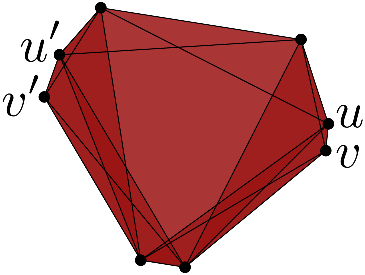

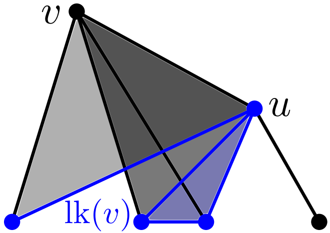

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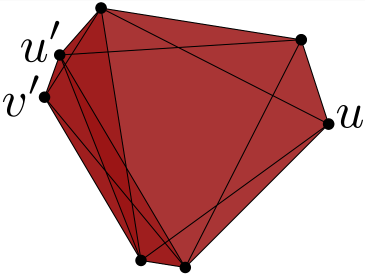

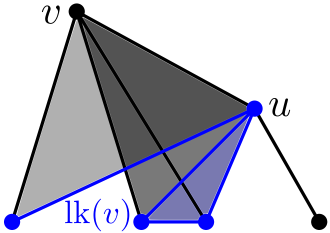

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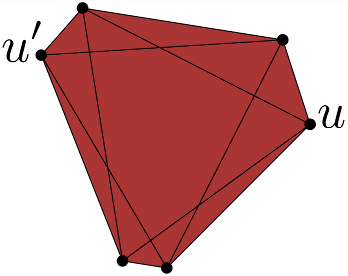

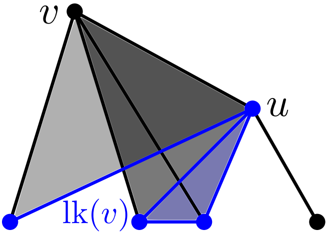

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Note $\mathrm{lk}(v) = \mathrm{VR}(N(v), r)$. We'll show $\tilde{H}_*\,\mathrm{lk}(v) = 0$.
So i_* is an isomorphism.

Theorem (Adamaszek, HA, 2014)

$$\mathrm{VR}(S^1, r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N}.$$

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1. Show $\mathrm{VR}(X, r) \simeq S^{2\ell+1}$:
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   By Whitehead's Theorem, suffices to show

$$\pi_k\big(\mathrm{VR}(X, r), x_0\big) \xrightarrow{\cong} \pi_k\big(\mathrm{VR}(S^1, r), x_0\big) \quad \forall k.$$

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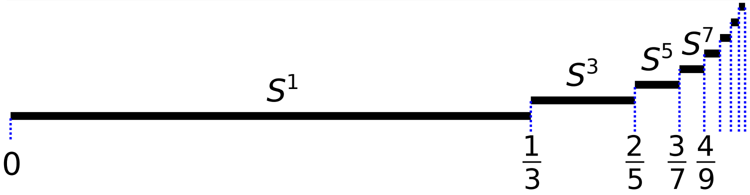

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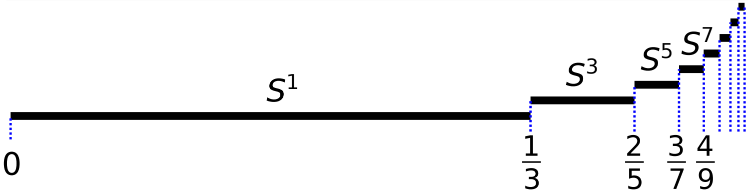

Can handle annuli, tori with the ℓ_∞ metric, and wedge sums.

