Last "real integration" example:

Previous example:

$$\int_0^\infty \frac{\sin(x)}{x}\,dx$$

(can't be evaluated using Calc 2 ideas)

Here is another such example:

For fixed $\alpha \in (0,1)$,

$$\int_0^\infty \frac{dx}{x^\alpha (1+x)}.$$

(Exercise: What happens when $\alpha = 0$ or 1 or when $\alpha > 1$.)

Note: Our usual even/odd trick to pass from $\int_0^\infty$ to $\int_{-\infty}^\infty$.

won't work here. So we need another type of contour:

Notice that as a function of z,

$$f(z) = \frac{1}{z^{\alpha}(1+z)}, \qquad 0 < \alpha < 1$$

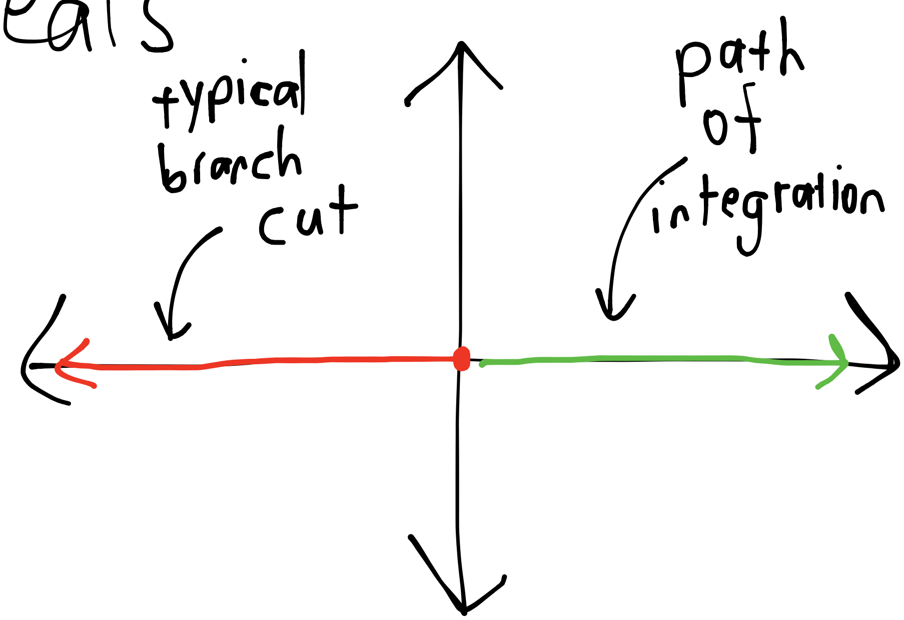
is multivalued!

