# Recap of Cauchy's residue thm:

If $f$ is holomorphic in a region of the shape

$$\{z : 0 < |z - z_0| < \varepsilon\}$$

but if is <u>not</u> holom. at $z_0$ itself, then $f$ has an isolated singularity at $z_0$.

## Cauchy Residue Thm:

Let $f$ be analytic on and interior to a simple closed contour $\gamma$ except possibly at finitely many isolated singularities

$z_1, z_2, \ldots, z_n,$

each lying interior to $\gamma$, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f,$$

where $\operatorname{Res}_{z=z_k} f$ is the $a_{-1}$ coefficient in the Laurent expansion

$$\sum_{n=-\infty}^{\infty} a_n (z-z_k)^n,$$

$$0 < |z-z_k| < \varepsilon,$$

where $\varepsilon$ is the radius of the punctured NBHD in $f$ is holom. near $z_k$.

This implies Cauchy-Goursat:
If $f$ is analytic on and interior to $\gamma$, then in a NBHD of any pt $z_0$ interior to $\gamma$, the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

will have $a_n = 0$ for $n \leq -1$.
This is because, by hypothesis, $f$ is analytic at $z_0$.

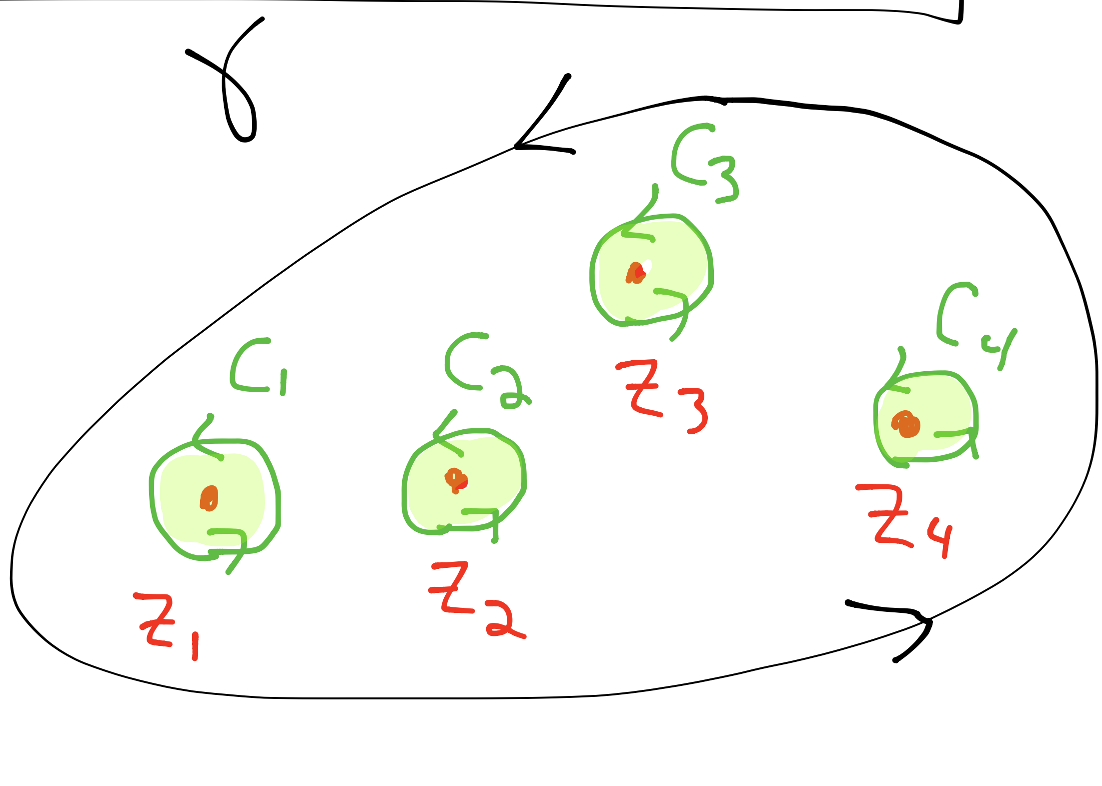
That is, $f$ has no

isolated singularities interior to $\gamma_1$ so

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f = 0$$

since the sum is empty.