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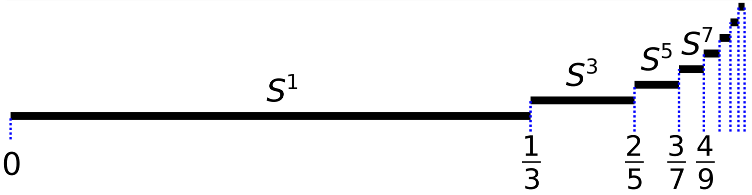

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Future work

- $\mathrm{VR}(S^n, r)$?
- $\mathrm{VR}(\text{ellipse} \subset \mathbb{R}^2, r)$? $\mathrm{VR}(X \approx S^1 \subset \mathbb{R}^2, r)$?
- Is $\mathrm{conn}(\mathrm{VR}(M, r))$ a non-decreasing function of r?
- Structure of the set of critical values for M compact?
- For r generic do we have $\mathrm{VR}(X_{\text{suff. dense}}, r) \xrightarrow{\simeq} \mathrm{VR}(M, r)$?

Uniformly random points

Theorem (Adamaszek, HA, 2014)

Let $\frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3}$ and $\delta = r - \frac{\ell}{2\ell+1}$. As $\delta \to 0$ we have

$$\mathrm{E}[\#\text{ points until } \vee^m S^{2\ell}] = \Theta\left(\delta^{-\frac{2\ell}{2\ell+1}}\right)$$
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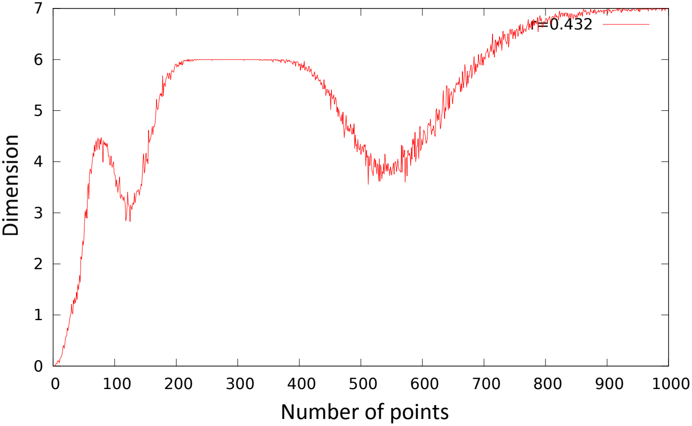

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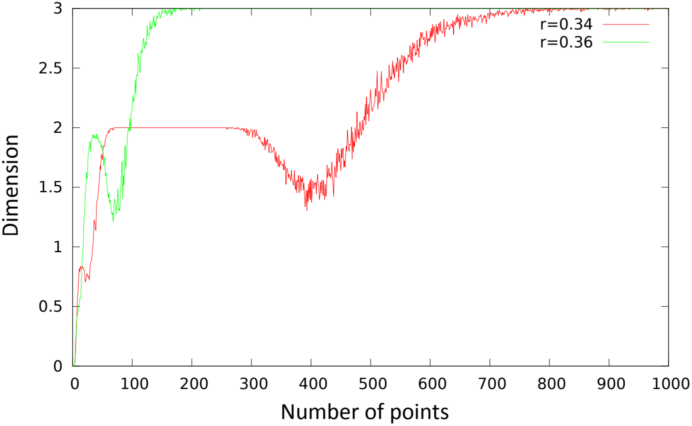


Definition

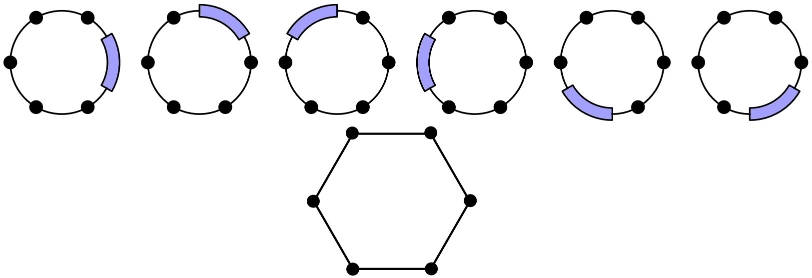
For $X \subseteq S^1$, the *ambient Čech complex* $\check{C}(X, r)$ has

- vertex set X
- simplex $[x_0, \ldots, x_k]$ when $\cap_{i=0}^k B(x_i, \frac{r}{2}) \neq 0$.

