Argument Principle

Let γ be a simple closed curve, and let $f(z)$

- be analytic and nonzero on γ
- be meromorphic interior to γ
- have $Z \geq 0$ zeros and $P \geq 0$ poles interior to γ.

Then

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z)$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

$$= Z - P.$$

$\boxed{\text{Ex}}$ Let $f(z) = \cos(z)$ and let γ be $\{z : |z| = 2\}$.

There are no poles of $\cos(z)$ interior to γ, and there are 2 zeros interior to γ: $z = \pm \pi/2$.

By the argument principal,

$$2 = Z - P$$

$$= \frac{1}{2\pi i} \int_{|z|=2} -\tan(z)\, dz$$

$$= \frac{1}{2\pi} \Delta_\gamma \arg(\cos(z)).$$

<u>Exercise</u> Try computing the integral using other methods.

Pf of the argument principal:

We have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

Parametrizing γ as $z(t)$ $(a \le t \le b)$, this equals

$$\frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt} f(z(t))}{f(z(t))}\,dt.$$

Since $f(z(t)) \neq 0$ for all $a \le t \le b$, we may write

. .

$f(z(t)) = \rho(t) e^{i\phi(t)}$.

($\rho(t)$, $\phi(t)$ differentiable.)

Thus

$$\frac{d}{dt} f(z(t)) =$$

$$\rho'(t) e^{i\phi(t)} + i\rho(t)\phi'(t) e^{i\phi(t)}$$

Thus
$$\frac{\frac{d}{dt} f(z(t))}{f(z(t))}$$

$$= \frac{\frac{d}{dt}\rho(t)}{\rho(t)} + i\phi'(t).$$

Thus

$$\frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt} f(z(t))}{f(z(t))}\, dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt}\rho(t)}{\rho(t)}\, dt$$

$$+ \frac{1}{2\pi \cancel{i}}\int_a^b \cancel{i}\,\frac{d}{dt}\phi(t)\, dt$$

$$= \cancel{\frac{1}{2\pi i}\ln(\rho(t))\Big|_a^b} \to 0$$

since γ is closed

$$+ \frac{1}{2\pi}\phi(t)\Big|_a^b$$

$$= \frac{1}{2\pi} \Delta_\gamma \arg f(z).$$

Now, we want to show that $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z - P.$

If f has a zero z_0 of order h interior to γ, then

$$f(z) = (z - z_0)^h H(z),$$

where $H(z)$ is analytic and

nonzero in a NBHD of z_0. Thus for $z \neq z_0$,

$$\frac{f'(z)}{f(z)} = \frac{h(z-z_0)^{h-1} H(z) + (z-z_0)^h H'(z)}{(z-z_0)^h H(z)}$$

$$= \frac{h}{z-z_0} + \frac{H'(z)}{H(z)}.$$

Thus $\operatorname*{res}_{z=z_0} \frac{f'(z)}{f(z)} = h$.

If f has a pole

of order m at $z = p_0$,

then $$f(z) = (z - p_0)^{-m} G(z),$$

where $G(z)$ is nonzero and analytic in a NBHD of p_0.

Thus in a NBHD of p_0,

$$\frac{f'(z)}{f(z)} = \frac{-m(z-p_0)^{-m-1} G(z) + (z-p_0)^{-m} G'(z)}{(z-p_0)^{-m} G(z)}$$

$$= \frac{-m}{z-p_0} + \frac{G'(z)}{G(z)},$$

provided that $z \neq p_0$.

Thus $\underset{z=p_0}{\operatorname{res}} \dfrac{f'(z)}{f(z)} = -m$.

We do this for each pole and zero of f interior to γ, and we sum the all of the contributions to achieve

$$Z - P$$

Z: zeros of f interior to γ, counted with

P: poles of f interior to γ, counted with multiplicity multiplicity

$\square$

Application of the argument principle:

Thm (Rouché's thm)

Let γ be a simple closed curve, and suppose that

1) f, g are analytic on and interior to γ,

2) $|f(z)-g(z)| < |f(z)|$ for all z on γ.

Then f and g have exactly the same number of zeros interior to γ.

<u>Next Time</u>: Proof of Rouché's thm and give an example.