Cauchy - Goursat:

If f is analytic on and interior to a simple closed contour γ, then

$$\int_\gamma f(z)\,dz = 0.$$

Q] What can we say when f is not always analytic interior to γ?

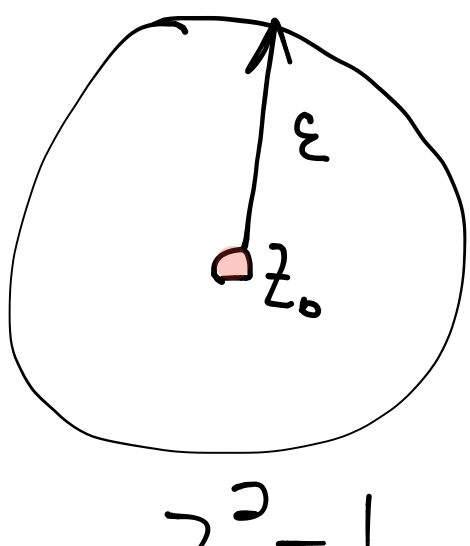
Ex]: $f(z) = \frac{1}{z}$, $\gamma: |z| = 1$

singularity at $z = 0$, so

Cauchy-Goursat does not apply.

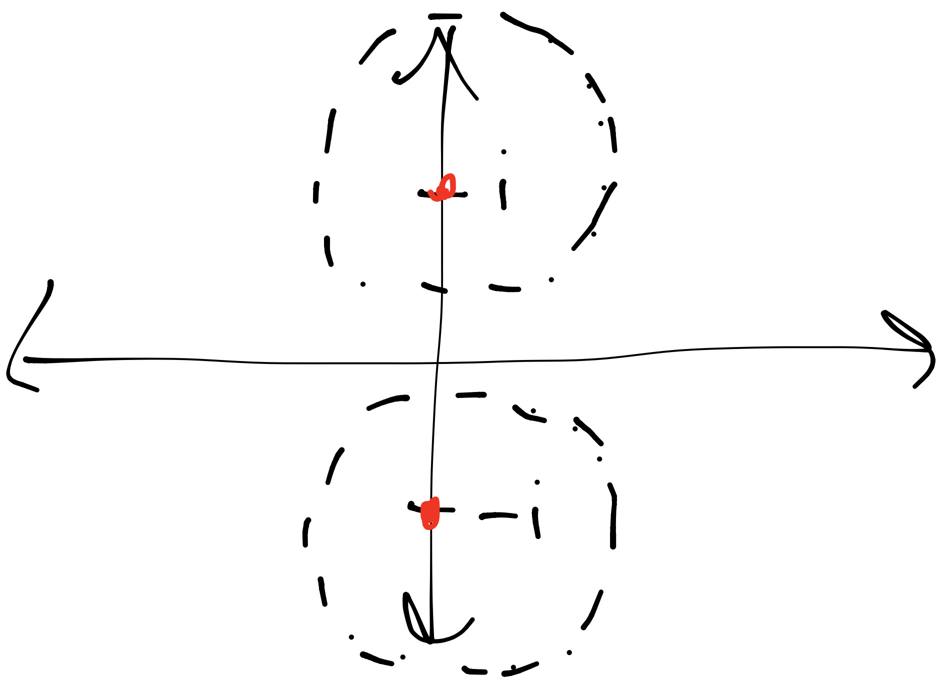
<u>Recall</u> If f is analytic at z_0, then f is analytic in a NBHD of z_0. If f is not analytic at z_0, but f is analytic at some pt in every NBHD of z_0, then f has a <u>singularity</u> at z_0.

Def] <u>An isolated singularity</u> of f is a singularity z_0 such that for some $\varepsilon > 0$, f is analytic in $\{z : 0 < |z - z_0| < \varepsilon\}$, but f is not analytic at z_0.