## Pf of Residue Thm



# Principle of deformation of curves :

$$\int_{\gamma} f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

Recall that the residue at $z = z_k$ is given by

$$2\pi i \int_{C_k} f(z)\,dz.$$

(Recall that this is $a_{-1}$ in $\sum_{n=-\infty}^{\infty} a_n (z-z_k)^n$.)

Thus

$$\sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f.$$

$\square$

**Ex** $$\int_{|z|=2} \frac{4z-5}{z(z-1)}\,dz.$$

Note: Integrand is not holom. at 2 points interior to $\gamma$ ($|z|=2$):

$$z=0, \quad z=1.$$

By the residue thm,

$$\int_{|z|=2} \frac{4z-5}{z(z-1)}\,dz$$

$$= 2\pi i \left( \operatorname*{Res}_{z=0} \frac{4z-5}{z(z-1)} + \operatorname*{Res}_{z=1} \frac{4z-5}{z(z-1)} \right).$$

Residue at $z=0$:

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z} \cdot \frac{1}{z-1}$$

$$= -\frac{4z-5}{z} \cdot \frac{1}{1-z}$$

$$= -\frac{(4z-5)}{z} \cdot \sum_{n=0}^{\infty} z^n \quad (|z|<1)$$

$$= \left(\frac{5}{z} - 4\right) \cdot \sum_{n=0}^{\infty} z^n$$

$$= \frac{5}{z} \cdot \sum_{n=0}^{\infty} z^n - 4\sum_{n=0}^{\infty} z^n$$

$$= \frac{5}{z} + 5 + 5z + 5z^2 + \cdots$$
$$- 4 - 4z - 4z^2 \cdots$$

$$= \frac{5}{z} + 1 + z + z^2 + \cdots$$

$$= \frac{5}{z-0} + 1 + (z-0) + (z-0)^2 + \cdots$$

.

Thus $\operatorname*{Res}_{z=0} \dfrac{4z-5}{z(z-1)} = 5.$

$$= a_{-1}.$$

## Residue at $z=1$:

$$\frac{4z-5}{z(z-1)} = \frac{4(z-1)-1}{(1+(z-1))(z-1)}$$

$$= \frac{4(z-1)-1}{z-1} \cdot \frac{1}{1+(z-1)}$$

$$= \frac{4(z-1)-1}{z-1} \cdot \frac{1}{1-(-(z-1))}$$

$$= \frac{4(z-1)-1}{z-1} \cdot \sum_{n=0}^{\infty} (-(z-1))^n,$$

$$0 < |z-1| < 1.$$

Now, I can proceed like we did for the residue at $z=0$, but now our residue is the coefficient of

$$\frac{1}{z-1}.$$

A small calculation shows

that this coefficient, our residue at $z=1$, equals

$$\operatorname*{Res}_{z=1} \frac{4z-5}{z(z-1)} = -1.$$

Now,

$$\int_{|z|=2} \frac{4z-5}{z(z-1)}\,dz = 2\pi i \left( \operatorname*{Res}_{z=0} \frac{4z-5}{z(z-1)} + \operatorname*{Res}_{z=1} \frac{4z-5}{z(z-1)} \right)$$

$$= 2\pi i\,(5 + (-1))$$

$$= 2\pi i\,(4) = 8\pi i.$$

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Computing contour integrals in the context of the residue thm boils down to understanding Laurent Series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$= T(z) + P(z),$$

Where

$$T(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

(Taylor Series) and

$$P(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0)^n,$$

(principal part).

Principal parts come in three flavors;

| Principal part | Type of isolated singularity |
| --- | --- |
| zero terms | removable singularity |
| m terms, $1 \leq m < \infty$ | pole of order m |
| infinitely many terms | essential singularity |

See the online notes for examples of

each type of singularity.