The Argument Principle

Recall: $\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{z}=1.$

$z(t)=e^{it}$, $0\le t\le 2\pi$

But if $z(t)=e^{int}$, $0\le t\le 2\pi$, $n\in\mathbb{Z}$, then with γ parametrized by $z(t)$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{ine^{int}}{e^{int}}\,dt$$

$$= \frac{in}{2\pi i} \int_0^{2\pi} dt$$

$= n$

$=$ # of times $z(t)$ circles around zero.

Def A function f is <u>meromorphic</u> on a domain D if f is holomorphic on D except possibly for finitely many poles.

Let γ be a simple closed contour, and let f be

- analytic on γ, and
- meromorphic interior to γ,
- $f(z) \neq 0$ on γ.

γ parametrized by $z(t)$, $a \leq t \leq b$.

When we think about $\int_\gamma f(z)\,dz$, we are really thinking about

$$\int_a^b \underline{f(z(t))} z'(t)\,dt$$

Let Γ (capital gamma) be the image of $f(z(t))$ for $a \leq t \leq b$.

<u>Note</u>: $f(z(t))$ might not be a positively oriented simple closed curve.

<u>Note</u>: Since $f(z) \neq 0$

for z on γ, Γ never touches 0.

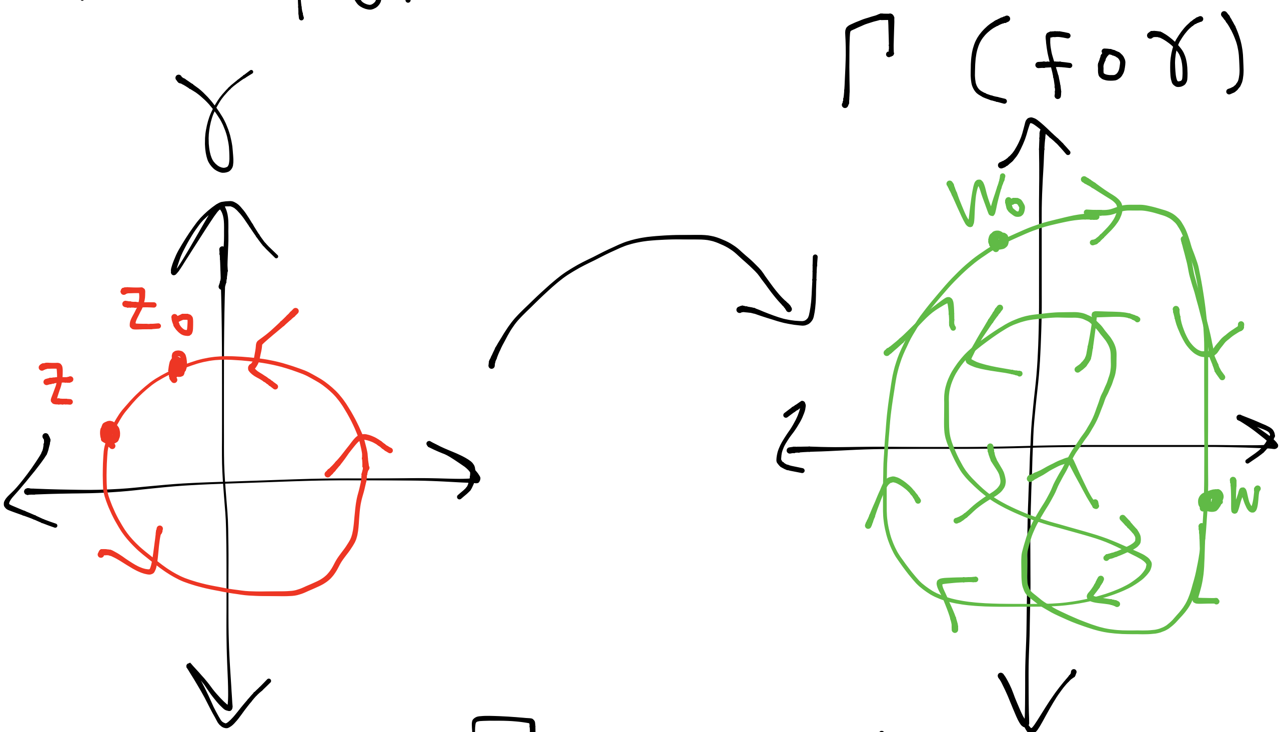
Let w, w_0 lie on Γ.

Let $\phi_0 \in \arg w_0$.

We will consider w as moving along Γ, starting at w_0.

<u>Idea</u>: Track how $\arg w$ varies along Γ relative

to Φ_0.



Since Γ is closed but possibly not simple closed, as w moves along Γ starting from w_0 and then ending up back at w_0, it might be the case that arg w changes a lot from the time we start at w_0 and then end up at w_0 again. If we call ϕ_1 the ending argument once we circle back to w_0, then we can define

$$(*) \quad \Delta_\gamma \arg f(z) = \phi_1 - \phi_0$$

This will be an integer

multiple of 2π, and it detects the number of times that w circles the origin along Γ (accounting for orientation) as w starts at w_0 and traverses all of Γ, ending back at w_0.

We call $\Delta_\gamma \arg f(z)$ the <u>winding number</u> of $\Gamma = f \circ \gamma$ (with respect to the origin).

Thm (Argument principle)

Let γ be a simple closed positively-oriented contour. Let f be analytic and nonzero on γ and meromorphic interior to γ. Then

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

which equals $Z-P$,

where Z is the number of

zeros of f interior to γ
and P is the number
of poles of f interior to γ.

We will prove this
next time.
This uses the residue
thm.