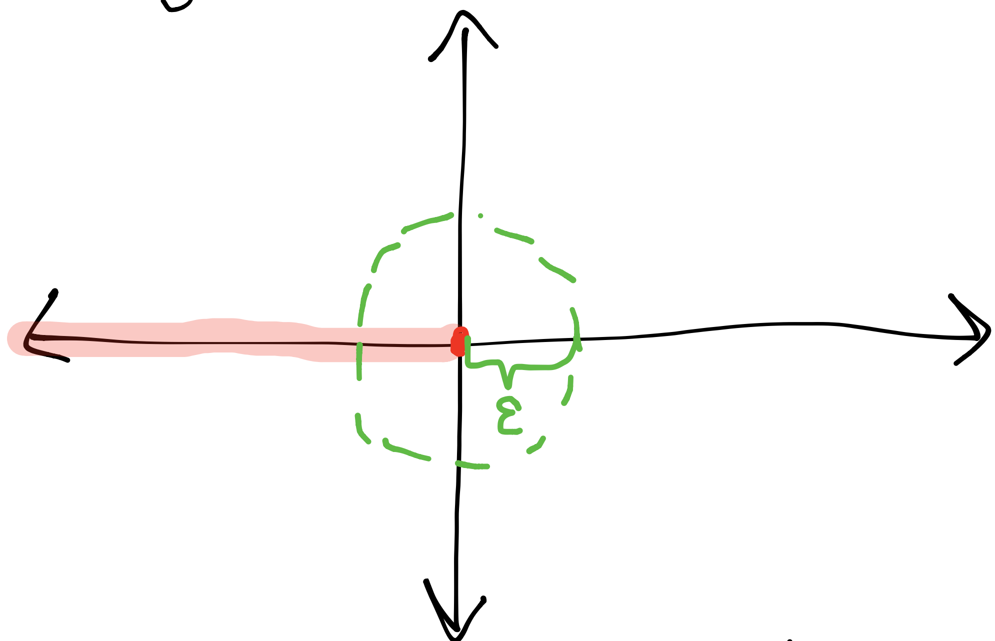


Ex] $f(z) = \dfrac{z^2 - 1}{z^2 + 1}$

Not analytic at $z = \pm i$, but f is analytic in some punctured NBHD of $z = i$ and some punctured NBHD of $z = -i$.



Ex) The principal branch $\mathrm{Log}(z)$ of $\log(z)$ has a singularity at $z=0$,



but not an isolated singularity.

Suppose we have $f(z)$ which has an isolated singularity at z_0. Let $R_2 > 0$, and consider the Laurent expansion

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

$A: 0 < |z-z_0| < R_2$.

Let γ be a positively-oriented simple closed curve in A. Then

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

$(-\infty < n < \infty)$.

When $n=-1$, we have

$$a_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)\,dz.$$

Def] We call a_{-1} the residue of f at z_0.

Ex] Consider

$$\int_{|z|=1} \frac{e^z}{z^4}\,dz.$$

Note that

$$\frac{e^z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= \frac{1}{z^4}\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right)$$

$$= \frac{1}{z^4}+\frac{1}{z^3}+\frac{1}{2z^2}+\frac{1}{6z}+\frac{1}{24}+\cdots$$

$$a_{-4}(z-0)^{-4} \qquad a_{-1}(z-0)^{-1} \qquad a_0(z-0)^0$$

Thus $a_{-1} = \frac{1}{6} = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^z}{z^4}\,dz$

Hence the integral equals

$\pi i/3$.

Ex] $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

$$\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, z^{2n}},$$

$$z \neq 0.$$

Coefficient of $\frac{1}{z}$ is 0, so

$$\int_{|z|=1} \cos\left(\frac{1}{z}\right) dz = 0.$$

(more examples on my website)

$\boxed{\text{Thm}}$ (Cauchy's residue theorem)

Let γ be a simple closed curve (positively oriented). If f is analytic on γ, and f is analytic interior to γ except at isolated singularities $z_1, z_2, \ldots, z_n$, then

$$\int_{\gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

where $\operatorname*{Res}_{z=z_k} f(z)$ is the residue of f at z_k.

Pf] Next time.