Last Time:

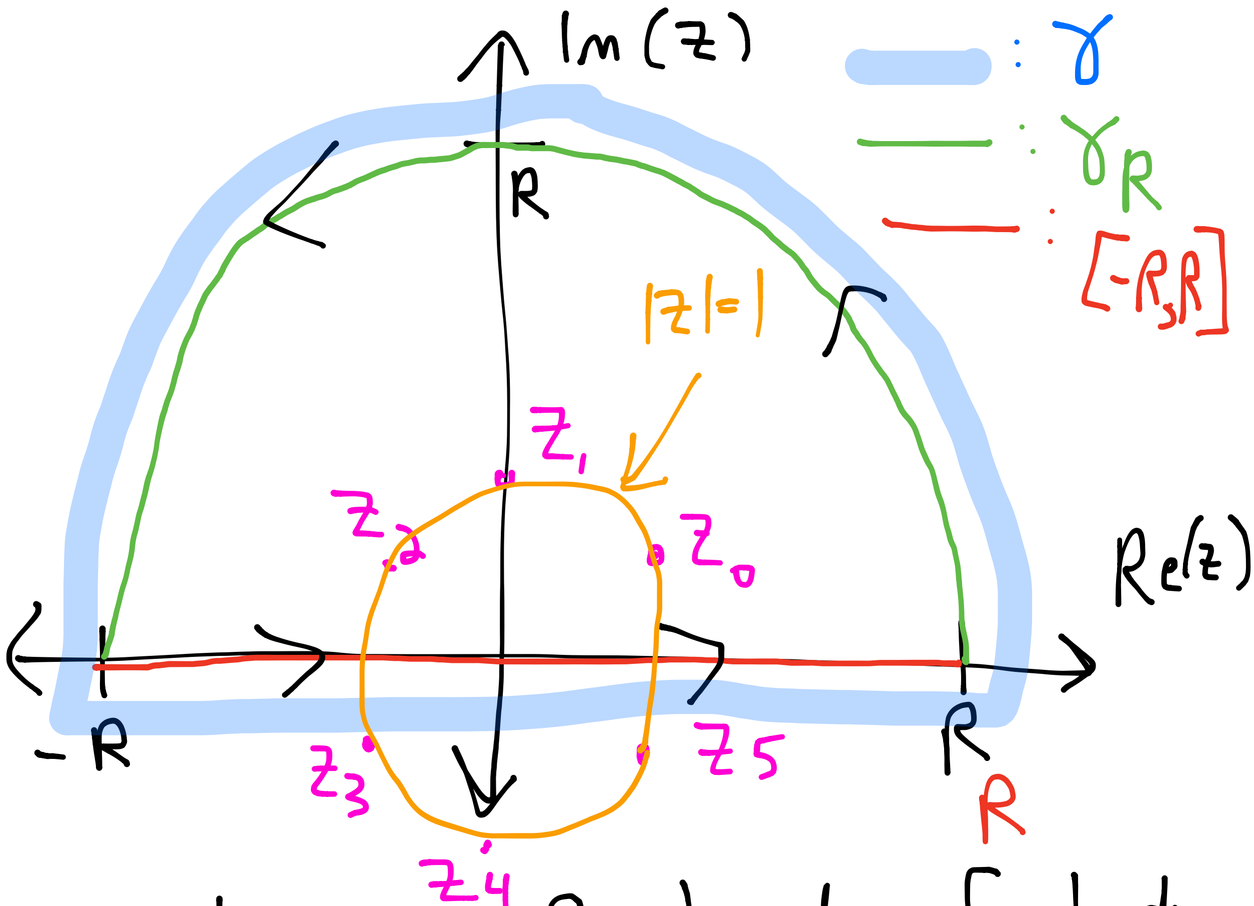
We used residue thm to compute an integral of a single real variable

More examples:

1) $I = \int_0^\infty \frac{1}{x^6+1}\,dx$ (integrand is even)

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^6+1}\,dx$$



$$\int_{\gamma} \frac{1}{z^6+1}\,dz = \int_{\gamma_R} \frac{1}{z^6+1}\,dz + \int_{-R}^{R} \frac{1}{x^6+1}\,dx$$

Residue Thm: The singularities of $\frac{1}{z^6+1}$ are the 6-th roots of -1:

$$z^6 = -1 = e^{\pi i + 2\pi i k} \quad (k \in \mathbb{Z})$$

$$\hookrightarrow z = e^{\frac{\pi i}{6} + \frac{2\pi i k}{6}}, \quad k = 0, 1, \dots, 5.$$

Call these roots $z_0, z_1, \dots, z_5$.

Thus by the residue thm,

$$\int_\gamma \frac{1}{z^6+1}\,dz = 2\pi i \cdot \sum_{k=0}^{2} \operatorname*{Res}_{z=z_k} \frac{1}{z^6+1}$$

(Note: z_0, z_1, z_2 have positive imaginary parts, while z_3, z_4, z_5 do not.)

By a lemma from last time,

$$2\pi i \cdot \sum_{k=0}^{2} \operatorname*{Res}_{z=z_k} \frac{1}{z^6+1}$$

$$= 2\pi i \sum_{k=0}^{2} \frac{1}{6z_k^5} \quad \left(\text{lemma from last time}\right)$$

$$= 2\pi i \sum_{k=0}^{2} \frac{z_k}{6z_k^6} \quad \left(z_k^6 = -1\right)$$

$$= -2\pi i \sum_{k=0}^{2} \frac{z_k}{6}$$

$$= 2\pi/3. \quad \left(= \int_\gamma \frac{1}{z^6+1}\,dz\right).$$

Goal: Show that $\lim_{R\to\infty} \int_{\gamma_R} = 0$.

$$\left|\int_{\gamma_R} \frac{1}{z^6+1}\,dz\right| \leq \text{length of } \gamma_R$$

$$\text{times } \max_{z\in\gamma_R} \left|\frac{1}{z^6+1}\right|$$

$$\text{length} = \frac{2\pi R}{2} = \pi R$$

Note: on γ_R, $|z| = R$. So,

$$\max_{z\in\gamma_R} \left|\frac{1}{z^6+1}\right| \leq \max_{z\in\gamma_R} \frac{1}{\left||z|^6-1\right|}$$

$$= \frac{1}{R^6-1}$$

Now, length $\times$ max $\leq \dfrac{\pi R}{R^6-1}$,

which $\longrightarrow 0$ as $R \longrightarrow \infty$.

Thus $\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{dz}{z^6+1} \right| = 0$,

so $\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^6+1}\, dz = 0$.

In summary,

$$\lim_{R\to\infty} \frac{1}{2} \int_{-R}^{R} \frac{1}{x^6+1}\, dx$$

$$= \lim_{R\to\infty} \frac{1}{2} \int_{\underbrace{[-R,R] \cup \gamma_R}_{\gamma}} \frac{1}{z^6+1}\, dz$$

$$= \frac{1}{2} \cdot \left(2\pi/3 \right) = \frac{\pi}{3}.$$

This idea is valid when computing $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$,

where $p(x), q(x)$ are polynomials, $q(x)$ has no real zeros, and

$$\lim_{|z|\to\infty} z\cdot\frac{p(z)}{q(z)} = 0$$

(to show that $\int_{\gamma_R} \to 0$ as $R\to\infty$).

Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} = 2\pi i \cdot \sum_{\substack{z_k \\ \operatorname{Im}(z_k)>0}} \operatorname*{Res}_{z=z_k} \frac{p(z)}{q(z)}$$

where z_k runs through the zeros of $q(x)$.