

# Argument Principle

Let  $\gamma$  be a simple closed curve, and let  $f(z)$

- be analytic and nonzero on  $\gamma$
- be meromorphic interior to  $\gamma$
- have  $Z \geq 0$  zeros and  $P \geq 0$  poles interior to  $\gamma$ .

Then

$$\frac{1}{2\pi} \Delta_{\gamma} \arg f(z)$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

$$= \sum -P.$$

Ex) Let  $f(z) = \cos(z)$  and let  $\gamma$  be  $\{z : |z| = a\}$ .

There are no poles of  $\cos(z)$  interior to  $\gamma$ , and

there are 2 zeros interior to  $\gamma$ :  $z = \pm \pi/2$ .

By the argument principle,

$$2 = Z - P$$

$$= \frac{1}{2\pi i} \int_{|z|=2} -\tan(z) dz$$

$$= \frac{1}{2\pi} \Delta_{\gamma} \arg(\cos(z)).$$

Exercise] Try computing the integral using other methods.

Pf of the argument principle:

We have

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz.$$

Parametrizing  $\gamma$  as  $z(t)$  ( $a \leq t \leq b$ ), this

equals

$$\frac{1}{2\pi i} \int_a^b \frac{\frac{d}{dt} f(z(t))}{f(z(t))} dt.$$

Since  $f(z(t)) \neq 0$  for all  $a \leq t \leq b$ , we may write

$$f(z(t)) = \rho(t) e^{i\phi(t)}.$$

( $\rho(t), \phi(t)$  differentiable)

Thus

$$\frac{d}{dt} f(z(t)) =$$

$$\rho'(t) e^{i\phi(t)} + i\rho(t)\phi'(t) e^{i\phi(t)}$$

$$\text{Thus } \frac{\frac{d}{dt} f(z(t))}{f(z(t))}$$

$$=$$

$$\frac{\frac{d}{dt} \rho(t)}{\rho(t)} + i\phi'(t).$$

Thus

$$\frac{1}{2\pi i} \int_a^b \frac{\frac{d}{dt} f(z(t))}{f(z(t))} dt$$

$$= \frac{1}{2\pi i} \int_a^b \frac{\frac{d}{dt} \rho(t)}{\rho(t)} dt$$

$$+ \frac{1}{2\pi i} \int_a^b \frac{d}{dt} \phi(t) dt$$

$$= \frac{1}{2\pi i} \ln(\rho(t)) \Big|_a^b$$

since  $f \circ \gamma$  is closed

$$+ \frac{1}{2\pi} \phi(t) \Big|_a^b$$

$$= \frac{1}{2\pi} \Delta_{\gamma} \arg f(z).$$

Now, we want to show  
that  $\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = Z - P.$

If  $f$  has a zero  $z_0$   
of order  $h$  interior to  $\gamma$ ,  
then

$f(z) = (z - z_0)^h H(z),$   
where  $H(z)$  is analytic and

nonzero in a NBHD of  $z_0$ . Thus for  $z \neq z_0$ ,

$$\frac{f'(z)}{f(z)} = \frac{h(z-z_0)^{h-1} H(z) + (z-z_0)^h H'(z)}{(z-z_0)^h H(z)}$$

$$= \frac{h}{z-z_0} + \frac{H'(z)}{H(z)}$$

Thus  $\operatorname{res}_{z=z_0} \frac{f'(z)}{f(z)} = h$ .

If  $f$  has a pole



of order  $m$  at  $z = p_0$ ,  
then  $f(z) = (z - p_0)^{-m} G(z)$ ,  
where  $G(z)$  is nonzero and  
analytic in a NBHD of  $p_0$ .

Thus in a NBHD of  $p_0$ ,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-m(z-p_0)^{-m-1} G(z) + (z-p_0)^{-m} G'(z)}{(z-p_0)^{-m} G(z)} \\ &= \frac{-m}{z-p_0} + \frac{G'(z)}{G(z)},\end{aligned}$$

provided that  $z \neq p_0$ .

Thus  $\operatorname{res}_{z=p_0} \frac{f'(z)}{f(z)} = -m$ .

We do this for each pole and zero of  $f$  interior to  $\gamma$ , and we sum the all of the contributions to achieve

$$\sum - P$$

↙

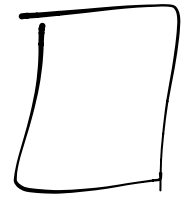
zeros of  $f$   
interior to  $\gamma$ ,  
counted with

↘

poles of  $f$   
interior to  $\gamma$ ,  
counted with

multiplicity

multiplicity



Application of the  
argument principle:

Thm (Rouche's thm)

Let  $\gamma$  be a simple  
closed curve, and  
suppose that

1)  $f, g$  are analytic  
on and interior to  $\gamma$ ,

$$2) |f(z) - g(z)| < |f(z)|$$

for all  $z$  on  $\gamma$ .

Then  $f$  and  $g$  have exactly the same number of zeros interior to  $\gamma$ .

Next Time: Proof of Rouché's thm and give an example.