Uniform vs. regular convergence

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$|z-z_0| < R$$

$$S_N(z) = \sum_{n=0}^{N} a_n (z-z_0)^n$$

"Regular convergence":

For all $\varepsilon > 0$, there exists $\mathcal{N} > 0$ such that

$|S(z) - S_N(z)| < \varepsilon$ whenever $N > \mathcal{N}$.

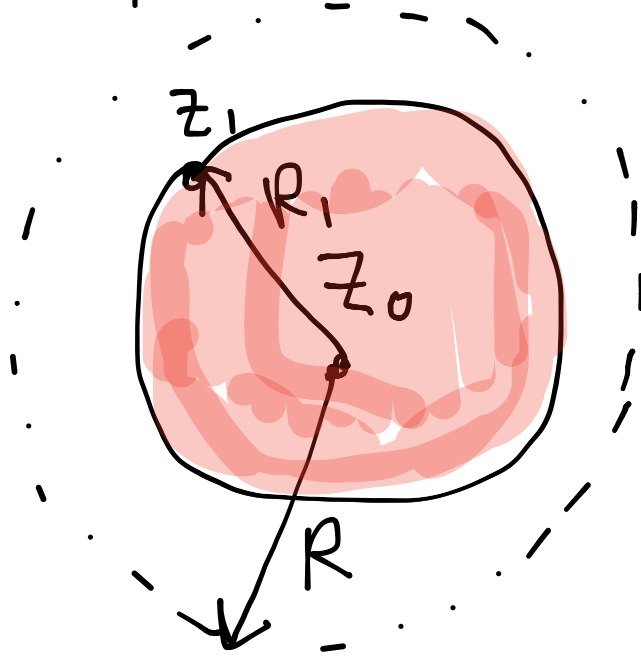
Uniform convergence:

Thinking of $S(z)$ and z_0 as being fixed from the onset, we have that N to depend only on ε, not on z.

Last Time: If z_1 lies interior to the circle of convergence $|z-z_0|=R$, then $S(z)$ converges uniformly in the closed disc

$$\{z : |z - z_0| \le R_1\},$$

where $R_1 = |z_1 - z_0|$.



Thm | If z_1 is interior to the circle of convergence $|z - z_0| = R$, then $S(z)$ represents a continuous function in the disc

$$\{z : |z - z_0| \le R_1\},$$
$$R_1 = |z_1 - z_0|.$$

Put another way, $S(z)$ is continuous for z interior to the circle of convergence $|z - z_0| = R$.

<u>Pf</u> Let z, z_1 lie interior to the circle of convergence.

<u>Goal</u>: $\lim_{z \to z_1} S(z) = S(z_1)$

that is, $\lim_{z \to z_1} \sum_{n=0}^{\infty} a_n (z - z_0)^n$

$$= \sum_{n=0}^{\infty} \lim_{z \to z_1} a_n (z-z_0)^n$$

<u>Rephrased goal</u> (ε-δ version)

We want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$|S(z) - S(z_1)| < \varepsilon$ whenever $|z - z_1| < \delta$.

To begin, pick $\varepsilon > 0$.

For $|z - z_0| < R$, we have

$$S(z) = S_N(z) + \rho_N(z).$$

Now,

$$|S(z) - S(z_1)|$$

$$= |S_N(z) + \rho_N(z) - (S_N(z_1) + \rho_N(z_1))|$$

$$= |(S_N(z) - S_N(z_1)) + (\rho_N(z) - \rho_N(z_1))|$$

$$\leq |S_N(z) - S_N(z_1)| + |\rho_N(z) - \rho_N(z_1)|.$$

Since $S_N(z)$ is a polynomial, which is continuous, we have that $|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3}$ when

$|z-z_1| < \delta$. (δ-ε def of continuity for $S_N(z)$).

By uniform convergence,

$$|\rho_N(z)| < \varepsilon/3$$
$$|\rho_N(z_1)| < \varepsilon/3$$

whenever $N > \mathcal{N}$, where $\mathcal{N}$ depends <u>only</u> on ε and <u>not</u> on z or z_1.

Putting this together:

$$|S_N(z)-S_N(z_1)|+|\rho_N(z)-\rho_N(z_1)|$$
$$\leq |S_N(z)-S_N(z_1)|+|\rho_N(z)|+|\rho_N(z_1)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon. \quad \square$$

The proof work similarly for Laurent series

$$L(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

$$R_1 < |z-z_0| < R_2.$$

Thm If γ is a contour lying interior to the circle of convergence for $S(z)$, and $g(z)$ is continuous on γ, then

$$\int_\gamma g(z) S(z)\, dz$$

$$\hookrightarrow \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n \int_\gamma g(z) (z-z_0)^n dz.$$

Pf Similar to the

situation with continuity
(book).

In my notes, there are some <u>important</u> examples of how unif. continuity can be applied.