COMPLEX ANALYSIS MISCELLANY

Abstract. I will use this to record proofs, examples, and explanations that I might have planned to give in class but was not able to. It also may contain other odds and ends.

1. The complex numbers form a field (Jan 6)

Let us quickly recall some basic properties of the real numbers, which we denote by $\mathbb{R}$.

Proposition 1.1. *Let a, b, c be real numbers.*

(1) $a + b$ and ab are also real numbers (closure).
(2) Addition is associative: $a + (b + c) = (a + b) + c$.
(3) Addition is commutative: $a + b = b + a$.
(4) There exists a real number, named 0, such that $a + 0 = a = 0 + a$.
(5) There exists a real number $-a$ such that $a + (-a) = 0 = (-a) + a$. For shorthand, we write $a + (-b)$ as $a - b$.
(6) Multiplication is associative: $a(bc) = (ab)c$.
(7) Multiplication is commutative: $ab = ba$.
(8) There exists a real number, named 1, such that $a \cdot 1 = a = 1 \cdot a$.
(9) If $a \neq 0$, then there exists a real number a^{-1} (the reciprocal of a) such that $a \cdot a^{-1} = 1 = a^{-1} \cdot a$. We sometimes write $a \cdot b^{-1}$ as a/b or $\frac{a}{b}$.
(10) The distributive property holds: $a(b + c) = ab + ac$.

Remark. These properties imply that $\mathbb{R}$ is a **field**.

A **complex number** is an ordered pair $z = (a, b)$ of real numbers (ordered: (a, b) does not necessarily equal (b, a)). We call a the **real part**, written as $\mathrm{Re}(z)$, and b the **imaginary part** of z, written as $\mathrm{Im}(z)$. Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ be complex numbers. We write

$$x = y \qquad \text{if and only if} \qquad a_1 = a_2 \text{ and } b_1 = b_2.$$

We define addition and multiplication on complex numbers by

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2), \qquad z_1 z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1). \tag{1.1}$$

We observe that if $b = d = 0$, then

$$z_1 + z_2 = (a_1 + a_2, 0), \qquad z_1 z_2 = (a_1 a_2, 0),$$

so we can think of z_1 and z_2 as being just like real numbers when their imaginary parts equal zero.

Observe that $(a, b) = (a, 0) + (0, b)$ and $(0, 1)(b, 0) = (0, b)$. Thus we can write

$$z = (a, b) = (a, 0) + (0, 1)(b, 0).$$

It will be convenient to introduce the shorthand $i = (0, 1)$ and $(a, 0) = a$. Therefore, we can represent any complex number $z = (a, b)$ as

$$z = a + ib.$$

With these conventions, we will show that the properties of $\mathbb{R}$ outlined in Proposition 1.1 also holds for the set of complex numbers, which we denote by $\mathbb{C}$.

Proposition 1.2. *Let z_1, z_2, z_3 be complex numbers.*

(1) $z_1 + z_2$ and $z_1 z_2$ are also complex numbers (closure).
(2) Addition is associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
(3) Addition is commutative: $z_1 + z_2 = z_2 + z_1$.
(4) There exists a complex number, named $0 = (0, 0)$, such that $z_1 + 0 = z_1 = 0 + z_1$.
(5) There exists a complex number $-z_1$ such that $z_1 + (-z_1) = 0 = (-z_1) + z_1$. For shorthand, we write $z_1 + (-z_2)$ as $z_1 - z_2$.
(6) Multiplication is associative: $z_1(z_2 z_3) = (z_1 z_2)z_3$.
(7) Multiplication is commutative: $z_1 z_2 = z_2 z_1$.
(8) There exists a complex number, named $1 = (1, 0)$, such that $z_1 \cdot 1 = z_1 = 1 \cdot z_1$.
(9) If $z_1 \neq 0$, then there exists a complex number z_1^{-1} (the reciprocal of z_1) such that $z_1 \cdot z_1^{-1} = 1 = z_1^{-1} \cdot z_1$. We sometimes write $z_1 \cdot z_2^{-1}$ as z_1/z_2 or $\frac{z_1}{z_2}$.
(10) The distributive property holds: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Proof. I'll verify (2) and (6). I leave the rest to you as an exercise. We will discuss (9) at length next class.

