LINEAR ALGEBRA QUIZ 1 SOLUTION

Problem. Let V be a vector space and U be a finite subset of V. Prove that $span(U)$ is a subspace of V.

Proof. Let F be the field over which V is a vector space. Since U is a finite subset of V, let $U = \{u_1, ..., u_n\} \subset V$. In order to prove $span(U)$ is a subspace of V, theorem 1.3 in the textbook tells us we must prove 1) $0 \in span(U)$, 2) $span(U)$ is closed under addition, and 3) $span(U)$ is closed under scalar multiplication.

1) Recall that $span(U) = \{a_1u_1 + ...a_nu_n \mid a_1, ..., a_n \in F\}$ so that

$$\sum_{i=1}^{n} 0u_i = 0 \in span(U).$$

Note that this also works when U is empty since the above sum would be the empty sum, to which we assign the value 0 by convention.

2) Let $x, y \in span(U)$. Then $x = a_1u_1 + ... + a_nu_n$ and $y = b_1u_1 + ... + b_nu_n$ for some $a_1, ..., a_n, b_1, ..., b_n \in F$ by definition of $span(U)$. Now it follows that

$$x + y = a_1u_1 + ... + a_nu_n + b_1u_1 + ... + b_nu_n = (a_1 + b_1)u_1 + ... + (a_n + b_n)u_n.$$

Therefore, $x + y \in span(U)$ since each $a_i + b_i \in F$.

3) Let $c \in F$ and $x \in span(U)$. Then $x = a_1u_1 + ...a_nu_n$ for some $a_1, ..., a_n \in F$. Now it follows that

$$cx = c\,(a_1u_1 + ... + a_nu_n) = ca_1u_1 + ... + ca_nu_n.$$

Therefore, $cx \in span(U)$ since each $ca_i \in F$. $\square$