Today is the last lecture with material on Midterm 2.

Last time, we looked at how to use the residue thm to compute

$$\int_0^\infty \frac{1}{x^6+1}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^6+1}\,dx.$$

How do we know that

$$\int_{-\infty}^{\infty} = \lim_{R\to\infty} \int_{-R}^{R} \; ?$$

In this case, it follows since $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$ converges absolutely:

$$\int_{-\infty}^{\infty} \left| \frac{1}{x^6+1} \right| dx < \infty.$$

More generally, if $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges absolutely, then $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx.$$

Q) What if $\int_{-\infty}^{\infty} f(x)\,dx$ does NOT converge absolutely?

A) (In a special case)

Let $g(z)$ be holomorphic in the regin $\operatorname{Im}(z) \geq 0$, apart from finitely many isolated singularities. The <u>Cauchy Principal Value</u> of $\int_{-\infty}^{\infty} g(x)e^{ix}\,dx$ is

$$\lim_{R\to\infty} \int_{-R}^{R} g(x)e^{ix}\,dx.$$

This is defined regardless of whether

$$\int_{-\infty}^{\infty} g(x)e^{ix}\,dx$$

converges absolutely.

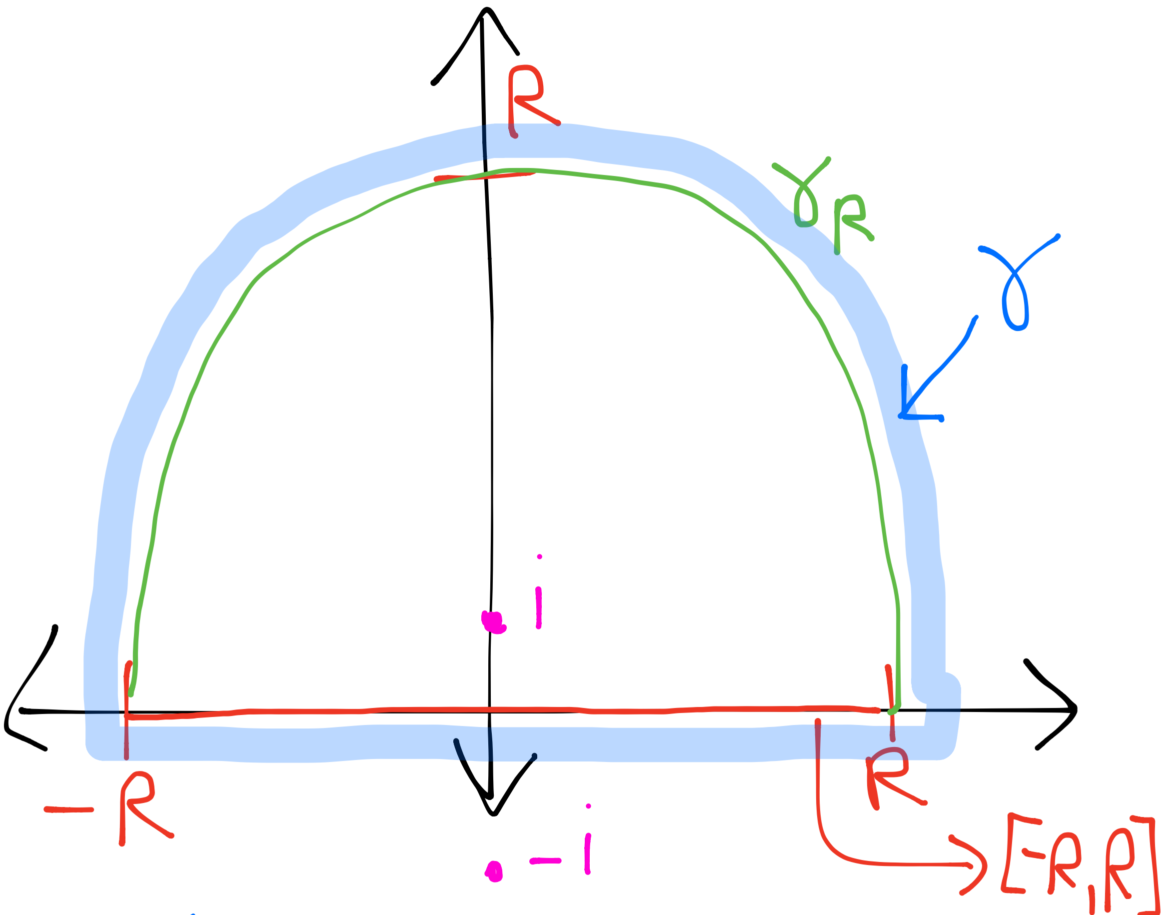
$\boxed{\text{Ex}}$ $\displaystyle\int_0^\infty \frac{\cos(x)}{x^2+1}\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos(x)}{x^2+1}\,dx$$

