Residue Thm

Let f be analytic on and interior to a simple closed curve γ, except possibly at isolated singularities $z_1, z_2, \ldots, z_n$.

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f,$$

where $\operatorname*{Res}_{z=z_k} f$ is the notation for the residue of f at $z=z_k$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_k)^n,$$

$0 < |z - z_k| < R,$

and the residue at $z = z_k$ is precisely a_{-1}.

Isolated singularities come in 3 flavors:

1) Removable singularities

ex: $f(z) = \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \dots$

$$= \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

$$= \begin{cases} \frac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

Thus $\frac{e^z - 1}{z}$ has a removable singularity at $z=0$.

More generally, if

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_k)^n}_{T(z)} + \underbrace{\sum_{n=-\infty}^{-1} a_n (z-z_k)^n}_{P(z)}$$

(principal part)

and $P(z)$ is identically zero, then $f(z)$ has a removable singularity at z_k.

2) Essential Singularity

ex) $1 + \underbrace{\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots}_{p(z)}$

$= e^{1/z}$ at $z = 0$.

Here, the principal part has infinitely many terms.

3) Pole of finite order m, with $1 \leq m < \infty$.

ex) $\underbrace{\frac{1}{z}}_{p(z)} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots$

$= \frac{e^z}{z}$

..

at $z=0$. Since there is <u>one</u> term in the principal part, e^z/z has a pole of order <u>one</u>.

Order m of the pole = the largest degree of any term in the principal part.

<u>Ex</u> $f(z) = \frac{1}{z^3} + 1 + z + \ldots$

has a pole of order 3.

Q) Given $f(z)$ with a pole of order m at $z = z_0$, how do we calculate the residue?

2 approaches

1. Use traditional power series manipulations, much like the HW so far. This gives the full Laurent series, and the residue is a_{-1}.

2. Do the following:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{a_{-1}}{z-z_0} + \dots + \frac{a_{-m}}{(z-z_0)^m}$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

$$+ a_{-1}(z-z_0)^{m-1} + \dots + a_{-m}$$

$$\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

$$= \sum_{n=0}^{\infty} (n+m)(n+m-1)\cdots(n+2)\, a_n (z-z_0)^{n+1}$$

$n+m-(m-1) = n+1$

$$+ a_{-1}(m-1)!$$

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

$$= a_{-1}(m-1)!$$

Conclusion:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z).$$

(always holds for computing residues at poles of order m)

This fails for essential singularities.

The above calculation shows:

Thm Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of order $m \geq 1$ of f if and only if there exists a function $\phi(z)$, holom. and nonzero in a NBHD of z_0, such that

$$f(z) = \phi(z)(z-z_0)^{-m},$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

<u>Ex</u> Let $m \geq 1$ be an integer. Then

$$\operatorname*{Res}_{z=0} \frac{e^z}{z^m}$$

$$= \frac{1}{(m-1)!} \lim_{z \to 0} \frac{d^{m-1}}{dz^{m-1}} \cancel{(z-0)^m} \frac{e^z}{\cancel{z^m}}$$

$$= \frac{1}{(n-1)!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} e^z$$

$$= \frac{1}{(n-1)!} \lim_{z \to 0} e^z$$

$$= \frac{1}{(n-1)!}.$$

Please have a look at the notes for some additional exposition on these themes.