Note $\mathrm{Cl}(\check{C}(X, r)) = \mathrm{VR}(X, r)$.

Example

$\check{C}(X_6, \frac{1}{6}) = S^1$.



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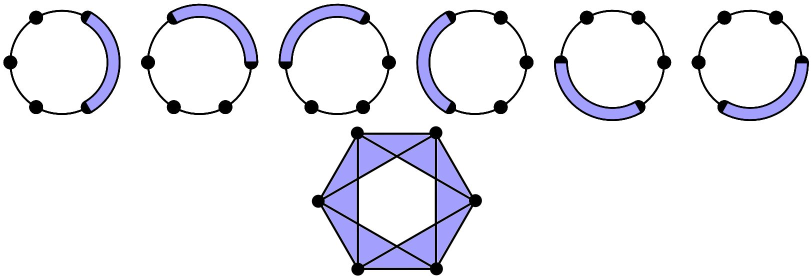

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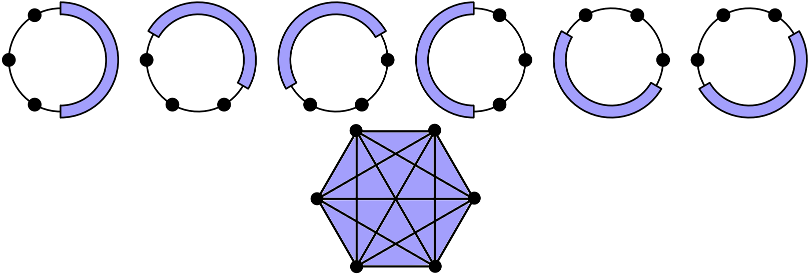

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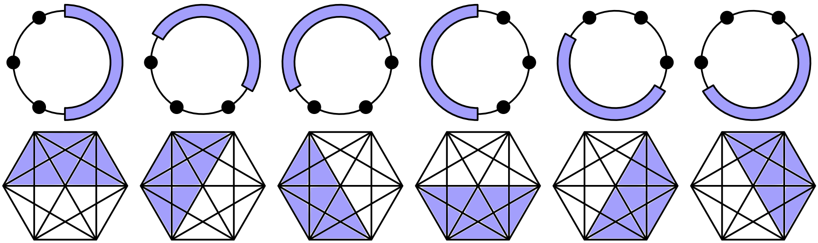
Example

$\check{C}(X_6, \frac{3}{6}) \simeq \bigvee^2 S^2$.



Definition

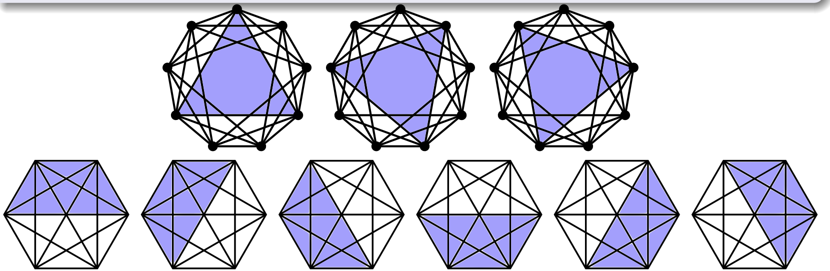
For $X \subseteq S^1$, the *ambient Čech complex* $\check{C}(X, r)$ has

- vertex set X
- simplex $[x_0, \ldots, x_k]$ when $\cap_{i=0}^{k} B(x_i, \frac{r}{2}) \neq 0$.

Note $\mathrm{Cl}(\check{C}(X, r)) = \mathrm{VR}(X, r)$.

Example

$\check{C}(X_6, \frac{3}{6}) \simeq \bigvee^2 S^2 \simeq \mathrm{VR}(X_9, \frac{3}{9})$.