– (2): Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$. Using (1.1), we find that

$$z_1+z_2 = (a_1, b_1)+(a_2, b_2) = (a_1+a_2, b_1+b_2), \qquad z_2+z_1 = (a_2, b_2)+(a_1, b_1) = (a_2+a_1, b_2+b_1).$$

Since $a_1 + a_2 = a_2 + a_1$ and $b_1 + b_2 = b_2 + b_1$ by Proposition 1.1 (3), we have the desired equality.

– (6): Using (1.1), we first compute

$$\begin{aligned} z_1(z_2 z_3) &= (a_1, b_1)((a_2, b_2) \cdot (a_3, b_3)) \\ &= (a_1, b_1)(a_2 a_3 - b_2 b_3, a_2 b_3 + a_3 b_2) \\ &= (a_1(a_2 a_3 - b_2 b_3) - b_1(a_2 b_3 + a_3 b_2), a_1(a_2 b_3 + a_3 b_2) + b_1(a_2 a_3 - b_2 b_3)) \end{aligned}$$

A similar computation yields

$$(z_1 z_2)z_3 = ((a_1 a_2 - b_1 b_2)a_3 - (a_1 b_2 + a_2 b_1)b_3, (a_1 a_2 - b_1 b_2)b_3 + a_3(a_1 b_2 + a_2 b_1)).$$

We are now left with the (tedious) task of using Proposition 1.1 to check that these complex numbers are equal. Give it a try, **only using Proposition 1.1**. □

Note that with these properties,

$$i^2 = (0, 1)^2 = (-1, 0) = -1, \qquad i^3 = (i^2)i = -i, \qquad i^4 = i^3 \cdot i = -i \cdot i = -(-1) = 1.$$

2. "Reverse" triangle inequality proof (Jan 8)

I stated this in class and did not prove it. Here is a proof. It is different from the book's but it uses more of the arithmetic of the complex numbers that we went over in class today.

Lemma 2.1. *If* z_1, z_2 *are complex numbers, then* $|z_1 + z_2| \geq ||z_1| - |z_2||$.

Proof. Since $|z_1| - |z_2|$ is a real number, we compute

$$\begin{aligned} ||z_1| - |z_2||^2 = (|z_1| - |z_2|)^2 &= |z_1|^2 + |z_2|^2 - 2|z_1| \cdot |z_2| \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| \\ &\leq |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1} = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1 + z_2|^2. \end{aligned}$$

Thus $||z_1| - |z_2||^2 \leq |z_1 + z_2|^2$. Hence $||z_1| - |z_2|| \leq |z_1 + z_2|$, as desired. ∎

3. de Moivre's formula example (Jan 10)

At the end of class, we arrived at de Moivre's formula: If n is an integer and θ is a real number, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

To give you an idea of how this might be used, consider the following example.

Example 3.1. Consider de Moivre's formula with $n = 2$. Then

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta).$$

We expand the left-hand side and obtain

$$(\cos\theta)^2 + 2i(\sin\theta)(\cos\theta) - (\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta). \tag{3.1}$$

(We used the fact that $i^2 = -1$.) Recall that $z_1 = x_1 + iy_1$ equals $z_2 = x_2 + iy_2$ precisely when the two equalities $x_1 = x_2$ and $y_1 = y_2$ hold. Thus (3.1) tells us that

$$(\cos\theta)^2 - (\sin\theta)^2 = \cos(2\theta) \qquad \text{and} \qquad 2(\sin\theta)(\cos\theta) = \sin(2\theta).$$

Recall that $(\cos\theta)^2 + (\sin\theta)^2 = 1$. This leaves us with

$$2(\cos\theta)^2 - 1 = \cos(2\theta) \qquad \text{and} \qquad 2(\sin\theta)(\cos\theta) = \sin(2\theta).$$

These are trigonometric identities that arise often in geometry and calculus.

Here is another example.

Example 3.2. Let us compute $(1-i)^8$. On one hand, you could expand out this 8th power using the binomial formula, but that is quite painful! On the other hand, we can realize that the principal argument of $1 - i$ lies in the 4th quadrant and satisfies

$$\tan\Theta = \frac{-1}{1} = -1.$$

Thus the principal argument Θ of $1 - i$ is $-\pi/4$, and the modulus is $\sqrt{2}$. Thus

$$1 - i = \sqrt{2}\Big(\cos\Big(-\frac{\pi}{4}\Big) + i\sin\Big(-\frac{\pi}{4}\Big)\Big).$$

