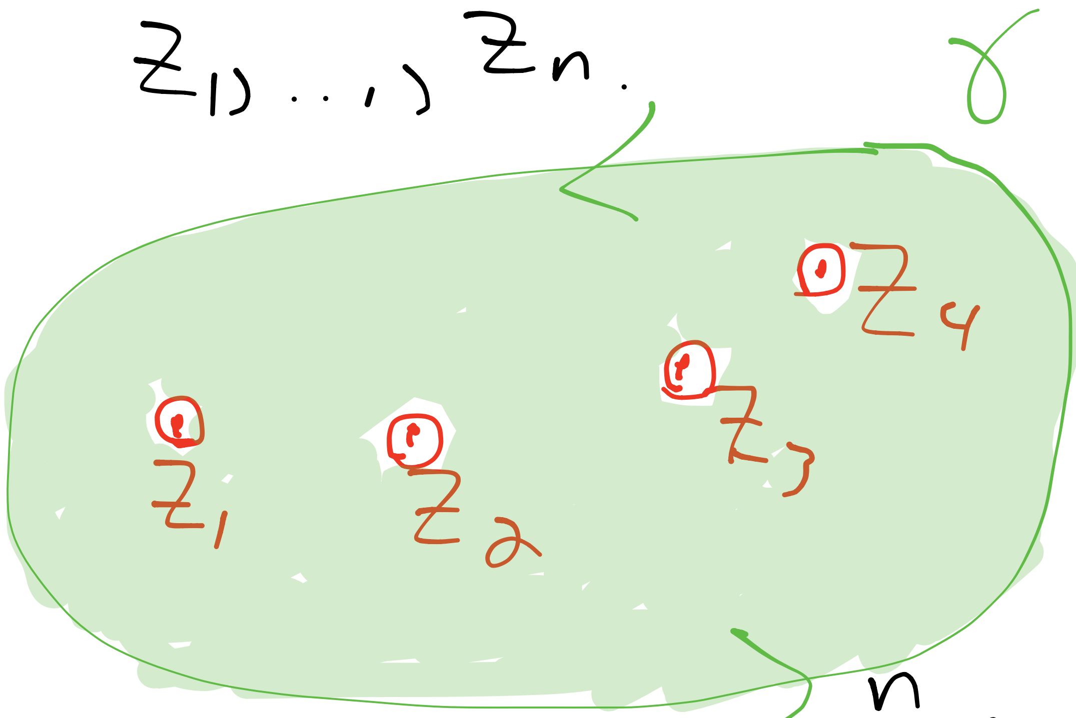
Residue Thm

Suppose that f is analytic on and interior to a simple closed curve γ except at isolated singularties $z_1, \dots, z_n$.



Then $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f.$

Application: Compute integrals of a single real variable.

Lemma Suppose that f has an isolated singularity z_0, and in a deleted NBHD of z_0, we have

$$f(z) = \frac{g(z)}{h(z)},$$

$h(z_0) = 0$, $h'(z_0) \neq 0$.

then $\displaystyle\operatorname*{Res}_{z=z_0} f = \frac{g(z_0)}{h'(z_0)}$.

Pf] Follows from l'Hopital's rule. See the notes.

Example] $I = \displaystyle\int_0^{2\pi} \frac{dt}{2+\sin(t)}$.

Change of variables:

$$z = e^{it}$$

$$dz = ie^{it}\,dt = iz\,dt$$

$$\frac{dz}{iz} = dt$$

$$z = e^{it} \longrightarrow \operatorname{Im}(z) = \sin(t)$$

$$\longrightarrow \frac{z - \bar{z}}{2i} = \sin(t)$$

since $\bar{z} = e^{-it} = \dfrac{1}{e^{it}} = \dfrac{1}{z}$,

$$\longrightarrow \frac{z - z^{-1}}{2i} = \sin(t)$$

With this change of variables, we see that we are integrating (with respect to z) along the unit circle centered at $z=0$:

$$I = \int_{|z|=1} \frac{1}{2 + \frac{z - z^{-1}}{2i}} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{2iz + \frac{z^2 - 1}{2}} dz$$

$$= \int_{|z|=1} \frac{2}{z^2 + 4iz - 1} dz$$

$$= 2 \int_{|z|=1} \frac{1}{z^2 + 4iz - 1} dz$$

Quadratic formula:

$$z^2 + 4iz - 1 = (z - (\frac{-4i + \sqrt{-16+4}}{2}))$$

$$\cdot\left(z-\left(\frac{-4i-\sqrt{-16+4}}{2}\right)\right)$$

$$=\left(z-\left(-2i+i\sqrt{3}\right)\right)$$

$$\cdot\left(z-\left(-2i-i\sqrt{3}\right)\right)$$

The first root is interior to $|z|=1$, and the second is not.

Thus $I=$

$$2\cdot 2\pi i\cdot \operatorname*{Res}_{z=-2i+i\sqrt{3}} \frac{1}{z^2+4iz-1}.$$

<u>Option 1:</u> Compute Laurent

series and find the residue

<u>Option 2</u>: Use the lemma from the beginning of the lecture: $g(z)=1$ $h(z)= z^2+4iz-1$

$$z_0=-2i+i\sqrt{3}.$$

Thus

$$2\cdot 2\pi i \operatorname*{Res}_{z=-2i+i\sqrt{3}} \frac{1}{z^2+4iz-1}$$

$$= \cancel{2}\cdot 2\pi i \cdot \frac{1}{\cancel{2z+4i}}\Bigg|_{z=-2i+i\sqrt{3}}$$

$$= 2\pi i \cdot \frac{1}{z+2i}\Bigg|_{z=-2i+i\sqrt{3}}$$

$$= 2\pi i \cdot \frac{1}{\cancel{-2i}+i\sqrt{3}+\cancel{2i}}$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}\sqrt{3}}$$

$$= 2\pi/\sqrt{3} = I .$$

This is a special case of a more general principle:

Let $R(x,y) = \dfrac{p(x,y)}{q(x,y)}$

be a rational function given by a ratio of two polynomials $p(x,y)$ and $q(x,y)$, where $R(x,y)$ has no poles on the circle $x^2+y^2=1$.

Then

$$\int_0^{2\pi} R(\cos(t), \sin(t))\, dt$$

$$= \int_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$

Residue Thm

$$= 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right),$$

where $z_1, \ldots, z_n$ are the isolated singularities of $\frac{1}{iz} \cdot R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ inside the contour $|z|=1$.