(Integral converges absolutely, so integral equals its Cauchy principal value)

$$= \frac{1}{2}\,\mathrm{Re}\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x^2+1}\,dx\right)$$



$\gamma = [-R,R] \cup \gamma_R$

$\gamma_R: \ z(t) = Re^{it}, \ 0 < t < \pi.$

Like last time,

$$\int_\gamma \frac{e^{iz}}{z^2+1}\,dz = \cancel{\int_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dx} \to 0$$

$$+ \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\,dx$$

isolated singularities: $z = \pm i$.

<u>Residue thm:</u>

$$\int_{\gamma} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iz}}{z^2+1}$$

$$= 2\pi i \left.\frac{e^{iz}}{2z}\right|_{z=i}$$

$$= 2\pi i \cdot \frac{e^{-1}}{2i}$$

$$= \pi/e.$$

Goal) Show that as $R \to \infty$, $\int_{\gamma_R} \frac{e^{iz}}{z^2+1} dz \to 0$.

To show this, let

$$M(R) = \max_{0 \leq \theta \leq \pi} \left| \frac{1}{(Re^{i\theta})^2+1} \right|.$$

Then

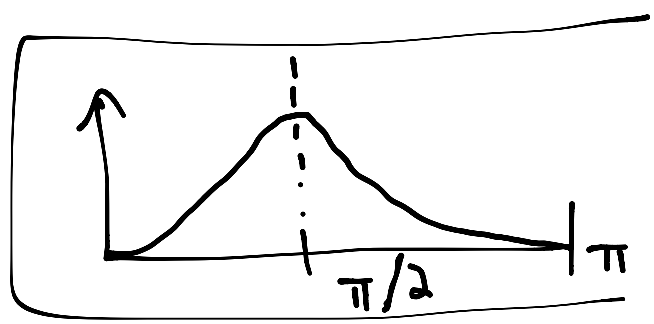
$$\left| \int_{\gamma_R} \frac{e^{iz}}{z^2+1} dz \right| \leq \int_{\gamma_R} \left| \frac{e^{iz}}{z^2+1} \right| dz$$

$$\leq M(R) \int_{\gamma_R} |e^{iz}| |dz|$$

$$= M(R)\int_0^{\pi} e^{Re(iRe^{i\theta})} R\,d\theta$$

$$= M(R)\int_0^{\pi} e^{-R\sin\theta} R\,d\theta.$$

Since $\sin\theta$ is symmetric about $\theta = \pi/2$ when $0 \leq \theta \leq \pi$,



$$= 2M(R)\int_0^{\pi/2} e^{\underline{-R\sin\theta}} R\,d\theta.$$

Note: If $0 \leq \theta \leq \pi/2$,

then $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$.

$\hookrightarrow -\frac{2\theta}{\pi} \geq -\sin\theta \geq -\theta$

$$\leq 2M(R)R\int_0^{\pi/2} e^{-R\frac{2\theta}{\pi}}\,d\theta$$

$$= M(R)\cdot\pi.$$

$$= \pi\cdot\max_{0\leq\theta\leq\pi}\left|\frac{1}{(Re^{i\theta})^2+1}\right|$$

$$\leq \frac{\pi}{R^2-1} \xrightarrow{R\to\infty} 0.$$

(Reverse triangle inequality)

Conclusion: $\int_{\gamma_R} \to 0$ as $R \to \infty$.

So

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\,dx$$

$$= \lim_{R\to\infty} \int_{\gamma} \frac{e^{iz}}{z^2+1}\,dz$$

$$= \pi/e.$$

To finish, we note that

$$\int_0^{\infty} \frac{\cos(x)}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\mathrm{Re}\left(\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{x^2+1}\,dx\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{e}$$

$$= \frac{\pi}{2e}. \quad \checkmark$$