Last Time:

Laurent Series

Taylor Series: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$|z-z_0| < R$

Laurent Series: $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$R_1 < |z-z_0| < R_2$

Q] If $f(z) = \sum a_n (z-z_0)^n$, what is $f'(z)$?

Guess: $f'(z) = \sum a_n n (z-z_0)^{n-1}$

Similarly, what is the antiderivative of f?

These are two instances where we have two limits and we want to interchange them:

$$\underset{(\text{deriv})}{\lim_{h\to 0}} \; \underset{(\text{sum})}{\lim_{N\to\infty}} \sum_{n=0}^{N}$$

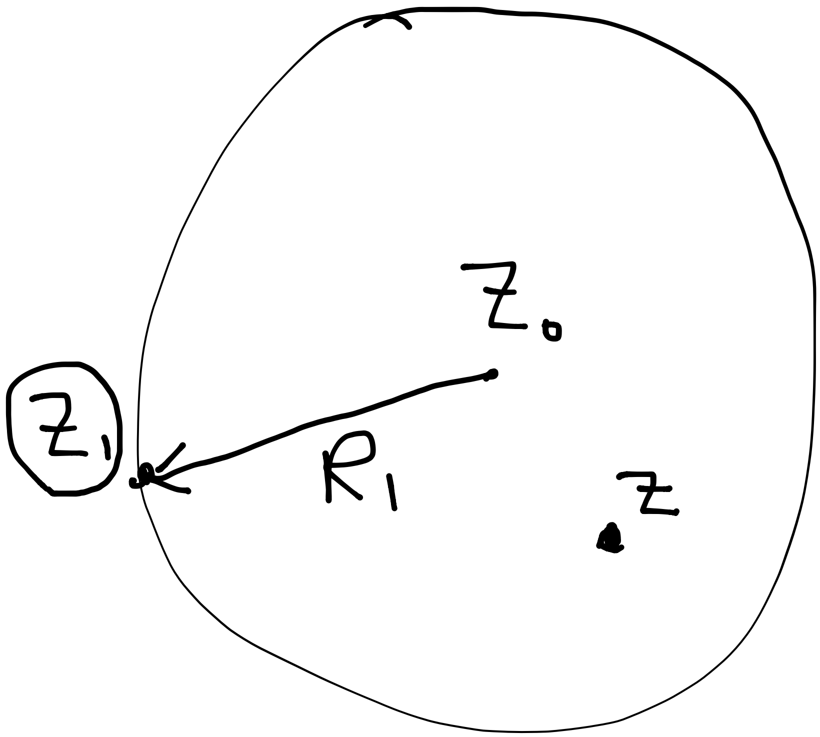
$$\overset{?}{=} \lim_{N\to\infty} \overset{(\text{sum})}{\sum_{n=0}^{N}} \overset{(\text{deriv})}{\lim_{h\to 0}}$$

<u>Thm</u> Let

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

If $f(z)$ converges when $z=z_1 \neq z_0$, then $f(z)$ converges absolutely when

$$|z-z_0|<R_1 := |z_1-z_0|.$$



$|z-z_0|<R_1$: converg absolutely

$|z-z_0|=R_1$, $z\neq z_1$: we don't know

$z=z_1$: converge in some unspecified way

The circle of greatest radius centered at z_0 inside which $f(z)$ convgs absolutely is called the <u>circle of convergence</u>.

Inside: absolute convgc

Outside: divergence

On boundary: Anything could happen

New notion of convergence:

Suppose we have

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

$$|z - z_0| < R.$$

$$S(z) = \lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} a_n (z - z_0)^n}_{S_N(z)}$$

Define $\rho_N(z) = S(z) - S_N(z)$

$S_N(z) \to S(z)$ if and only if $\rho_N(z) \to 0$.

Interior to the circle of convergence for $S(z)$, $(|z-z_0|<R)$, we have

$$\lim_{N\to\infty} \rho_N(z) = 0.$$

ε- definition:

For all $\varepsilon>0$, there exists $\eta>0$ such that

$|\rho_N(z)-0|<\varepsilon$ whenever $N>\underline{\eta}$.

Q] What could N depend on?

A] z_0, ε, z (where you lie in the circle of convergence)

Def] When N depends only on ε and is independent of the point z in the circle of convergence, we say that the convergence

of $\rho_N(z)$ to zero
(or the convergence of
$S_N(z)$ to $S(z)$) is
<u>uniform</u>.

<u>Thm</u> If z lies interior
to the circle of
convergence for $S(z)$,
we have that $S_N(z)$
Converges uniformly to
$S(z)$.

This notion of uniform convergence will allow us to swap limit operations.