DICE GAMES AND EXPECTED VALUE

UF MATH CIRCLE

Reminder: if you're playing a game with outcomes $x_1, x_2, \ldots x_r$ that occur with probability $p_1, p_2, \ldots, p_r$, the expected value of the outcome is

$$E[G] = p_1 \cdot x_1 + p_2 \cdot x_2 + \ldots + p_r \cdot x_r$$

Example 1. The game has you roll a six-sided die and gain the number of chips shown on the die. As all six numbers on the die have an equal chance of occurring, you have a $1/6$ chance of gaining 1 chip, a 1/6 chance of gaining 2 chips, and so forth up to 6. The expected value of this game is

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

Here are a bunch of different games. Play some of them with a friend. Your friend should charge you some chips to play - how much are you willing to pay? What is the expected value of each game? For those that have choices, which option should you choose?

1. BASIC GAMES

(1) Roll a six-sided die. If it is an odd number, you gain 1 chip. If it is an even number, you gain 2 chips.

(2) You choose odd or even, and then roll one die. If the number on the die is even and you chose even, you gain 2 chips. If the number on the dice is odd and you said odd, you gain 1 chip. (What should you name? What is the expected value?)

(3) Roll a six-sided die. If it is 1 or 2, gain 3 chips. If it is $3, 4, 5$, of 6, gain 1 chip.

(4) Choose "small" or "large", then roll one dice. If you chose small and the dice is 1 or 2, gain 3 chips. If you chose large and the dice is $3, 4, 5$, or 6, gain 1 chip.

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(5) Roll a six-sided die. Gain chips depending on the value of the dice as follows:

die	1	2	3	4	5	6
chips	20	4	3	2	1	0

What is the expected value? If you had to pay 4 chips to play, would you still play?

(6) Roll two six-sided die. Gain 3 chips if both numbers are even or both numbers are odd.

(7) Roll one six-sided die. You can either gain n chips, where n is the number you got on the first role, or reroll the die and gain $m - 1$ chips, where m is the number showing the second time.

(8) Same as the previous game, except if you reroll you may continue to reroll until you get a number different from your first role.

2. Many Rolls

(9) If you roll a six-sided die 1200 times, what is the expected number of times you will get a multiple of 3?

(10) Roll two six-sided dice, and gain chips equal to the largest number shown. What is the expected value? (What about more than two dice?)

(11) Roll a six-sided die, and let N be the number shown. Then roll N six-sided dice and gain stones equal to the sum of all numbers shown on the N dice you rolled.

3. Multiple Players

(12) You and your opponent both roll six-sided dice. If your roll is odd, your opponent gives you 1 chip. If your roll is even, your opponent gives you 2 chips. If your opponent's roll is 1 or 2, you give your opponent 2 chips. If your opponent rolls $3 - 6$, you give your opponent 1 chip. Is this fair? Who will accumulate more chips if you play repeatedly?

(13) You and your opponent both roll six-sided dice. The person with the lower number gives 1 chip to the person with the higher number (do nothing with a tie). Is this fair? Who will accumulate more chips if you play repeatedly?

(14) You and your opponent both roll six-sided dice. If both numbers are even or both numbers are odd, your opponent gives you 3 chips. If your opponent's roll is even, you give your opponent 2 chips. Is this fair? Who will accumulate more chips if you play repeatedly?

4. Other Kinds of Dice

(15) Roll an eight-sided die and gain chips equal to the number shown. What is the expected value? (Similarly, roll a twenty sided die and gain chips equal to the number shown.)

(16) Roll a twenty-sided die. Gain 2 chips if the resulting number is a multiple of 2. Gain 3 chips if it is a multiple of 3. Gain 5 chips if it is a multiple of 5. (Example: 15 gets you 8 chips.)

(17) Roll a four-sided die and an eight-sided die. Gain 10 chips if they show the same number.

(18) Roll a four-sided die and a six-sided die. Gain 10 chips if they show the same number.

5. Advanced Question

(19) Come up with your own game. What is the fair price to play it?

(20) You roll a dice repeatedly until you get an odd number. I will give you 2^n chips, where n is the number of rolls you made. If you had to pay me 4 chips to play, would you do so? How much would you pay to play this game?

(21) I have ten cards - five hearts and five spades. I shuffle them and deal you three. You gain 2^n chips, where n is the number of hearts I deal you. What is the expected value of this game?

(22) On a geological gameshow, there are three doors. Behind one is a diamond, behind the other two are hunks of granite. You pick one of the doors, hoping for the diamond. Before you open the door, the host opens a different door, showing you a hunk of granite, then gives you an option to change which door you picked. Should you?