$\mathrm{VR}(X_n, \frac{k}{n})$

	$k=1$	2	3	4	5
$n=4$	S^1	$*$	$*$	$*$	$*$
5	S^1	$*$	$*$	$*$	$*$
6	S^1	S^2	$*$	$*$	$*$
7	S^1	S^1	$*$	$*$	$*$
8	S^1	S^1	S^3	$*$	$*$
9	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$
10	S^1	S^1	S^1	S^4	$*$
11	S^1	S^1	S^1	S^3	$*$
12	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5
13	S^1	S^1	S^1	S^1	S^3
14	S^1	S^1	S^1	S^1	S^3
15	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$
16	S^1	S^1	S^1	S^1	S^1
17	S^1	S^1	S^1	S^1	S^1

$\check{\mathrm{C}}(X_n, \frac{k}{n})$

	$k=1$	2	3	4	5
$n=3$	S^1	$*$	$*$	$*$	$*$
4	S^1	S^2	$*$	$*$	$*$
5	S^1	S^1	S^3	$*$	$*$
6	S^1	S^1	$\bigvee^2 S^2$	S^4	$*$
7	S^1	S^1	S^1	S^3	S^5
8	S^1	S^1	S^1	$\bigvee^3 S^2$	S^3
9	S^1	S^1	S^1	S^1	S^3
10	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$
11	S^1	S^1	S^1	S^1	S^1
12	S^1	S^1	S^1	S^1	S^1
13	S^1	S^1	S^1	S^1	S^1
14	S^1	S^1	S^1	S^1	S^1
15	S^1	S^1	S^1	S^1	S^1
16	S^1	S^1	S^1	S^1	S^1

$\mathrm{VR}(X_n, \frac{k}{n})$

	$k=6$	7	8	9	10
$n=14$	S^6	$*$	$*$	$*$	$*$
15	$\bigvee^2 S^4$	$*$	$*$	$*$	$*$
16	S^3	S^7	$*$	$*$	$*$
17	S^3	S^5	$*$	$*$	$*$
18	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$
19	S^1	S^3	S^5	$*$	$*$
20	S^1	S^3	$\bigvee^3 S^4$	S^9	$*$
21	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	$*$
22	S^1	S^1	S^3	S^5	S^{10}
23	S^1	S^1	S^3	S^3	S^7
24	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
25	S^1	S^1	S^1	S^3	$\bigvee^4 S^4$
26	S^1	S^1	S^1	S^3	S^3
27	S^1	S^1	S^1	$\bigvee^8 S^2$	S^3

$\check{\mathrm{C}}(X_n, \frac{k}{n})$

	$k=6$	7	8	9	10
$n=6$	$*$	$*$	$*$	$*$	$*$
7	$*$	$*$	$*$	$*$	$*$
8	S^6	$*$	$*$	$*$	$*$
9	$\bigvee^2 S^4$	S^7	$*$	$*$	$*$
10	S^3	S^5	S^8	$*$	$*$
11	S^3	S^3	S^5	S^9	$*$
12	$\bigvee^5 S^2$	S^3	$\bigvee^3 S^4$	$\bigvee^2 S^6$	S^{10}
13	S^1	S^3	S^3	S^5	S^7
14	S^1	$\bigvee^6 S^2$	S^3	S^3	S^5
15	S^1	S^1	S^3	S^3	$\bigvee^4 S^4$
16	S^1	S^1	$\bigvee^7 S^2$	S^3	S^3
17	S^1	S^1	S^1	S^3	S^3
18	S^1	S^1	S^1	$\bigvee^8 S^2$	S^3
19	S^1	S^1	S^1	S^1	S^3

