Problem

Suppose that f is <u>entire</u> and suppose that there exists a constant $c>0$ such that

$$|f(z)| \leq c|z|$$

for all z with $|z|>100$.

Prove that f is linear. (i.e., $f(z) = az+b$).

Pf] Note that in order for f to be linear, we need to show that $\underline{f^{(m)}(0) = 0}$ for all $m \geq 2$. Then we would have

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m$$

$$= f(0) + f'(0)z + 0 + 0 + 0 + \dots$$

Strategy similar to Liouville:

Take γ to be $\{z : |z| = R\}$ for some $R > 100$.

Cauchy Int. Formula: For $m \geq 2$, we have

$$|f^{(m)}(0)| = \left| \frac{m!}{2\pi i} \int_\gamma \frac{f(w)}{w^{m+1}} dw \right|$$

$$\leq \frac{m!}{2\pi} \int_\gamma \left| \frac{f(w)}{w^{m+1}} \right| |dw|$$

$$\leq \frac{m!}{2\pi} \int_\gamma \frac{c|w|}{|w|^{m+1}} \cdot |dw|$$

$$= \frac{m!\,C}{2\pi} \frac{1}{R^m} \int_\gamma |dw|$$

$$= \frac{m!\,C}{2\pi} \frac{1}{R^m} \cdot 2\pi R$$

$$= \frac{m!\,C}{R^{m-1}} \quad (m \geq 2)$$

$$\leq \frac{m!\,C}{R}.$$

We can let $R \to \infty$, and we conclude that $|f^{(m)}(0)| = 0$ for all $m \geq 2$. $\square$

Ex) Show that if we change $|f(z)| \leq c|z|$ to $|f(z)| \leq c|z|^2$, then f must be linear or quadratic.

$$(f(z) = dz^2 + fz + g)$$

Problem) Compute

$$\int_{|z|=1} \frac{dz}{z \sin(z)}$$

1) Find singularities

Note: $z\sin(z) = 0$ when $z = 0$ or $\sin(z) = 0$.

$\sin(z) = 0 \rightarrow z = n\pi,\ n \in \mathbb{Z}$.

Only singularity within $|z| = 1$ is $z = 0$.

2) Apply <u>residue thm</u>:

$$\int_{|z|=1} \frac{dz}{z\sin(z)} = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z\sin(z)}$$

3) Compute the residue

Option 1 $\lim_{z\to 0}\left(\frac{d}{dz} z^2 \cdot \frac{1}{z\sin(z)}\right)$

Option 2 $\boxed{\text{Laurent Series}}$

$$\frac{1}{z\sin(z)} = \frac{1}{z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)}$$

$$= \frac{1}{z^2\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)}$$

$$= \frac{1}{z^2} \cdot \boxed{\frac{1}{\left(1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)\right)}}$$

Geometric sums: If $\left|\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right|<1$ (this will happen when $|z|<1$), we have that

$$\frac{1}{z\sin(z)}=\frac{1}{z^2}\cdot\frac{1}{1-\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)}$$

$$=\frac{1}{z^2}\left(1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)^2+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)^3+\cdots\right)$$

$$= \frac{1}{z^2}\left(1 + \frac{z^2}{3!} + \left(-\frac{1}{5!} + \left(\frac{1}{3!}\right)^2\right)z^4 + \dots\right)$$

$$= \frac{1}{z^2} + \frac{1}{3!} + \left(-\frac{1}{5!} + \left(\frac{1}{3!}\right)^2\right)z^2 + \dots$$

$$= \frac{1}{z^2} + \frac{0}{z} + \frac{1}{3!} + \left(-\frac{1}{5!} + \left(\frac{1}{3!}\right)^2\right)z^2 + \dots$$

$$(0 < |z| < 1)$$

Thus $\operatorname*{Res}_{z=0} \frac{1}{z\sin(z)} = 0$,

So $\int_{|z|=1} \frac{dz}{z\sin(z)} = 0$.

Ex)

1) Compute a Laurent series for $\frac{1}{\mathrm{Log}(z)}$ (principal branch of $\log z$) centered at $z=1$.

2) Determine the annular region in which your Laurent expansion is valid.

3) Use this Laurent expansion to compute

$$\operatorname*{Res}_{s=1} \frac{1}{\mathrm{Log}(z)}$$

and

$$\int_{|z-1|=\frac{1}{2}} \frac{dz}{\mathrm{Log}(z)}.$$