More examples of integrals to compute using the residue thm:

Ex) $$\int_0^\infty \frac{\sin(x)}{x}\,dx$$

This integral does not converge absolutely:

$$\int_0^\infty \left|\frac{\sin(x)}{x}\right| dx \text{ diverges.}$$

Strategy: Proceed like we did

for $\int_0^\infty \frac{\cos(x)}{x^2+1}\,dx$:

- Associate the integral with its Cauchy principal value:

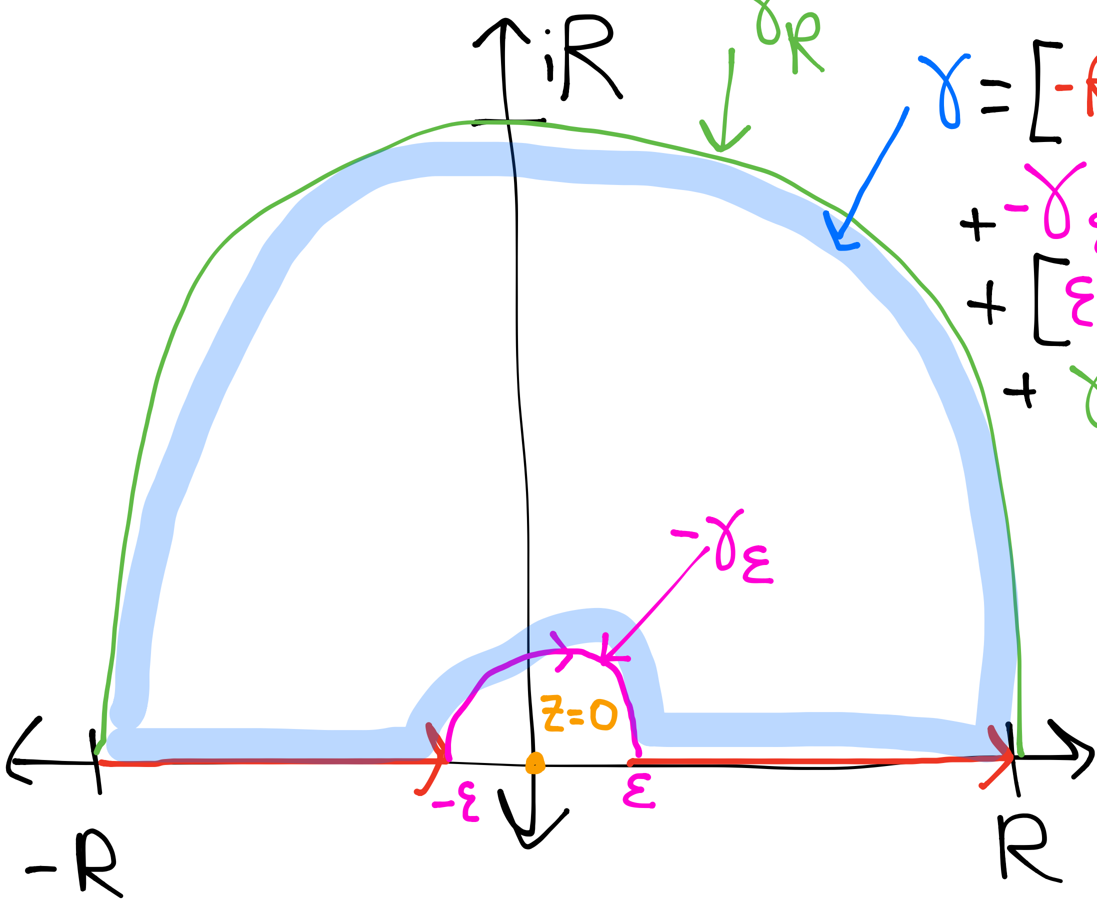
$$\int_0^\infty \frac{\sin(x)}{x}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{\sin(x)}{x}\,dx$$

$$= \frac{1}{2} \operatorname{Im}\left(\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x}\,dx\right)$$

- <u>Semi-circular contour</u>



$$\gamma = [-R, -\varepsilon] + -\gamma_\varepsilon + [\varepsilon, R] + \gamma_R$$

Idea: Proceed as

before, but include the semicircle around zero, and take $\varepsilon \to 0^+$.

- The integral along γ_R tends to zero as $R \to \infty$ by arguing similarly as we did with $\int_0^\infty \frac{\cos(x)}{x^2+1}\,dx$ (triangle ineq. for integrals).