Now, by de Moivre (3.1),

$$(1-i)^8 = (\sqrt{2})^8\Big(\cos\Big(-\frac{\pi}{4}\Big) + i\sin\Big(-\frac{\pi}{4}\Big)\Big)^8 = 16\Big(\cos\Big(-\frac{\pi}{4}\cdot 8\Big) + i\sin\Big(-\frac{\pi}{4}\cdot 8\Big)\Big).$$

This simplifies to

$$(1-i)^8 = 16(\cos(-2\pi) + i\sin(-2\pi)).$$

Since $\cos(-2\pi) = 1$ and $\sin(-2\pi) = 0$, we are left with

$$(1-i)^8 = 16.$$

4. Interior, exterior, and boundary points of unit circle (Jan 15)

Let $S = \{z \colon |z| < 1\}$.

Theorem 4.1. *The set S is open.*

To complement the geometrically intuitive discussion I gave in class, I will give some rigorous proofs here. There are a few pieces that are required.

Lemma 4.2. *If z_0 is a complex number such that $|z_0| < 1$, then z_0 is an interior point of S.*

Proof. If z_0 lies in S, then one can take $\varepsilon = 1 - |z_0|$ (which is positive since $|z_0| < 1$), and then the ε-neighborhood $\{z \colon |z - z_0| < \varepsilon\}$ is a neighborhood of z_0 containing only points in S. Thus z_0 satisfies the definition for an interior point. □

Lemma 4.3. *If z_0 is a complex number such that $|z_0| > 1$, then z_0 is an exterior point of S.*

Proof. If $|z_0| > 1$, then we choose $\varepsilon = |z_0| - 1$ (which is positive since $|z_0| > 1$). Then the neighborhood $\{z \colon |z - z_0| < \varepsilon\}$ contains only points outside of S. Hence if $|z_0| > 1$, then z_0 is an exterior point □

Lemma 4.4. *If z_0 is a complex number such that $|z_0| = 1$, then z_0 is a boundary point of S.*

Proof. I will prove that $z_0 = 1$ is a boundary point, and I leave it to you to handle the other cases. To begin, fix $\varepsilon > 0$. We have two cases: $\varepsilon > 1$ or $\varepsilon \leq 1$. If $\varepsilon > 1$, then the ε-neighborhood $\{z \colon |z - 1| < \varepsilon\}$ contains both $\frac{1}{2}$ (interior to S) and $\frac{3}{2}$ (exterior to S).

Now, consider the case where $0 < \varepsilon < 1$. Note that the set $\{z \colon |z - 1| < \varepsilon\}$ contains both $1 - \frac{\varepsilon}{2}$ (which is interior to S) and $1 + \frac{\varepsilon}{2}$ (which is exterior to S).

By combining the two cases, we see that no ε-neighborhood of $z_0 = 1$ contains either only points interior to S or only points exterior to S. Thus $z_0 = 1$ is neither an interior nor an exterior point. Thus $z_0 = 1$ is a boundary point.

(Again, I leave it to you to complete the proof for other choices of z_0 with $|z_0| = 1$.) □

Proof of Theorem 4.1. We have identified the boundary of S to be $\partial S = \{z \colon |z| = 1\}$, and S does not contain any point in ∂S. Thus S is open. □

5. Another $\delta - \varepsilon$ limit example (Jan. 22)

Proposition 5.1. *Pick an integer $n \geq 0$ and a complex number z_0. We have that*

$$\lim_{z \to z_0} z^n = z_0^n.$$

My discussion here will be long, because I will provide lots of details so that you have a sense of how these arguments work. Again, because we our work is taking place in the plane instead of on the real line, we really need this $\delta - \varepsilon$ definition. Our work here will also use mathematical induction. The book states this result in Section 16 but does not give any details.

Case 0: $n = 0$. If $n = 0$, the claimed result reduces to

$$\lim_{z \to z_0} 1 = 1.$$

Verifying this is purely a matter of knowing the definition of the limit. I leave this straightforward step to you.

Case 1: $n = 1$. If $n = 1$, then the claimed result reduces to

$$\lim_{z \to z_0} z = z_0.$$

Let $\varepsilon > 0$, and let $\delta' = \varepsilon$. If $|z - z_0| < \delta'$, then $|z - z_0| < \varepsilon$. This is simply unraveling the definition of the limit, and in this case it's pretty straightforward. We will use the number δ' later.

