

Problem

Suppose that f is entire and suppose that there exists a constant $c > 0$ such that

$$|f(z)| \leq c|z|$$

for all z with $|z| > 100$.

Prove that f is linear.

(i.e., $f(z) = az + b$).

Pf] Note that in order for f to be linear, we need to show that

$f^{(m)}(0) = 0$ for all $m \geq 2$. Then we would have

$$\begin{aligned} f(z) &= \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m \\ &= f(0) + f'(0)z + 0 + 0 + 0 + \dots \end{aligned}$$

Strategy similar to Liouville:

Take γ to be $\{z: |z|=R\}$
for some $R > 100$.

Cauchy Int. Formula:

For $m \geq 2$, we have

$$\begin{aligned} |f^{(m)}(0)| &= \left| \frac{m!}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{m+1}} dw \right| \\ &\leq \frac{m!}{2\pi} \int_{\gamma} \left| \frac{f(w)}{w^{m+1}} \right| |dw| \\ &\leq \frac{m!}{2\pi} \int_{\gamma} \frac{c|w|}{|w|^{m+1}} \cdot |dw| \end{aligned}$$

$$\begin{aligned}
&= \frac{m!c}{2\pi} \frac{1}{R^m} \int_{\gamma} |dw| \\
&= \frac{m!c}{2\pi} \frac{1}{R^m} \cdot 2\pi R \\
&= \frac{m!c}{R^{m-1}} \quad (m \geq 2) \\
&\leq \frac{m!c}{R}.
\end{aligned}$$

We can let $R \rightarrow \infty$,
and we conclude that
 $|f^{(m)}(0)| = 0$ for all
 $m \geq 2$. \square

Ex) Show that if we change $|f(z)| \leq c|z|$ to $|f(z)| \leq c|z|^2$, then f must be linear or quadratic.

$$(f(z) = dz^2 + fz + g)$$

Problem Compute

$$\int_{|z|=1} \frac{dz}{z \sin(z)}$$

1) Find singularities

$$\text{Note: } z \sin(z) = 0$$

When $z=0$ or $\sin(z)=0$.

$$\sin(z)=0 \rightarrow z=n\pi, n \in \mathbb{Z}.$$

Only singularity within $|z|=1$ is $z=0$.

2) Apply residue thm:

$$\int_{|z|=1} \frac{dz}{z \sin(z)} = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z \sin(z)}$$

3) Compute the residue

Option 1) $\lim_{z \rightarrow 0} \left(\frac{d}{dz} z^2 \cdot \frac{1}{z \sin(z)} \right)$

Option 2) Laurent Series

$$\frac{1}{z \sin(z)} = \frac{1}{z \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)}$$

$$= \frac{1}{z^2 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right)}$$

$$= \frac{1}{z^2} \cdot \left(1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right) \right)$$

Geometric sums: If

$$\left| \frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right| < 1 \quad (\text{this})$$

will happen when $|z| < 1$),

we have that

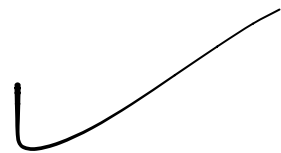
$$\frac{1}{z \sin(z)} = \frac{1}{z^2} \cdot \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right)}$$

$$= \frac{1}{z^2} \left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right)^2 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots \right)^3 + \dots \right)$$

$$\begin{aligned}
&= \frac{1}{z^2} \left(1 + \frac{z^2}{3!} + \left(\frac{-1}{5!} + \left(\frac{1}{3!} \right)^2 \right) \frac{z^4}{4} + \dots \right) \\
&= \frac{1}{z^2} + \frac{1}{3!} + \left(\frac{-1}{5!} + \left(\frac{1}{3!} \right)^2 \right) z^2 + \dots \\
&= \frac{1}{z^2} + \frac{0}{z} + \frac{1}{3!} + \left(\frac{-1}{5!} + \left(\frac{1}{3!} \right)^2 \right) z^2 + \dots \\
&\quad (0 < |z| < 1)
\end{aligned}$$

Thus $\operatorname{Res}_{z=0} \frac{1}{z \sin(z)} = 0,$

So $\int_{|z|=1} \frac{dz}{z \sin(z)} = 0.$



Ex

1) Compute a
Laurent series
for $\frac{1}{\text{Log}(z)}$

(principal branch of $\log z$)
centered at $z=1$.

2) Determine the
annular region in which
your Laurent expansion
is valid.

3) Use this
Laurent expansion
to compute

$$\operatorname{Res}_{s=1} \frac{1}{\operatorname{Log}(z)}$$

and $\int_{|z-1|=\frac{1}{2}} \frac{dz}{\operatorname{Log}(z)}$.