$\mathrm{VR}(X_n, \frac{k}{n})$

	$k=6$	7	8	9	10
$n=14$	S^6	*	*	*	*
15	$\bigvee^2 S^4$	*	*	*	*
16	S^3	S^7	*	*	*
17	S^3	S^5	*	*	*
18	$\bigvee^5 S^2$	S^3	S^8	*	*
19	S^1	S^3	S^5	*	*
20	S^1	S^3	$\bigvee^3 S^4$	S^9	*
21	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	*
22	S^1	S^1	S^3	S^5	S^{10}
23	S^1	S^1	S^3	S^3	S^7
24	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
25	S^1	S^1	S^1	S^3	$\bigvee^4 S^4$
26	S^1	S^1	S^1	S^3	S^3
27	S^1	S^1	S^1	$\bigvee^8 S^2$	S^3

$\check{\mathrm{C}}(X_n, \frac{k}{n})$

	$k=6$	7	8	9	10
$n=6$	*	*	*	*	*
7	*	*	*	*	*
8	S^6	*	*	*	*
9	$\bigvee^2 S^4$	S^7	*	*	*
10	S^3	S^5	S^8	*	*
11	S^3	S^3	S^5	S^9	*
12	$\bigvee^5 S^2$	S^3	$\bigvee^3 S^4$	$\bigvee^2 S^6$	S^{10}
13	S^1	S^3	S^3	S^5	S^7
14	S^1	$\bigvee^6 S^2$	S^3	S^3	S^5
15	S^1	S^1	S^3	S^3	$\bigvee^4 S^4$
16	S^1	S^1	$\bigvee^7 S^2$	S^3	S^3
17	S^1	S^1	S^1	S^3	S^3
18	S^1	S^1	S^1	$\bigvee^8 S^2$	S^3
19	S^1	S^1	S^1	S^1	S^3

Theorem (Adamaszek, HA, Frick, Peterson, Previte–Johnson, 2014)

$\mathrm{VR}(X_{n+k}, \frac{k}{n+k}) \xrightarrow{\simeq} \check{\mathrm{C}}(X_n, \frac{k}{n})$ *via* $i \mapsto i \mod n$.

$\mathrm{VR}(X_n, \frac{k}{n})$

	$k=1$	2	3	4	5
$n=4$	S^1	$*$	$*$	$*$	$*$
5	S^1	$*$	$*$	$*$	$*$
6	S^1	S^2	$*$	$*$	$*$
7	S^1	S^1	$*$	$*$	$*$
8	S^1	S^1	S^3	$*$	$*$
9	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$
10	S^1	S^1	S^1	S^4	$*$
11	S^1	S^1	S^1	S^3	$*$
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13	S^1	S^1	S^1	S^1	S^3
14	S^1	S^1	S^1	S^1	S^3
15	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$
16	S^1	S^1	S^1	S^1	S^1
17	S^1	S^1	S^1	S^1	S^1

$\check{\mathrm{C}}(X_n, \frac{k}{n})$

	$k=1$	2	3	4	5
$n=3$	S^1	$*$	$*$	$*$	$*$
4	S^1	S^2	$*$	$*$	$*$
5	S^1	S^1	S^3	$*$	$*$
6	S^1	S^1	$\bigvee^2 S^2$	S^4	$*$
7	S^1	S^1	S^1	S^3	S^5
8	S^1	S^1	S^1	$\bigvee^3 S^2$	S^3
9	S^1	S^1	S^1	S^1	S^3
10	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$
11	S^1	S^1	S^1	S^1	S^1
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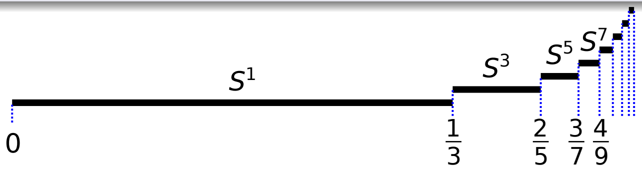
Theorem (Adamaszek, HA, Frick, Peterson, Previte–Johnson, 2014)

$$\mathrm{VR}(X_{n+k}, \tfrac{k}{n+k}) \xrightarrow{\simeq} \check{\mathrm{C}}(X_n, \tfrac{k}{n}) \textit{ via } i \mapsto i \mod n.$$