Case 1 provides us with the base case in our use of mathematical induction. Now, suppose that for some $n \geq 1$, we have that

$$\lim_{z \to z_0} z^n = z_0^n. \tag{5.1}$$

Goal: Prove that $\lim\limits_{z \to z_0} z^{n+1} = z_0^{n+1}$.

We start with (5.1) and unravel what this means using the definition of the limit: Given $\varepsilon > 0$ (the same as in Case 1), we can find some $\delta'' > 0$ such that

$$|z^n - z_0^n| < \varepsilon \qquad \text{whenever} \quad |z - z_0| < \delta''. \tag{5.2}$$

We know that $|z^n - z_0^n|$ will be small by our inductive hypothesis (5.2) and $|z - z_0|$ will be small as we can see from the base case $n = 1$. So we want to try to express $|z^{n+1} - z_0^{n+1}|$ in terms of $|z^n - z_0^n|$ and $|z - z_0|$. Observe that

$$z^{n+1} - z_0^{n+1} = z \cdot (z^n - z_0^n) + z_0^n(z - z_0).$$

Now, if $\delta < \min\{\delta', \delta'', 1\}$, and $|z - z_0| < \delta$, then by the triangle inequality,

$$\begin{aligned} |z^{n+1} - z_0^{n+1}| &= |z \cdot (z^n - z_0^n) + z_0^n(z - z_0)| \\ &\leq |z| \cdot |z^n - z_0^n| + |z_0|^n \cdot |z - z_0| \\ &< |z| \cdot \varepsilon + |z_0|^n \cdot \varepsilon \\ &< (|z_0| + \delta) \cdot \varepsilon + |z_0|^n \cdot \varepsilon \\ &\leq (|z_0| + 1 + |z_0|^n)\varepsilon. \end{aligned}$$

This is very close to what we want, but there is a problem: The limit definition requires that we have

$$|z^{n+1} - z_0^{n+1}| < \varepsilon,$$

not

$$|z^{n+1} - z_0^{n+1}| < (|z_0| + 1 + |z_0|^n)\varepsilon.$$

There is a simple fix for this: We simply rescale ε. Since ε was chosen arbitrarily anyway and z_0 is a constant chosen at the beginning of the problem (it is not varying as z varies), we can replace every instance of ε with $\varepsilon/(|z_0| + 1 + |z_0|^n)$. Since the denominator is always a positive constant, we are in good shape, and by mathematical induction, we have finished our work.

HOWEVER...

We "worked backwards" to find the $\delta > 0$ that works for every $\varepsilon > 0$. Proofs almost always read better when they have a "forward flow". So we will tidy up our work and make the proof a little more presentable.

Proposition 5.2. *Pick an integer $n \geq 0$ and a complex number z_0. We have that*

$$\lim_{z \to z_0} z^n = z_0^n.$$

Proof. The case when $n = 0$ is immediate from the definition of the limit. We proceed by mathematical induction for $n \geq 1$. Our base case, $n = 1$, is proved as follows:

$$\text{Let } \varepsilon > 0, \text{ and let } \delta' = \tfrac{\varepsilon}{|z_0|+1+|z_0|^n}.$$
$$\text{Whenever } |z - z_0| < \delta', \text{ we also have } |z - z_0| < \tfrac{\varepsilon}{|z_0|+1+|z_0|^n} < \varepsilon.$$

Now, let $n \geq 1$ be an integer, and suppose that $\lim_{z \to z_0} z^n = z_0^n$. This means that:

$$\text{Given an } \varepsilon > 0, \text{ there exists } \delta'' > 0 \text{ such that}$$
$$|z^n - z_0^n| < \tfrac{\varepsilon}{|z_0|+1+|z_0|^n} \text{ whenever } |z - z_0| < \delta''.$$

We choose $\delta = \min\{\delta', \delta'', 1\}$. If $\varepsilon > 0$ and $|z - z_0| < \delta$, then by the triangle inequality, we have

$$\begin{aligned}
|z^{n+1} - z^n| &= |z \cdot (z^n - z_0^n) + z_0^n(z - z_0)| \\
&\leq |z| \cdot |z^n - z_0^n| + |z_0|^n \cdot |z - z_0| \\
&< (|z_0| + \delta)\frac{\varepsilon}{|z_0| + 1 + |z_0|^n} + |z_0|^n \cdot \frac{\varepsilon}{|z_0| + 1 + |z_0|^n} \\
&\leq (|z_0| + 1)\frac{\varepsilon}{|z_0| + 1 + |z_0|^n} + |z_0|^n \cdot \frac{\varepsilon}{|z_0| + 1 + |z_0|^n} \\
&= \frac{(|z_0| + 1 + |z_0|^n)\varepsilon}{(|z_0| + 1 + |z_0|^n)} \\
&= \varepsilon.
\end{aligned}$$

$\square$

6. A limit non-example (Jan 24)

The end of class was a bit rushed, so I thought I would take the time to spell out the details of the last example.