$$\left(z^{\alpha} = e^{\alpha \log z} \right)$$

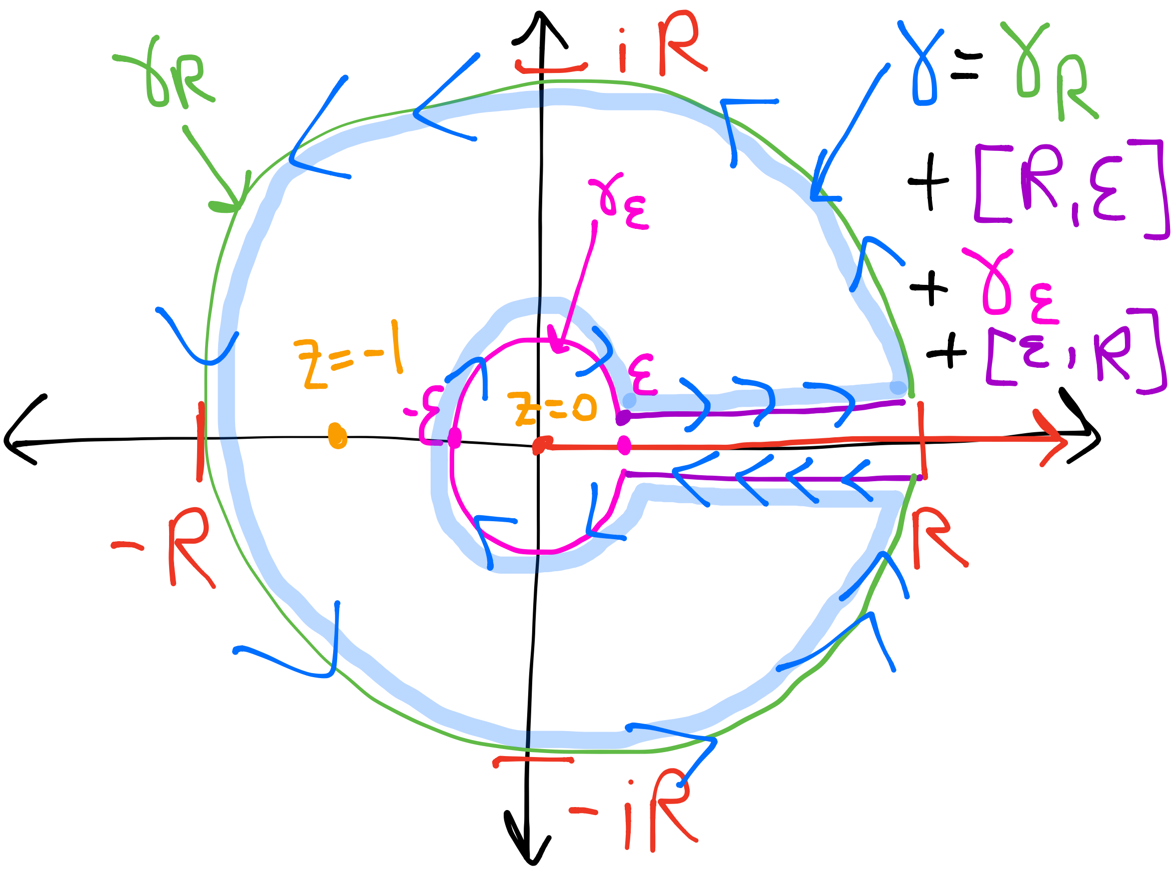
..

Need to pick a branch!!

Typical choice for log: remove the nonpositive reals:



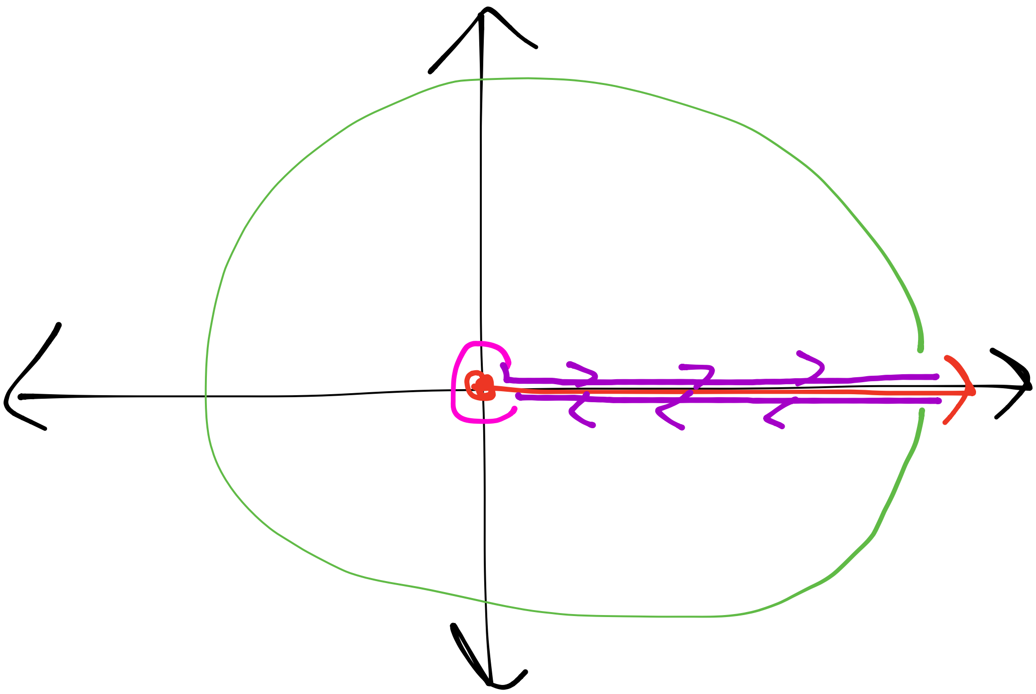
Our choice here: remove the nonnegative reals.