- Integral along γ equals zero by Cauchy's thm ($\frac{e^{iz}}{z}$ analytic when $z \neq 0$).

- Summary so far:

$$\cancelto{0}{\int_{\gamma}} = \int_{-R}^{-\varepsilon} + \int_{-\gamma_\varepsilon} + \int_{\varepsilon}^{R} + \cancelto{0}{\int_{\gamma_R}}$$

(when $R \to \infty$)

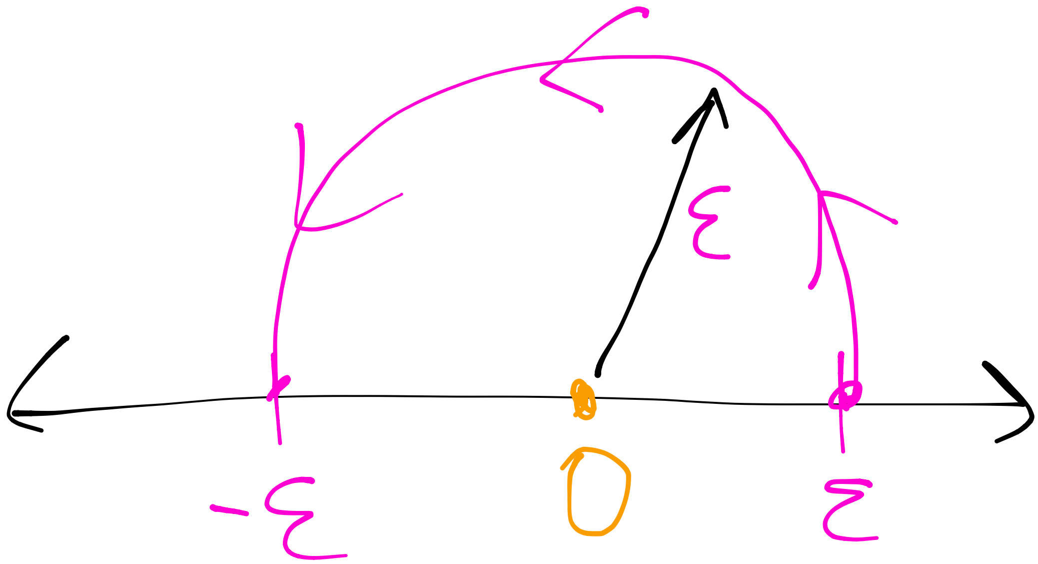
Thus

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx$$

$$= -\int_{-\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz$$

$$= \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz$$

counterclockwise direction now!!!



Lemma: If $z=0$ is a pole of order 1 for a function $g(z)$, then

$$\lim_{\varepsilon\to 0^+} \int_{\gamma_\varepsilon} g(z)\,dz = \pi i \cdot \operatorname*{Res}_{z=0} g(z).$$

Pf) The Laurent series for $g(z)$ at $z=0$ is

$$g(z) = \frac{a_{-1}}{z} + \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{h(z)}.$$

If ε is small enough, then

$$\int_{\gamma_\varepsilon} g(z)\,dz = \int_{\gamma_\varepsilon} \frac{a_{-1}}{z}\,dz + \int_{\gamma_\varepsilon} h(z)\,dz$$

$= \pi i a_{-1}$ + something bounded by $\pi \varepsilon \cdot \max_{|z| \le \varepsilon} |h(z)|$

(by parametrization) $z(t) = \varepsilon e^{it}$, $0 \le t \le \pi$

(triangle inequality for integrals).

This $\to 0$ as $\varepsilon \to 0^+$ because $h(z)$ is analytic at $z = 0$.

Thus as $\varepsilon \to 0^+$,
we are left with $\pi i a_{-1}$.

$\square$

Thus if R is big, then

$$\lim_{\varepsilon \to 0^+} \left(\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx \right)$$

$$= \lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz$$

(lemma)

$$= \pi i \operatorname*{Res}_{z=0} \frac{e^{iz}}{z} = \pi i.$$

Now, recall from before that

$$\frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x)}{x}\,dx$$

$$= \frac{1}{2}\operatorname{Im}\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x}\,dx\right)$$

$$= \frac{1}{2}\operatorname{Im}\left(\pi i\right)$$

$$= \pi/2.$$

Next Time:

Last integral example