Proposition 6.1. *The function $f(z) = |z|^2$ is differentiable at $z_0 = 0$ and nowhere else.*

Proof. To recap from class: If $z_0 = 0$ and $z \neq 0$, then

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{|z|^2}{z} = \frac{z\overline{z}}{z} = \overline{z}.$$

It is then clear that for $z_0 = 0$, we have

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{z \to 0} \overline{z} = 0.$$

Thus $f'(0) = 0$.

To give more detail about the remaining cases: Let $z_0 \neq 0$ and $z \neq z_0$. Then

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z\overline{z} - z_0\overline{z_0}}{z - z_0}.$$

We add zero in the form $z\overline{z_0} - z\overline{z_0}$ to the numerator:

$$\frac{z\overline{z} - z_0\overline{z_0}}{z - z_0} = \frac{z\overline{z} + z\overline{z_0} - z\overline{z_0} - z_0\overline{z_0}}{z - z_0} = \frac{z(\overline{z} - \overline{z_0}) - \overline{z_0}(z - z_0)}{z - z_0} = z\frac{\overline{z} - \overline{z_0}}{z - z_0} - \overline{z}_0.$$

(Notice that $\frac{\overline{z} - \overline{z_0}}{z - z_0}$ has modulus 1. The numerator is the conjugate of the denominator, and $|\overline{w}/w| = |\overline{w}|/|w| = |w|/|w| = 1$ whenever $w \neq 0$.)

If z approaches z_0 by traveling along the complex numbers such that $\mathrm{Im}(z) = \mathrm{Im}(z_0)$ (this means that the imaginary part of z is fixed, which gives us a horizontal line in the complex plane), then

$$z\frac{\overline{z} - \overline{z_0}}{z - z_0} - \overline{z}_0 \qquad \text{simplifies to} \qquad \mathrm{Re}(z) + \mathrm{Re}(z_0),$$

and so we combine our work from above to obtain

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{z \to z_0} \left(z\frac{\overline{z} - \overline{z_0}}{z - z_0} - \overline{z}_0 \right) = \lim_{z \to z_0} (\mathrm{Re}(z) + \mathrm{Re}(z_0)) = 2\mathrm{Re}(z_0).$$

On the other hand, if z approaches z_0 by traveling along the complex numbers such that $\mathrm{Re}(z) = \mathrm{Re}(z_0)$ (this means that the real part of z is fixed, which gives us a vertical line in the complex plane), then

$$z\frac{\overline{z} - \overline{z_0}}{z - z_0} - \overline{z}_0 \qquad \text{simplifies to} \qquad -i(\mathrm{Im}(z) + \mathrm{Im}(z_0)),$$

and so we combine our work from above to obtain

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{z \to z_0} \left(z\frac{\overline{z} - \overline{z_0}}{z - z_0} - \overline{z}_0 \right) = -i \lim_{z \to z_0} (\mathrm{Im}(z) + \mathrm{Im}(z_0)) = -2i\mathrm{Im}(z_0).$$

As we discussed a little while ago, limits, when they exist, are unique—they remain the same regardless of the directioin in which z approaches z_0. We have shown that from one direction, the limit equals $2\mathrm{Re}(z_0)$, and in a different direction, the limit equals $-2i\mathrm{Im}(z_0)$. The only complex number z_0 which satisfies $2\mathrm{Re}(z_0) = -2i\mathrm{Im}(z_0)$ is $z_0 = 0$ (check this), but we assumed that $z_0 \neq 0$ (we handled this case earlier). So we must conclude that the limit only exists when $z_0 = 0$. $\square$

7. The Cauchy–Riemann converse (Jan 27)

I'll expound a bit on the idea behind the result proved on Jan 27:

Theorem 7.1. *Let $z = x+iy$, and let $f(z) = u(x, y)+iv(x, y)$ be defined in an ε-neighborhood of $z_0 = x_0 + iy_0$. Suppose that*

(1) the partial derivatives u_x, u_y, v_x, and v_y exist and continuous everywhere in the neighborhood, and

(2) these partial derivatives satisfy the Cauchy–Riemann equations at (x_0, y_0).