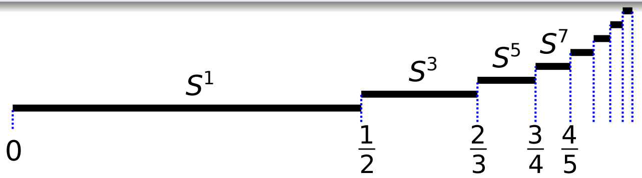
Theorem (Adamaszek, HA, 2014)

$$\mathrm{VR}(S^1, r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^\infty S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N}.$$



Theorem (Adamaszek, HA, 2014)

$$\check{\mathrm{C}}(S^1, r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{\ell+1} < r < \frac{\ell+1}{\ell+2} \\ \bigvee^\infty S^{2\ell} & \text{if } r = \frac{\ell}{\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N}.$$



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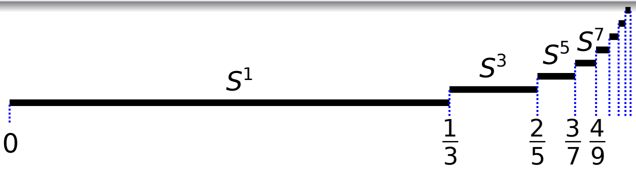

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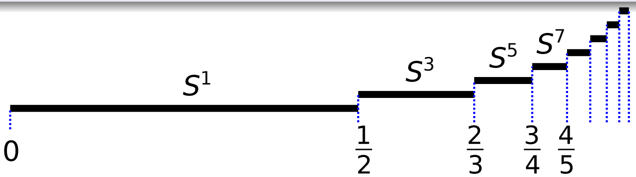

Relationship between $\mathrm{VR}(M, r)$ and $\check{\mathrm{C}}(M, r)$ for more general M?

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Thank you!

Theorem (Hausman, 1995)

Let M be a Riemannian manifold with $r(M) > 0$.
If $0 < r \leq r(M)$, then $\mathrm{VR}(M, r) \simeq M$.

Definition

Let $r(M)$ be the largest satisfying:
(a) If $d(x, y) < 2r(M)$, then $\exists!$ shortest geodesic between x and y.

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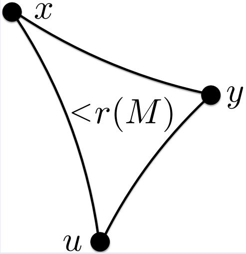
(b) 

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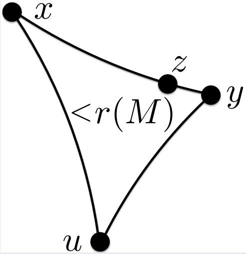
(b) 

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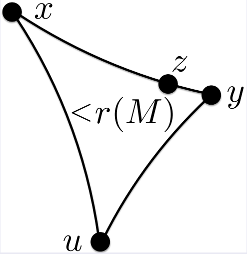
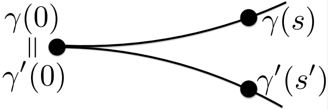
(b) 

(c) 

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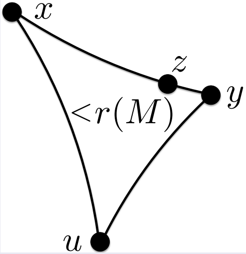
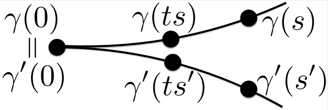
(b) 

(c) 

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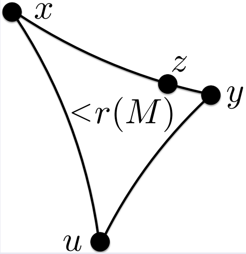
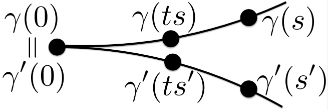
(b) 

(c) 

- The n-sphere with great circle circumference 1 has $r(S^n) = \frac{1}{4}$.
- $r(M) > 0$ if M has positive injectivity radius and bounded sectional curvature (in particular if M compact).