Our closed loop for integrating with the residue thm cannot cross the branch cut!

<u>Idea</u>: Take $R \to \infty$
$\varepsilon \to 0^+$.

This leads to



$$2\pi i \sum_{z_k} \operatorname{Res}_{z=k} f = \int_{\gamma} f(z)\,dz$$

$$= \cancelto{0}{\int_{\gamma_R} f(z)\,dz}$$

$$+ \cancelto{0}{\int_{[R,\varepsilon]} f(x)\,dx}$$

$$+\cancel{\int_{\gamma_\varepsilon} f(z)\,dz}$$

$$\cancel{+\int_\varepsilon^R f(x)\,dx}$$ Goal

On one side of the branch cut, z^α is

$$(|z|e^{i\theta})^\alpha = |z|^\alpha e^{i\theta\alpha}.$$

On the other side,

$$z^\alpha = \left(|z|e^{i(\theta+2\pi)}\right)^\alpha$$

$$= z^{\alpha} e^{2\pi i \alpha}.$$

Thus

$$\int_{[\varepsilon, R]} \frac{1}{x^{\alpha}(1+x)}\,dx + \int_{[R, \varepsilon]} \frac{1}{x^{\alpha}(1+x)}\,dx$$

$$= \int_{[\varepsilon, R]} \frac{dx}{x^{\alpha}(1+x)} - \frac{1}{e^{2\pi i \alpha}} \int_{\varepsilon}^{R} \frac{dx}{x^{\alpha}(1+x)}$$

$$= \left(1 - e^{-2\pi i \alpha}\right) \int_{\varepsilon}^{R} \frac{dx}{x^{\alpha}(1+x)}.$$

The integrals $\int_{\gamma_R}$ and $\int_{\gamma_\varepsilon}$

tend to zero as $R \to \infty$ & $\varepsilon \to 0$

(triangle inequality, parametrization).

We are left with

$$(1-e^{-2\pi i \alpha}) \int_{\varepsilon}^{R} \frac{dx}{x^{\alpha}(1+x)}$$

$$= \int_{\gamma} \frac{dz}{z^{\alpha}(1+z)},$$

$$= 2\pi i \operatorname*{Res}_{z=-1} \frac{1}{z^{\alpha}(1+z)}.$$

Take $R \to \infty$, $\varepsilon \to 0^+$:

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)} = \frac{2\pi i}{1-e^{-2\pi i\alpha}} \operatorname*{Res}_{z=-1} \frac{1}{z^\alpha(1+z)}$$

$$= \frac{2\pi i}{1-e^{-2\pi i\alpha}} \frac{1}{\frac{d}{dz}\left(z^\alpha(1+z)\right)\Big|_{z=-1}}$$

$$= \frac{2\pi i}{1-e^{-2\pi i\alpha}} \cdot \frac{1}{(-1)^\alpha}$$

(branch cut)

$$= \frac{2\pi i}{1-e^{-2\pi i\alpha}} \cdot \frac{1}{e^{\pi i\alpha}}$$

$$= \pi \frac{2i}{e^{\pi i \alpha} - e^{-\pi i \alpha}}$$

$$= \frac{\pi}{\sin(\pi\alpha)}.$$