Then $f'(z_0)$ exists and equals $u_x(x_0, y_0) + iv_x(x_0, y_0)$.

7.1. **Notation.** We write

$$\Delta x = x - x_0, \qquad \Delta y = y - y_0, \qquad \Delta z = z - z_0 = \Delta x + i\Delta y.$$

7.2. **Philosophy: Functions of a single real variable.** Suppose that $r(x)$ is a function of a real variable x which is differentiable on (a, b), and let $a < x_0 < b$. Then

$$\lim_{x\to x_0} \frac{r(x) - r(x_0)}{x - x_0} = r'(x_0).$$

In other words, when $x \neq x_0$ is really close to x_0 (perhaps within an ε-neighborhood...), then we have the very close approximation

$$\frac{r(x) - r(x_0)}{x - x_0} \approx r'(x_0),$$

in which case

$$r(x) - r(x_0) \approx r'(x_0)(x - x_0).$$

This is a first-order **Taylor expansion**. Now, if you want to take the limit of $r(x)$ as $x \to x_0$, we see pretty clearly that the limit is $r(x_0)$ (since $r(x) - r(x_0) \to 0$ because $r'(x_0)(x - x_0) \to 0$).

7.3. **Philosophy: Functions of two real variables.** We take the linear approximation philosophy for computing limits of differentiable functions. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable in an ε-neighborhood of (x_0, y_0). When F_x and F_y are continuous in a neighborhood of (x_0, y_0), we have the *multi-variable* first-order Taylor expansion

$$F(x, y) - F(x_0, y_0) \approx F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0).$$

7.4. **Idea of the proof of Theorem 7.1.** Assume that u and v satisfy (1) and (2) in the theorem statement. Since (2) is satisfied, we have the first-order approximation

$$\begin{aligned} f(z) - f(z_0) &= u(x, y) - u(x_0, y_0) + i(v(x, y) - v(x_0, y_0)) \\ &\approx u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y). \end{aligned}$$

Now, by (1), we have $u_x = v_y$ and $u_y = -v_x$. Thus

$$\begin{aligned} &u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y) \\ &= u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + i(v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y) \\ &= u_x(x_0, y_0)(\Delta x + i\Delta y) + i(v_x(x_0, y_0)(\Delta x + i\Delta y)) \\ &= (u_x(x_0, y_0) + iv_x(x_0, y_0))(\Delta x + i\Delta y) \end{aligned}$$

Piecing these together, we find that

$$\begin{aligned} f(z) - f(z_0) &\approx (u_x(x_0, y_0) + iv_x(x_0, y_0))(\Delta x + i\Delta y) \\ &= (u_x(x_0, y_0) + iv_x(x_0, y_0))(z - z_0). \end{aligned}$$

Since we are taking a limit as $z \to z_0$ (so $z \neq z_0$), we conclude that as z comes to within an ε-neighborhood of z_0, we have that

$$\frac{f(z) - f(z_0)}{z - z_0} \approx u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Thus we can infer that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0, y_0) + iv_x(x_0, y_0),$$

in which case (a) the derivative exists, and (b) it equals what we claimed.

7.5. **Last remarks.** This is **NOT** a proof! This is simply the *idea* of the proof. Notice that never have we made clear what the symbol $\approx$ means! Notice that since the derivative is a limit, we have to find for all $\varepsilon > 0$ that there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - (u_x(x_0, y_0) + iv_x(x_0, y_0)) \right| < \varepsilon \qquad \text{whenever} \quad |z - z_0| < \delta.$$

We didn't touch δ's and ε's at all! The idea is that the δ's and the ε's that arise in the limit definition for the partial derivatives u_x, u_y, v_x, and v_y allow us to say very specifically what $\approx$ means at each step. This is spelled out in Section 23 of the book, but their approach is motivated by the linear approximation philosophy that we described earlier.

8. Complex differentiability vs. being holomorphic (uploaded on Feb 9)

There seemed to be some confusion on last week's quiz regarding the (important!) difference between being holomorphic at a point and being complex-differentiable at a point.

In order for a function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

to be **complex-differentiable** at z_0 means that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Remember: You need to approach the same value as you approach z_0 from *any* direction. In the (special!) case when the first-order partial derivatives u_x, u_y, v_x, and v_y are continuous in a neighborhood of z_0 (the *neighborhood* part of this is important!!!), then complex-differentiability is the same as u and v satisfying the Cauchy–Riemann equations at $z_0 = x_0 + iy_0$.

In order for f to be **holomorphic** at z_0, we need to establish that f his complex-differentiable **everywhere in some ε-neighborhood of** z_0. This is a very important distinction from complex-differentiability! On Quiz 4, you saw an example of a function $f(z)$ that is complex-differentiable only along a line $\mathrm{Re}(z) = \mathrm{Im}(z)$ in the complex plane. When you draw an ε-neighborhood around any point on a line, it must contain points off the line! Therefore, you can't prove complex-differentiability of f in any neighborhood of any point on the line. Therefore, f is holomorphic nowhere, even though it is complex-differentiable at many places!

9. Midterm Review (Feb 9)

We have a review session in Little Hall, Room 368, on Monday, February 10, from 4:15pm–6:15pm. **The doors of Little Hall will lock at 4:30pm.** If you cannot make it to the review session, try to copy notes from someone who could come.

9.1. **Problems.** I gave lots of suggested problems. If you did all of the suggested problems, great! Now you can do all of the other problems that I didn't suggest in the sections that we covered. If you did not do all of the suggested problems, finish those first. These are all good preparation for the midterm.

9.2. **Topics.**

(1) Different ways to interpret complex numbers (ordered pairs, real part + $i\cdot$ imaginary part, exponential form, point in the plane / vectors, point on Riemann sphere, etc.)
(2) δ-ε definition of the limit
(3) Full definitions of continuity, complex-differentiability
(4) Cauchy–Riemann equations, significance for the derivative
(5) What are sufficient conditions for differentiability?
(6) Holomorphic functions vs. complex-differentiable functions
(7) Finding arguments / principal argument. $\arg z$ as a set. What does $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ mean?
(8) Solving equations with complex solutions / computing complex n-th roots
(9) Proof of triangle inequality (and the "reverse" triangle inequality)
(10) Proof of: If $f = u + iv$ is holomorphic on a domain D (viewed as a subset of the complex plane), then u and v are harmonic on D (viewed as a subset of $\mathbb{R}^2$)
(11) Multi-valued functions (log, power function, etc.). $\log(z)$ as a set vs. $\mathrm{Log}(z)$ as a function.
(12) Branches / branch cuts
(13) Exponential, trig, polynomial (etc.) functions of a complex variable
(14) Hyperbolic trigonometric functions (definitions, basic properties)
(15) Rules of taking derivatives (product, quotient, chain, etc.)

10. Green's theorem and complex integrals (Feb 17)

Recall Green's theorem from multivariable calculus: If D is a region bounded by a closed curve C and $L = L(s, y)$ and $M = M(x, y)$ are functions defined on an open region containing D and having continuous partial derivatives there, then

$$\int_C (Ldx + Mdy) = \int \int_D (M_x - L_y) dxdy,$$

where C is positively oriented. Now, suppose that $f(z) = u(x, y) + iv(x, y)$. Then

$$\int_C f(z)dz = \int_C (udx - vdy) + i \int_C (vdx + udy).$$

By Green's theorem, if the first order partial derivatives are continuous, then f being holomorphic implies, via the Cauchy–Riemann equations, that

$$\int_C f(z)dz = \int \int_D (-v_x - u_x)dxdy + i \int_C (u_x - v_y)dxdy = 0.$$

We have thus concluded a preliminary version of Cauchy's theorem.

Theorem 10.1. *If f is analytic on a region R with piecewise differentiable boundary ∂R, and f' is continuous there, then*

$$\int_{\partial R} f dz = 0.$$

11. Cauchy's Theorem for Simple Closed Curves

We used Goursat's theorem to prove that if f is holomorphic along and inside of a disc D, then $\int_\gamma f dz = 0$ for every simple closed curve contained in D. In this note, we show how one can use Goursat to go beyond discs. Recall that a **simple closed curve** is a loop with no self-intersections except at its starting and ending points (because the starting and ending points equal each other).

Theorem 11.1. *Let f be holomorphic on an open set D, and let γ is a simple closed curve in the interior of D. Then $\int_\gamma f dz = 0$.*

We require a little preliminary work.

Lemma 11.2. *Let P be a simple polygon (that is, let P be a simple closed curve which is the union of line segments which does not self-intersect). If f is holomorphic on P and on the interior of P, then $\int_{P_n} f dz = 0$.*

Sketch of proof. Take P oriented in the positive direction. Once we dissect P into a union of triangles $T_1, T_2, \ldots$ and express $\int_{P_n} f dz$ as a sum of $\int_{T_1} f dz, \int_{T_2} f dz, \ldots$, each oriented in the positive direction. Since the integral around each of the triangles is zero by Goursat, the full integral $\int_{P_n} f dz$ will also be zero. □

For a subset $U \subseteq \mathbb{C}$, let $\overline{U}$ be the **closure** of U (the union of the interior of U and the boundary of U).

Lemma 11.3. *Let $B(z,\rho) = \{w \in \mathbb{C} \colon |z - w| < \rho\}$. There exists a constant $\rho > 0$ such that*

$$R = \bigcup_{z \in \gamma} \overline{B(z,\rho)} \subseteq D.$$

Moreover, R is itself closed and bounded.

Sketch of proof. Since γ is on the interior of D, D is open, and γ is closed, we have that the minimum distance between any point on γ and the boundary of U must be positive. Whatever this positive distance is, we let ρ be $1/2$ of that distance. Since R clearly contains its boundary points (this can be seen geometrically since R is a "ribbon" contained in D which envelopes γ), R is closed. Boundedness is clear. □

Proof of Cauchy's Theorem for Simple Closed Curves. Parametrize our simple closed curve γ (oriented in the positive direction) as $z(t)$ with $0 \leq t \leq 1$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ (notice that $z(0) = z(1)$ since γ is closed), and let $z_j = z(t_j)$. The Riemann sum for $\int_\gamma f(z) dz$ is

$$S_n = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

Let P_n be the polygon formed by connecting z_0 to z_1, z_1 to z_2,...,z_{n-2} to z_{n-1}, z_{n-1} to $z_n = z_0$. Observe that

$$\int_{P_n} f dz - S_n = \sum_{j=1}^{n} \Big(\int_{z_{j-1}}^{z_j} f dz - f(z_j)(z_j - z_{j-1}) \Big).$$

Note that by construction, we have

$$length(P_n) \leq length(\gamma).$$

Since R is closed and bounded by the lemma and f is continuous on D, we can find, for all $\varepsilon > 0$, a $\delta > 0$ (depending *only* on ε) such that

$$|f(z) - f(w)| < \delta \qquad \text{whenever } |z - w| < \varepsilon/length(\gamma),$$

the point being that this one δ works for *any* pair $z, w \in D$. (This is called **uniform continuity**.)

By increasing n, we can add more points to our polygon P so that $|z_j - z_{j-1}| < \delta$ for each $1 \leq j \leq n$. We are now set up for our final calculation:

$$\begin{aligned}
\Big|\int_{P_n} f(z)dz - S_n\Big| &= \Big|\sum_{j=1}^{n}\int_{z_{j-1}}^{z_j}(f(z) - f(z_j))dz\Big| \\
&\leq \sum_{j=1}^{n}\Big|\int_{z_{j-1}}^{z_j}(f(z) - f(z_j))dz\Big| \\
&\leq \sum_{j=1}^{n}\int_{z_{j-1}}^{z_j}|f(z) - f(z_j)| \cdot |dz| \\
&\leq \sum_{j=1}^{n}\int_{z_{j-1}}^{z_j}\max_{w\in[z_{j-1},z_j]}|f(w) - f(z_j)| \cdot |z_j - z_{j-1}| \\
&< \sum_{j=1}^{n}(\varepsilon/length(\gamma)) \cdot |z_j - z_{j-1}| \\
&\leq (\varepsilon/length(\gamma)) \cdot length(P_n) \\
&\leq \varepsilon.
\end{aligned}$$

Now, note that by the triangle inequality,

$$\Big|\int_{P_n} f dz - \int_\gamma f dz\Big| \leq \Big|\int_{P_n} f dz - S_n\Big| + \Big|\int_\gamma f dz - S_n\Big|.$$

The two absolute values on the right hand side can be made arbitrarily small when n is sufficiently large: The first absolute value by the above calculation, the second absolute value because S_n is the Riemann sum for $\int_\gamma f dz$. Thus if n is big enough, the right hand side will be bounded by 2ε, say. And we can take ε to be arbitrarily small. Since

$$\int_{P_n} f dz = 0$$

by one of our lemmas above (the mild extension of Goursat's theorem), we are left with

$$\int_\gamma f dz = 0.$$

$\square$